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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR FRACTIONAL p-LAPLACIAN EQUATIONS WITH SIGN-CHANGING POTENTIAL

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Abstract. In this article, we prove the existence of infinitely many solutions for the fractional p-Laplacian equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N$$

where $s \in (0,1)$, $2 \le p < \infty$. Based on a direct sum decomposition of a space E^s, we investigate the multiplicity of solutions for the fractional p-Laplacian equation in $\mathbb{R}^N$. The potential V is allowed to be sign-changing, and the primitive of the nonlinearity f is of super-p growth near infinity in u and allowed to be sign-changing. Our assumptions are suitable and different from those studied previously.

1. Introduction and statement of main results

We consider the fractional p-Laplacian equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $(-\Delta)_p^s$ denotes the fractional p-Laplacian operator, $0 < s < 1$, $2 \le p < \infty$, $f : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}$, $V : \mathbb{R}^N \mapsto \mathbb{R}$, f and V are allowed to be sign-changing. Equation (1.1) driven by the fractional Laplacian arises in various areas and different applications, such as phase transitions, finance, stratified materials, flame propagation, ultra-relativistic limits of quantum mechanics, and water waves. For more detailed introductions and applications, we refer the reader to [16, 17]. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for (1.1) have been extensively investigated in the literature over the past several decades. See e.g., [8, 9, 10, 11, 12, 28, 29, 32, 35] and the references quoted in them.

For $p = 2$, (1.1) reduces to the so-called fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Equation (1.2) arises in the study of the nonlinear fractional Schrödinger equation

$$i\frac{\partial \varphi}{\partial t} = (-\Delta)^s \varphi + W(x)\varphi - f(x,|\varphi|)\varphi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

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Most of the references deal with the (1.2) with the potential $\inf_{x\in\mathbb{R}^N} V(x)>0$, and the others handled the case where $V(x)$ is sign-changing. For the second case, the classical proof is based on the following classical condition which was introduced by Ambrosetti and Rabinowitz in [1]: (AR) there exists $\mu>2$ such that

$$0<\mu F(x,t)\le tf(x,t),\quad t\neq 0,$$

where $F(x,t)=\int_0^t f(x,\tau)d\tau$. In [28], the authors used the concentration compactness principle to show that (1.2) ($V(x)\equiv 1$) has at least two nontrivial radial solutions without the (AR) condition. In [10, 32, 35], the authors used variant Fountain Theorems and the $\mathbb{Z}_2$ version of Mountain Pass Theorem to establish some new existence theorems on infinitely many nontrivial high or small energy solutions for (1.2).

Furthermore, there are also many classical and fantastic studies on nonlocal fractional problems, see for example, [5, 6, 13, 14, 20, 21, 22, 23, 26, 27, 36, 37, 38]. In [21], the authors study the existence of multiple ground state solutions for a class of parametric fractional Schrödinger equations. In [36], the authors obtain the existence of infinitely many weak solutions for equations driven by nonlocal integrodifferential operators with homogeneous Dirichlet boundary conditions. Recently, some authors have been concerned about the general case (i.e., $p\neq 2$) of (1.1), see for example [7, 30, 33, 34]. In [33], by using Mountain Pass Theorem with Cerami condition, Torres established the existence of weak solutions for (1.1), in which the nonlinearity $f(x,u)$ is subcritical and p-superlinear, and the potential $V(x)$ satisfies coercive condition at infinity. For general case $p>2$, if $V(x)$ is a sign-changing potential, (1.1) is far more difficult as $((-\Delta)_p^s+V)$ is no longer a self-adjoint and so a complete description of its spectrum is not available. For these reasons, only a few papers have treated this case so far. In [7], by applying Mountain Pass Theorem, Cheng considered this case and established the existence of one nontrivial solutions for (1.1).

In a recent paper [2], Vincenzo ambrosio used a variant of the Fountain Theorem to prove the existence of infinitely many nontrivial weak solutions for (1.1), where the following the assumptions on V and f are introduced:

- (A1) $V\in C(\mathbb{R}^N,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x)>-\infty$;
- (A2) there exists a constant $d_0>0$ such that
$$\lim_{|y|\to+\infty}\operatorname{meas}\{x\in\mathbb{R}^N:|x-y|\le d_0, V(x)\le M\}=0,\quad \forall M>0,$$
where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;
- (A3) $f\in C(\mathbb{R}^N,\mathbb{R})$, and there exist constant $c_1,c_2>0$ and $q\in(p,p_s^*)$ such that
$$|f(x,t)|\le c_1|t|^{p-1}+c_2|t|^{q-1},\quad \forall (x,t)\in\mathbb{R}^N\times\mathbb{R},$$
where $p_s^*=\infty$ if $N\le sp$ and $p_s^*=\frac{Np}{N-sp}$ if $N>sp$, p_s^* is the fractional critical exponent;
- (A4) there exists $\sigma\ge 1$ such that
$$\sigma\mathcal{F}(x,t)\ge\mathcal{F}(x,\tau t),\quad \forall (x,t)\in\mathbb{R}^N\times\mathbb{R},\ \tau\in[0,1],$$
where $\mathcal{F}(x,t):=\frac{1}{p}f(x,t)t-F(x,t)$ and $F(x,t):=\int_0^t f(x,\tau)d\tau$;
- (A5) $F(x,0)\equiv 0, F(x,t)\ge 0$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$ and
$$\lim_{|t|\to+\infty}\frac{F(x,t)}{|t|^p}=+\infty\quad \text{uniformly in } x\in\mathbb{R}^N;.$$

(A6) $f(x,-t) = -f(x,t)$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Specifically, the author established the following theorem in [2].

