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NONEXISTENCE OF GLOBAL SOLUTIONS OF CAUCHY PROBLEMS FOR SYSTEMS OF SEMILINEAR HYPERBOLIC EQUATIONS WITH POSITIVE INITIAL ENERGY

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Abstract. In this paper we study the Cauchy problem for a system of semilinear hyperbolic equations. We prove a theorem on the nonexistence of global solutions with positive initial energy.

1. Introduction

We study the solution of some Cauchy problems for systems containing nonlinear wave equations, from mathematical physics problems in [4, 8, 25, 31]. We consider the system of nonlinear Klein-Gordon equations

$$u_{ktt} - \Delta u_k + u_k + \gamma u_{kt} = f_k(u_1, \dots, u_m) \quad k = 1, 2, \dots, m\,, \tag{1.1}$$

with initial conditions

$$u_k(0, x) = u_{k0}(x), \quad u_{kt}(0, x) = u_{k1}(x), \quad x \in R^n,\ k = 1, \dots, m, \tag{1.2}$$

where $f_k(u_1, \dots, u_k) = |u_1|^{\rho_{1k}}|u_2|^{\rho_{2k}} \dots |u_m|^{\rho_{mk}} u_k$, $\rho_{jk} = p_j + 1$, $\rho_{kk} = p_k - 1$, $k,\ j = 1, 2, \dots, m$, $(u_1, u_2, \dots, u_m)$ are real functions depending on $t \in R_+$ and $x \in \mathbb{R}^n$, $p_1, p_2, \dots, p_m$ are real numbers. System (1.1) describes the model of interaction of various fields with single masses [8]. The goal of this paper is to investigate nonexistence of global solutions of problem (1.1), (1.2).

Before going further, we briefly introduce some results for the wave equation

$$u_{tt} - \Delta u = f(u), \tag{1.3}$$

with

$$f(u) \ge (2 + \varepsilon)F(u), \tag{1.4}$$

where $F(u) = \int_0^u f(s)ds$. The general nonlinearity $f(u)$ satisfying (1.4) was firstly considered for some abstract wave equations in [12], where Levine proved the blow-up result when the initial energy is negative. But most results concerning the Cauchy problem of the wave equation were established for the typical form of nonlinearity as $f(u) = |u|^{p-1}u$ where $1 < p < \frac{n+2}{n+1}$ as $n \ge 3$ and $1 < p < \infty$ as $n = 1, 2$. Here we note that the above power satisfies the condition (1.4). For the

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nonlinearity satisfying (1.4), the wave equations with damping term were studied by many authors [5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 20, 22, 23, 32].

For existence and non-existence of global solutions for the Cauchy problem of equation (1.3) with a damping term, we refer the reader to [7, 14, 26, 27]. In particular, recently the wave equation with damping term was considered in [14], where Levine and Todorova showed that for arbitrarily positive initial energy there are choices of initial data such that the local solution blows up in finite time. Subsequently, Todorova and Vitillaro [26] established more precise result regarding the existence of initial values such that the corresponding solution blows up in finite time for arbitrarily high initial energy. More recently, Gazzola and Squassina [7] established sufficient conditions of initial data with arbitrarily positive initial energy such that the corresponding solution blows up in finite time for the wave equation with linear damping and in the mass free case on a bounded Lipschitz subset of R^n. A fairly comprehensive picture of the studies in this direction can be gained from the monograph [22].

In [15], [18] the authors obtained sufficient conditions on initial functions for which the initial boundary value problem for second-order quasilinear strongly damped wave equations blow up in a finite time. The nonexistence of global solutions of a generalized fourth-order Klein-Gordon equation with positive initial energy was analyzed in [11].

A mixed problem for systems of two semilinear wave equations with viscosity and with memory was studied in [10, 21, 24, 29], where the nonexistence of global solutions with positive initial energy was proved.

The nonexistence of global solutions of the problem

$$\begin{aligned} &u_{1tt} - \Delta u_1 + u_1 + \gamma u_{1t} = g_1(u_1, u_2), \\ &u_{2tt} - \Delta u_2 + u_2 + \gamma u_{2t} = g_2(u_1, u_2), \end{aligned} \tag{1.5}$$

with

$$u_i(0,x) = u_{i0}(x), \quad u_{it}(0,x) = u_{i1}(x), \quad x \in R^n,\ i = 1,2, \tag{1.6}$$

where

$$g_1(u_1,u_2) = |u_1|^{p-1}|u_2|^{p+1}u_1, \quad g_2(u_1,u_2) = |u_1|^{p+1}|u_2|^{p-1}u_2,$$

with negative initial energy was studied in [21], [27]. In the case when

$$g_1(u_1,u_2) = |u_1|^{p-1}|u_2|^{q+1}u_1, \quad g_2(u_1,u_2) = |u_1|^{p+1}|u_2|^{q-1}u_2.$$

The absence of global solutions for problem (1.5), (1.6) was investigated in [1, 2]. Recently, more investigations were carried out in this field [1, 7, 24, 29].

The absence of global solutions with positive arbitrary initial energy for systems of semilinear hyperbolic equations

$$u_{itt} - \Delta u_i + u_i + \gamma u_{it} = \sum_{i,j=1_{i\neq j}}^{m} |u_j|^{p_j}|u_i|^{p_i}u_i \quad i = 1,2,\dots,m$$

was investigated in [2], where $n \geq 2$, $p_j \geq 0$, $j = 1,2,\dots,m$, and $\sum_{k=1}^m p_k \leq \frac{2}{n-2}$ if $n \geq 3$.

2. Formulation of the problem and main results

To state our main results, we briefly mention some facts, notation, and well known results. We denote the norm on the space $L_2(R^n)$ by $|\cdot|$, the inner product on $L_2(R^n)$ by $\langle\cdot,\cdot\rangle$, and the norm on the Sobolev space $H^1 = W_2^1(R^n)$ by $\|u\| = [|\nabla u|^2 + |u|^2]^{\frac{1}{2}}$. The constants C and c used throughout this paper are positive generic constants that may be different in various occurrences.

Assume that

$$p_j > 0, \quad j = 1, 2, \dots, m,\ m = 2, 3, \dots; \tag{2.1}$$

$$\sum_{k=1}^{m} p_k + m - 2 \le \frac{2}{n-2} \quad \text{if } n \ge 3. \tag{2.2}$$

Let $E(t)$ be the energy functional

$$\begin{aligned} E(t) = \sum_{j=1}^{m} \frac{p_j+1}{2}\Big[|u'_{jt}(t,\cdot)|^2 + \|u_j(t,\cdot)\|^2 + 2\gamma\int_0^t |u'_{jt}(s,\cdot)|^2 ds\Big] \\ - \int_{R^n} \prod_{j=1}^{m} |u_j(t,x)|^{p_j+1} dx. \end{aligned}$$

We also set

$$I(u_1,\dots,u_m) = \sum_{j=1}^{m} \frac{p_j