Theorem 1.1 ([2, Theorem 1])**.** *Assume that* (A1)–(A6) *are satisfied. Then* (1.1) *has infinitely many nontrivial weak solutions.*

We note that the usual condition $f(x,u)/u \to 0$ as $u \to 0$ is not needed in Theorem 1.1. This is the highlights in [2]. Conditions like (A1) and (A2) have been given by Bartsch, Wang and Willem [4]. Condition (A4) is due to Jeanjean [15]. This condition is also used together with a Cerami type argument in singularly perturbed elliptic problems in $\mathbb{R}^N$ with autonomous nonlinearity. Moreover, there are many functions (e.g. $f(x,t) = a|t|^{p-2}t\ln(1+|t|), a > 0$) which satisfy (A4), but do not satisfy the following classical condition:

(AR) there exists $\mu > p$ such that

$$0 < \mu F(x,t) \le tf(x,t), \quad t \neq 0.$$

However, condition (AR) does not imply condition (A4); see the example in [31].

Motivated by the above works, we shall further study the infinitely many nontrivial solutions of (1.1) with sign-changing potential and subcritical p-superlinear nonlinearity. Moreover, we are interested in the case where the potential V and the primitive of f are both sign-changing, which is called a double sign-changing case and prevents us from applying a standard variational argument directly. For the above reasons, just a few papers dealt with such a double sign-changing case as regards (1.1) until now. We will give a direct sum decomposition of the fractional Sobolev space and establish some new theorems on the infinitely many nontrivial solutions of (1.1) with mild assumptions deeply different from those studied in previous related works. For any $\eta > 0$, the Sobolev embedding theorem implies $E^s(B_\eta) \hookrightarrow L^2(B_\eta)$, where $B_\eta = \{x \in \mathbb{R}^N : |x| < \eta\}$. Based on the above fact, we can construct a direct sum decomposition of $E^s(B_\eta)$. As far as we know, there were no such multiplicity results in this situation.

To state our results, we introduce the following assumptions:

(A5') $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty$, a.e. $x \in \mathbb{R}^N$, and there exists $r_0 \ge 0$ such that

$$F(x,t) \ge 0, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \ge r_0;$$

(A7) $\mathcal{F}(x,t) := \frac{1}{p}f(x,t)t - F(x,t) \ge 0$, and there exist $c_0 > 0$ and $\kappa > \max\{1, \frac{N}{ps}\}$ such that

$$|F(x,t)|^\kappa \le c_0|t|^{p\kappa}\mathcal{F}(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \ge r_0;$$

(A8) there exist $\mu > p$ and $\varrho > 0$ such that

$$\mu F(x,t) \le tf(x,t) + \varrho|t|^p, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(A9) there exist $\mu > p$ and $r_1 > 0$ such that

$$\mu F(x,t) \le tf(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \ge r_1;$$

(A10) $f(x,t) = o(|t|^{p-1})$, as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$.

It is easy to check that (A3) and (A9) imply (A8). Now we are ready to state the main results of this paper.

Theorem 1.2. *Assume that* (A1)–(A3), (A5'), (A6), (A7) *are satisfied. Then* (1.1) *possesses infinitely many nontrivial solutions*

Theorem 1.3. *Assume that* (A1)–(A3), (A5'), (A6), (A8) *are satisfied. Then* (1.1) *possesses infinitely many nontrivial solutions.*

Corollary 1.4. *Assume that* (A1)-(A3), (A5'), (A6), (A9) *are satisfied. Then* (1.1) *possesses infinitely many nontrivial solutions.*

Remark 1.5. It is easy to see that (A5') and (A7) are weaker than (A5) and (AR), respectively. In particular, we remove the usual condition (A10), and $F(x,t)$ is allowed to be sign-changing in Theorems 1.2, 1.3 and Corollary 1.4. The role of (AR) is to ensure the boundedness of the Palais-Smale (PS) sequences of the energy functional, it is also significant to construct the variational framework. This is very crucial in applying the critical point theory. However, there are many functions which are superlinear at infinity, but do not satisfy the condition (AR) for any $\mu > p$. For example, the superlinear function $f(x,t)$ for the case $p = 2$:

$$f(x,t) = a(x)t\ln(1+|t|), \tag{1.3}$$

where $0 < \inf_{x\in\mathbb{R}^N} a(x) \le \sup_{x\in\mathbb{R}^N} a(x) < +\infty$. Indeed, (AR) implies that $f(x,u) \ge C|u|^{\mu}$ for some $C > 0$, and so (1.3) not satisfies (AR). It is easy to check function $f(x,t) = b|t|^{p-2}t\ln(1+|t|)$ with $b > 0$ satisfies (A3), (A4), (A5), (A7) and (A10). However, the function

$$f(x,t) = 3t|t|\int_0^t |\tau|^{1+\sin\tau}\tau d\tau + |t|^{4+\sin t}t, \tag{1.4}$$

satisfies (A7) not (A4) for $p = 2$, see [31, Section 3]. In addition, the function

$$f(x,t) = d(x)|t|^{p-1}t[(p+3)t^2 - 2(p+2)t + p + 1] \tag{1.5}$$

satisfies (A3), (A5'), (A6), (A9) and (A10), where

$$0 < \inf_{x\in\mathbb{R}^N} d(x) \le \sup_{x\in\mathbb{R}^N} d(x) < +\infty.$$

One can see that (1.5) satisfies neither (AR) nor (A4), see [27, Section 1].

On the existence of ground state solutions we have the following result.

Theorem 1.6. *Assume* (A1)–(A3), (A5'), (A6), (A7) *and* (A10) *are satisfied. Then* (1.1) *has a ground state solution u_0 such that $J(u_0) = \inf_{u\in\mathcal{M}} J(u)$, where $\mathcal{M} = \{u \neq 0, J'(u) = 0\}$.*

2. Variational setting and proofs the main results

Throughout this section, we make the following assumption instead of (A1):

(A1') $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

Firstly, we give some notation related to the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$. For $0 < s < 1$ and $p \ge 2$, define the so-called Gagliardo seminorm by

$$[u]_{s,p} := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p},$$

where $u : \mathbb{R}^N \mapsto \mathbb{R}$ is a measurable function. Then the fraction Sobolev space $W^{s,p}(\mathbb{R}^N)$ is given by

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : u \text{ is measurable } [u] < \infty\},$$

which can be equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \Big([u]_{s,p}^p + \|u\|_{L^p}^p\Big)^{1/p},$$

where $\|u\|_{L^p}^p = \int_{\mathbb{R}^N} |u(x)|^p dx$. By condition (A1'), we define the fractional Sobolev space with potential $V(x)$ by

$$E^s := \big\{u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\big\}$$

equipped with the norm

$$\|u\|_{E^s} = ([u]_{s,p}^p + \|V^{\frac{1}{p}} u\|_{L^p}^p)^{\frac{1}{p}}.$$

Lemma 2.1 ([24, Theorem 6.5]). *Under assumption* (A1'), *for any $r \in [p, p_s^*]$, the embedding*

$$E^s \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$$

is continuous. In particular, there exist constants $\rho_r > 0$ and $\gamma_r > 0$ such that

$$\|u\|_{L^r} \le \rho_r \|u\|_{W^{s,p}} \le \gamma_r \|u\|_{E^s}, \quad \forall u \in E^s. \tag{2.1}$$

Lemma 2.2 ([2, Lemma 1]). *Under assumptions* (A1') *and* (A2)*, the embedding $E^s \hookrightarrow L^r(\mathbb{R}^N)$ is compact for any $r \in [p, p_s^*)$.*

Now we define a functional J on E^s by

$$\begin{aligned} &J(u)\\ &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &= \frac{1}{p}\|u\|_{E^s}^p - \Phi(u), \end{aligned} \tag{2.2}$$

where $\Phi(u) = \int_{\mathbb{R}^N} F(x,u)dx$, for all $u \in E^s$. We say that $u \in E^s$ is a weak solution for (1.1) if

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(h(x)-h(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uh dx - \int_{\mathbb{R}^N} f(x,u)h dx = 0, \quad \forall h \in E^s. \end{aligned}$$

Note that critical points of J correspond to weak solutions of equation (1.1).

Lemma 2.3 ([7]). *Under assumptions* (A1'), (A3). *The functional $\Phi \in C^1(E^s, \mathbb{R})$ and $\langle \Phi'(u), h\rangle = \int_{\mathbb{R}^N} f(x,u)h dx$ for all $u, h \in E^s$. Moreover $\Phi' : E^s \mapsto E^{s*}$ is weakly continuous.*

The proof of the above lemma is similar to the proof in [7], we omit the proof.

From the above facts, we know J is well defined in E^s and $J \in C^1(E^s, \mathbb{R})$. Moreover,

$$\begin{aligned} \langle J'(u), h\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(h(x)-h(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uh dx - \langle \Phi'(u), h\rangle \end{aligned} \tag{2.3}$$

for all $u, h \in E^s$. Obviously, solutions for equation (1.1) are correspond to critical points of the energy functional J.

Define the nonlinear operator $\Upsilon\colon E^s \times E^s \mapsto \mathbb{R}$ by

$$\Upsilon(u,h)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(h(x)-h(y))}{|x-y|^{N+ps}}\,dx\,dy,\quad \forall u,h\in E^s.$$

We say that $I\in C^1(X,\mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\}\subset X$ such that

$$I(u_n)\to c,\quad \|I'(u_n)\|(1+\|u_n\|)\to 0 \tag{2.4}$$

has a convergent subsequence.

Lemma 2.4 ([31, 3])**.** *Let X be an infinite dimensional Banach space, $X=Y\oplus Z$, where Y is finite dimensional. If $I\in C^1(X,\mathbb{R})$ satisfies the $(C)_c$-condition for all $c>0$, and*

1. *$I(0)=0$, $I(-u)=I(u)$ for all $u\in X$;*
2. *there exist constants ρ, $\alpha>0$ such that $I|_{\partial B_\rho\cap Z}\geq\alpha$;*
3. *for any finite dimensional subspace $\widetilde{X}\subset X$, there is $R=R(\widetilde{X})>0$ such that $I(u)\leq 0$ on $\widetilde{X}\backslash B_R$. Then I possesses an unbounded sequence of critical values.*

Lemma 2.5. *Under assumptions* (A1'), (A2), (A3), (A5'), (A7)*, any sequence of $\{u_n\}\subset E^s$ satisfying*

$$J(u_n)\to c>0,\quad \langle J'(u_n),u_n\rangle\to 0 \tag{2.5}$$

is bounded in E^s.

Proof. To achieve our goals, arguing by contradiction, suppose that $\|u_n\|_{E^s}\to\infty$. Let $v_n=\frac{u_n}{\|u_n\|_{E^s}}$, then $\|v_n\|_{E^s}=1$ and $\|v_n\|_{L^r}\leq\gamma_r\|v_n\|_{E^s}=\gamma_r$ for $p\leq r<p_s^*$. Observe that for n large

$$c+1\geq J(u_n)-\frac{1}{p}\langle J'(u_n),u_n\rangle=\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)dx, \tag{2.6}$$

where $\mathcal{F}(x,u_n)=\frac{1}{p}f(x,u_n)u_n-F(x,u_n)\geq 0$. By condition (A3), we obtain

$$|F(x,t)|\leq\frac{c_1}{p}|t|^p+\frac{c_2}{q}|t|^q. \tag{2.7}$$

For $0\leq a<b$, let

$$\Omega_n(a,b)=\{x\in\mathbb{R}^N: a\leq|u_n(x)|<b\}. \tag{2.8}$$

Going if necessary to a subsequence, we may assume that

$$\begin{gathered} v_n\rightharpoonup v\quad\text{in } E^s\\ v_n\to v\quad\text{in } L^r(\mathbb{R}^N),\ p\leq r<p_s^*\\ v_n(x)\to v(x)\quad\text{a.e. on } \mathbb{R}^N. \end{gathered} \tag{2.9}$$

Now, we consider two possible cases: $v=0$ or $v\neq 0$.

Case 1: if $v=0$, then $v_n\to v$ in $L^r(\mathbb{R}^N)$, $p\leq r<p_s^*$, and $v_n\to 0$ a.e. on $\mathbb{R}^N$. Note that

$$J(u_n)=\frac{1}{p}\|u_n\|_{E^s}^p-\int_{\mathbb{R}^N}F(x,u_n)dx.$$

This, together with (2.5), implies that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{|F(x,u_n)|}{\|u_n\|_{E^s}^p}dx\geq\frac{1}{p}. \tag{2.10}$$

On the other hand, by (2.7), one has

$$\begin{aligned}
\int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{\|u_n\|_{E^s}^p}dx &= \int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^p}\frac{|u_n|^p}{\|u_n\|_{E^s}^p}dx \\
&= \int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^p}|v_n|^p dx \\
&\leq \Big(\frac{c_1}{p}r_0 + \frac{c_2}{q}{r_0}^{q-p}\Big)\int_{\Omega_n(0,r_0)} |v_n|^p dx \\
&\leq \Big(\frac{c_1}{p}r_0 + \frac{c_2}{q}{r_0}^{q-p}\Big)\|v_n\|_{L^p}^p \to 0, \quad \text{as } n\to\infty.
\end{aligned} \tag{2.11}$$

Set $\kappa' = \kappa/(\kappa-1)$, $\kappa > \max\{1, N/ps\}$, then $p\kappa' \in [p, p_s^*)$. Hence, from (A7), (2.6) and (2.9), one has

$$\begin{aligned}
&\int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{\|u_n\|_{E^s}^p}dx \\
&= \int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{|u_n|^p}|v_n|^p dx \\
&\leq \Big[\int_{\Omega_n(r_0,+\infty)} \Big(\frac{|F(x,u_n)|}{|u_n|^p}\Big)^{\kappa} dx\Big]^{1/\kappa}\Big[\int_{\Omega_n(r_0,+\infty)} |v_n|^{p\kappa'}dx\Big]^{1/\kappa'} \\
&\leq {c_0}^{1/\kappa}\Big[\int_{\Omega_n(r_0,+\infty)} \mathcal{F}(x,u_n)dx\Big]^{1/\kappa}\Big[\int_{\Omega_n(r_0,+\infty)} |v_n|^{p\kappa'}dx\Big]^{1/\kappa'} \\
&\leq [c_0(c+1)]^{1/\kappa}\Big[\int_{\Omega_n(r_0,+\infty)} |v_n|^{p\kappa'}dx\Big]^{1/\kappa'} \\
&\leq [c_0(c+1)]^{1/\kappa}\|v_n\|_{L^{p\kappa'}}^p \to 0, \quad \text{as } n\to\infty.
\end{aligned} \tag{2.12}$$

Combining (2.11) with (2.12), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|_{E^s}^p}dx \\
&= \int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^p}|v_n|^p dx + \int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{|u_n|^p}|v_n|^p dx \to 0, \quad \text{as } n\to\infty,
\end{aligned}$$

which contradicts (2.10).

Case 2: $v \neq 0$. Set $A := \{x \in \mathbb{R}^N : v(x) \neq 0\}$, then $\text{meas}(A) > 0$. For a.e. $x \in A$, we have $\lim_{n\to\infty}|u_n(x)| = +\infty$. Hence $A \subset \Omega_n(r_0,+\infty)$ for large $n \in \mathbb{N}$. It follows from (A3), (A5'), (2.9) and Fatou's Lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|_{E^s}^p} = \lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|_{E^s}^p} \\
&= \lim_{n\to\infty} \Big[\frac{1}{p} - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|_{E^s}^p}\Big] \\
&\le \lim_{n\to\infty} \Big[\frac{1}{p} + \int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^p}|v_n|^p dx \\
&\quad - \int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{|u_n|^p}|v_n|^p dx\Big] \\
&\le \frac{1}{p} + \limsup_{n\to\infty} \int_{\Omega_n(0,r_0)} (\frac{c_1}{p} + \frac{c_2}{q}|u_n|^{q-p})|v_n|^p dx \\
&\quad - \liminf_{n\to\infty} \int_{\Omega_n(r_0,+\infty)} \frac{|F(x,u_n)|}{|u_n|^p}|v_n|^p dx \\
&\le \frac{1}{p} + (\frac{c_1}{p} + \frac{c_2}{q}{r_0}^{q-p})\gamma_p^p \\
&\quad - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{|u_n|^p}[\chi_{\Omega_n(r_0,+\infty)}(x)]|v_n|^p dx \\
&\le \frac{1}{p} + (\frac{c_1}{p} + \frac{c_2}{q}{r_0}^{q-p})\gamma_p^p \\
&\quad - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{|F(x,u_n)|}{|u_n|^p}[\chi_{\Omega_n(r_0,+\infty)}(x)]|v_n|^p dx = -\infty,
\end{aligned} \tag{2.13}
$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E^s. □

Lemma 2.6. *Under assumptions* (A1'), (A2), (A3), (A5'), (A7)*, any* $(C)_c$*-sequence of* J *has a convergent subsequence in* E^s*.*

Proof. Let $\{u_n\}$ be a $(C)_c$-sequence of J, then

$$
J(u_n) \to c, \quad \|J'(u_n)\|(1+\|u_n\|_{E^s}) \to 0, \quad \sup_{n\in\mathbb{N}} \|u_n\|_{E^s} < +\infty. \tag{2.14}
$$

Lemma 2.5 implies that $\{u_n\}$ is bounded in E^s. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E^s. By Lemma 2.2, $u_n \to u$ in $L^r(\mathbb{R}^N)$ for $p \le r < p_s^*$. By a calculation, it follows from (2.14) that

$$
\begin{aligned}
o(1) &= \langle J'(u_n) - J'(u), u_n - u\rangle \\
&= \Upsilon(u_n, u_n - u) - \Upsilon(u, u_n - u) \\
&\quad + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \\
&\quad - \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle.
\end{aligned} \tag{2.15}
$$

Firstly, to prove our results, we need recall the well-known Simion inequality

$$
(|a|^{p-2}a - |b|^{p-2}b)(a-b) \ge k_p|a-b|^p, \quad k_p > 0, \quad \forall a,\, b \in \mathbb{R} \tag{2.16}
$$

for $p \ge 2$. By (2.16), we have

$$
\Upsilon(u_n, u_n - u) - \Upsilon(u, u_n - u) \ge k_p \Upsilon(u_n - u, u_n - u) = k_p[u_n - u]_{s,p}^p, \tag{2.17}
$$

$$
\int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \ge k_p \int_{\mathbb{R}^N} V(x)|u_n - u|^p dx. \tag{2.18}
$$

Secondly, in view of (A3), by the Hölder inequality, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\left[f(x,u_n)-f(x,u)\right](u_n-u)dx\Big|\\
&\le (c_1+c_2)\int_{\mathbb{R}^N}\left[|u_n|^{p-1}+|u|^{p-1}+|u_n|^{q-1}+|u|^{q-1}\right]|u_n-u|dx\\
&\le (c_1+c_2)[(\|u_n\|_{L^p}^{p-1}+\|u\|_{L^p}^{p-1})\|u_n-u\|_{L^p}\\
&\quad+(\|u_n\|_{L^q}^{q-1}+\|u\|_{L^q}^{q-1})\|u_n-u\|_{L^q}],
\end{aligned}$$

which implies that

$$\langle \Phi'(u_n)-\Phi'(u),u_n-u\rangle\to 0,\quad \text{as } n\to\infty. \tag{2.19}$$

Finally, the combination of (2.15)-(2.19) implies

$$o(1)\ge k_p\|u_n-u\|_{E^s}^p+o(1). \tag{2.20}$$

Then

$$u_n\to u\quad \text{in } E^s,\quad \text{as } n\to\infty.$$

This completes the proof. □

Lemma 2.7. *Under assumptions* (A1'), (A2), (A3), (A5'), (A8)*, every* $(C)_c$*-sequence of* J *has a convergent subsequence in* E^s*.*

Proof. Employing Lemma 2.6, we only prove that $\{u_n\}$ is bounded in E^s. Arguing by contradiction, we suppose that $\|u_n\|_{E^s}\to\infty$. Let $v_n=\frac{u_n}{\|u_n\|_{E^s}}$, then $\|v_n\|_{E^s}=1$ and $\|v_n\|_{L^r}\le\gamma_r\|v_n\|_{E^s}=\gamma_r$ for $p\le r<p_s^*$. Passing to a subsequence, we may assume that $v_n\rightharpoonup v$ in E^s, by Lemma 2.2, $v_n\to v$ in $L^r(\mathbb{R}^N)$, $p\le r<p_s^*$, and $v_n\to v$ a.e. on $\mathbb{R}^N$. By (2.2), (2.3), (2.4) and (A8), one has

$$\begin{aligned}
c+1&\ge J(u_n)-\frac{1}{\mu}\langle J'(u_n),u_n\rangle\\
&=\frac{\mu-p}{p\mu}\|u_n\|_{E^s}^p+\int_{\mathbb{R}^N}\Big[\frac{1}{\mu}f(x,u_n)u-F(x,u_n)\Big]dx\\
&\ge\frac{\mu-p}{p\mu}\|u_n\|_{E^s}^p-\frac{\varrho}{\mu}\|u_n\|_{L^p}^p,
\end{aligned}$$

for large $n\in\mathbb{N}$, which implies

$$1\le\frac{p\varrho}{\mu-p}\limsup_{n\to\infty}\|v_n\|_{L^p}^p. \tag{2.21}$$

Hence, it follows from (2.21) that $v\ne 0$. By a similar process as in (2.13), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in E^s. The rest proof is the same as that in Lemma 2.6. □

Lemma 2.8. *Under assumptions* (A1'), (A2), (A3), (A5')*, for any finite dimensional subspace* $\widetilde{E}^s\subset E^s$*, we have*

$$J(u)\to-\infty,\quad \textit{as } \|u\|_{E^s}\to\infty,\quad u\in\widetilde{E}^s. \tag{2.22}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\}\subset\widetilde{E}^s$ with $\|u_n\|_{E^s}\to\infty$, there is $M>0$ such that $J(u_n)\ge -M$ for all $n\in\mathbb{N}$. Set $v_n=\frac{u_n}{\|u_n\|_{E^s}}$, then $\|v_n\|_{E^s}=1$. Passing to a subsequence, we may assume that $v_n\rightharpoonup v\in\widetilde{E}^s$. Since $\widetilde{E}^s$ is finite dimensional, then $v_n\to v\in\widetilde{E}^s$ in E^s, $v_n\to v$ a.e. on $\mathbb{R}^N$, and so $\|v\|_{E^s}=1$. Hence, we can deduce a contradiction in a similar way as (2.13). □

Corollary 2.9. *Under assumptions* (A1'), (A2), (A3), (A5')*, for any finite dimensional subspace* $\widetilde{E}^s \subset E^s$*, there is* $R = R(\widetilde{E}^s) > 0$ *such that*

$$J(u) \le 0, \quad \forall u \in \widetilde{E}^s, \quad \|u\|_{E^s} \ge R. \tag{2.23}$$

Lemma 2.10. *Let* $\Omega := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N : 0 \le x_i < \frac{d_0}{\sqrt{N}}, i = 1, 2, \dots, N\}$*. Then there exists a constant* $a_0 \ge 0$ *such that*

$$\Big(\int_\Omega |u|^{p\lambda} dx\Big)^{1/\lambda} \le a_0 \|u\|_{E^s}^p, \tag{2.24}$$

where $\lambda = 2$ *if* $sp < N$ *and* $\lambda = (2N - sp)/(2N - 2sp)$ *if* $sp \le N$*.*

The proof of the above lemma, follows from Lemma 2.1; it is easy to see that (2.24) holds.

Lemma 2.11. *Under assumptions* (A1') *and* (A2)*, for any* $r > 0$ *and* $M > 0$*,*

$$\int_{|x|>r} |u|^p dx \le \Big\{\frac{1}{M} + a_0[\varepsilon_r(M)]^{(\lambda-1)/\lambda}\Big\} \|u\|_{E^s}^p, \quad \forall u \in E^s, \tag{2.25}$$

where

$$\varepsilon_r(M) := \sup_{|y|\ge r} \text{meas}\{x \in \mathbb{R}^N : |x - y| \le d_0, V(x) \le M\}.$$

Proof. Let $\Omega := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N : 0 \le x_i < \frac{d_0}{\sqrt{N}}, i = 1, 2, \dots, N\}$. Then $\Omega \subset B_{d_0}$. Take a sequence $\{z^i\}_{i\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\mathbb{R}^N = \cup_{i\in\mathbb{N}}(z^i + \Omega), \quad (z^i + \Omega) \cap (z^j + \Omega) = \emptyset, \quad \forall i \ne j. \tag{2.26}$$

Let $\Lambda_i = z^i + \Omega$. For any $r > 0$ and $M > 0$, let

$$\begin{aligned} A(r, M) &:= \{x \in \mathbb{R}^N : |x| > r, V(x) > M\}, \\ B(r, M) &:= \{x \in \mathbb{R}^N : |x| > r, V(x) \le M\}. \end{aligned}$$

Then

$$\begin{aligned} \int_{A(r,M)} |u|^p dx &\le \frac{1}{M} \int_{A(r,M)} V(x)|u|^p dx \\ &\le \frac{1}{M} \int_{\mathbb{R}^N} V(x)|u|^p dx \le \frac{1}{M} \|u\|_{E^s}^p. \end{aligned} \tag{2.27}$$

On the other hand, from (2.24), (2.26) and the Hölder inequality, one has

$$\begin{aligned} \int_{B(r,M)} |u|^p dx &= \sum_{i\in\mathbb{N}} \int_{B(r,M)\cap\Lambda_i} |u|^p dx \\ &\le \sum_{i\in\mathbb{N}} [\text{meas}(B(r, M) \cap \Lambda_i)]^{(\lambda-1)/\lambda} \Big(\int_{B(r,M)\cap\Lambda_i} |u|^{p\lambda} dx\Big)^{1/\lambda} \\ &\le [\varepsilon_r(M)]^{(\lambda-1)/\lambda} \sum_{i\in\mathbb{N}} \Big(\int_{B(r,M)\cap\Lambda_i} |u|^{p\lambda} dx\Big)^{1/\lambda} \\ &\le a_0[\varepsilon_r(M)]^{(\lambda-1)/\lambda} \|u\|_{E^s}^p. \end{aligned} \tag{2.28}$$

Both (2.26) and (2.27) imply that (2.25) holds. $\square$

Since $\varepsilon_r(M) \to 0$ as $r \to \infty$, by Lemma 2.11, we can choose $\eta_0 > 0$ such that

$$\int_{|x|>\eta_0} |u|^p dx \le \frac{1}{2pc_1}\|u\|_{E^s}^p, \quad \forall u \in E^s. \tag{2.29}$$

Let $\{e_j\}$ be a total orthonormal basis of $L^2(B_{\eta_0})$ and define $X_j = \mathbb{R}e_j, j \in \mathbb{N}$,

$$Y_k = \oplus_{j=1}^k X_j,\ Z_k = \oplus_{j=k+1}^{\infty} X_j, \quad K \in \mathbb{N}. \tag{2.30}$$

Lemma 2.12. *Under assumptions* (A1') *and* (A2)*, for* $p \le r < p_s^*$,

$$\beta_k^s := \sup_{u\in Z_k, \|u\|_{E^s(B_{\eta_o})}=1} \|u\|_{L^r(B_{\eta_0})} \to 0, \quad \text{as } k \to \infty. \tag{2.31}$$

Proof. Note that $E^s(B_{\eta_0}) \hookrightarrow L^r(B_{\eta_0})$ for $1 \le r < p_s^*$. It is clear that $0 < \beta_{k+1}^s \le \beta_k^s$, and so that $\beta_k^s \to \beta^s \ge 0, k \to \infty$. For every $k \ge 0$, there exists $u_k \in Z_k$ such that $\|u_k\|_{E^s(B_{\eta_0})} = 1$ and $\|u_k\|_{L^r(B_{\eta_0})} > \beta_k^s/2$. By definition of Z_k, $u_k \rightharpoonup 0$ in $L^2(B_{\eta_0})$, and so $u_k \rightharpoonup 0$ in $E^s(B_{\eta_0})$. Lemma 2.2 implies that $u_k \to 0$ in $L^r(B_{\eta_0})$. Thus we have proved that $\beta^s = 0$. $\square$

By Lemmas 2.1 and 2.12, for all $u \in Z_m \cap W^{s,p}(B_{\eta_0})$, we can take an integer $m \ge 1$ such that

$$\begin{aligned}
&\int_{|x|\le\eta_0} |u|^p dx \\
&\le \frac{1}{2pc_1}\Big[\int_{|x|\le\eta_0}\int_{|x|\le\eta_0} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{|x|\le\eta_0} V(x)|u|^p dx\Big]
\end{aligned} \tag{2.32}$$

Let $\zeta(x) = 0$ if $|x| \le \eta_0$ and $\zeta(x) = 1$ if $|x| > \eta_0$. Define

$$Y = \{(1-\zeta)u : u \in E^s, (1-\zeta)u \in Y_m\}, \tag{2.33}$$

$$Z = \{(1-\zeta)u : u \in E^s, (1-\zeta)u \in Z_m\} + \{\zeta v : v \in E^s\}. \tag{2.34}$$

Then Y and Z are subspaces of E^s, and $E^s = Y \oplus Z$.

Lemma 2.13. *Under assumptions* (A1'), (A2), (A3)*, there exist constants* $\rho, \alpha > 0$ *such that* $J|_{\partial B_\rho \cap Z} \ge \alpha$.

Proof. By (2.29), (2.32) and (2.34), we have

$$\begin{aligned}
\|u\|_{L^p}^p &= \int_{|x|\le\eta_0} |u|^p dx + \int_{|x|>\eta_0} |u|^p dx \\
&\le \frac{1}{2pc_1}\Big[\int_{|x|\le\eta_0}\int_{|x|\le\eta_0} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad + \int_{|x|\le\eta_0} |V(x)^{1/p}u|^p dx\Big] + \frac{1}{2pc_1}\|u\|_{E^s}^p \\
&\le \frac{1}{2pc_1}\|u\|_{E^s}^p + \frac{1}{2pc_1}\|u\|_{E^s}^p \\
&\le \frac{1}{pc_1}\|u\|_{E^s}^p, \quad \forall u \in Z.
\end{aligned} \tag{2.35}$$

Hence, it follows from (A3), (2.1), (2.2) and (2.35) that

$$J(u) = \frac{1}{p}\|u\|_{E^s}^p - \Phi(u)$$

$$
\begin{aligned}
&\geq \frac{1}{p}\|u\|_{E^s}^p - \frac{c_1}{p}\|u\|_{L^p}^p - \frac{c_2}{q}\|u\|_{L^q}^q \\
&\geq \frac{1}{p}\|u\|_{E^s}^p - \frac{1}{p^2}\|u\|_{E^s}^p - \frac{c_2}{q}\gamma_q^q\|u\|_{E^s}^q \\
&\geq \frac{p-1}{p^2}\|u\|_{E^s}^p - \frac{c_2}{q}\gamma_q^q\|u\|_{E^s}^q .
\end{aligned}
$$

Since $p < q$, the assertion follows. □

By (A1), there exists a constant $V_0 > 0$ such that

$$
\widetilde{V}(x) = V(x) + V_0 \geq 1, \quad \forall x \in \mathbb{R}^N .
$$

Let

$$
\widetilde{f}(x,u) = f(x,u) + V_0|u|^{p-2}u.
$$

Then, it is easy to verify the following lemma.

Lemma 2.14. *Equation* (1.1) *is equivalent to the problem*

$$
(-\Delta)_p^s u + \widetilde{V}(x)|u|^{p-2}u = \widetilde{f}(x,u), \quad forall x \in \mathbb{R}^N . \tag{2.36}
$$

Proof of Theorem 1.2. Let $X = E^s$, Y and Z be defined by (2.33) and (2.34), clearly, $\widetilde{f}$ satisfies (A3), (A5'), (A6) and (A7), Lemmas 2.5, 2.6 and 2.13 and Corollary 2.9 imply that J satisfies all conditions of Lemma 2.4. Thus, (2.36) possesses infinitely many nontrivial solutions. By Lemma 2.14, Equation (1.1) also possesses infinitely many nontrivial solutions. □

Proof of Theorem 1.3. Let $X = E^s$, Y and Z be defined by (2.33) and (2.34), clearly, $\widetilde{f}$ satisfies (A3), (A5'), (A6) and (A8), Lemmas 2.7 and 2.13 and corollary 2.9 imply that J satisfies all conditions of Lemma 2.4. Thus, (2.36) possesses infinitely many nontrivial solutions. By Lemma 2.14, (1.1) also possesses infinitely many nontrivial solutions. □

Proof of Theorem 1.6. By (2.2), (2.3), and (A7), we have

$$
\begin{aligned}
J(u) &= \frac{1}{p}\|u\|_{E^s}^p - \int_{\mathbb{R}^N} F(x,u)dx \\
&= \int_{\mathbb{R}^N} \Big[\frac{1}{p}f(x,u)u - F(x,u)\Big] dx \geq 0,
\end{aligned}
$$

and so $m = \inf_{u\in\mathcal{M}} J(u) \geq 0$. We choose a sequence $\{u_i\} \subset \mathcal{M}$ such that $J(u_i) \to m$, as $i \to \infty$, and $\|J'(u_i)\|(1 + \|u_i\|) = 0$. Hence, $\{u_i\}$ is a Cerami sequence, there exists $u_0 \in E^s$ such that $u_i \to u_0$. Since $J \in C^1(E^s, \mathbb{R})$, one has

$$
J(u_0) = \lim_{i\to\infty} J(u_i) = m, \quad J'(u_0) = \lim_{i\to\infty} J'(u_i).
$$

Hence, we obtain that u_0 is also a critical point of J and $J(u_0) = \inf_{u\in\mathcal{M}} J(u)$. Furthermore, under assumptions (A10) and (A3), we have

$$
|f(x,u_i)| \leq \varepsilon|u_i|^{p-1} + C_\varepsilon|u_i|^{q-1}, \quad \forall \varepsilon > 0 \tag{2.37}
$$

for $2 \leq p < q < p_s^*$. By (2.37) and Lemma 2.1, we have

$$
\begin{aligned}
\|u_i\|_{E^s}^p &= \int_{\mathbb{R}^N} f(x,u_i)u_i dx \\
&\leq \varepsilon\|u_i\|_{L^p}^p + C_\varepsilon\|u_i\|_{L^q}^q \\
&\leq \varepsilon\gamma_p\|u_i\|_{E^s}^p + C_\varepsilon\gamma_q\|u_i\|_{E^s}^q .
\end{aligned} \tag{2.38}
$$

For sufficiently small $\varepsilon > 0$, (2.38) implies that there exists a constant $\omega > 0$ such that

$$\|u_0\|_{E^s} = \lim_{i\to\infty} \|u_i\|_{E^s} \geq \omega > 0.$$

Thus, $u_0 \neq 0$. $\square$

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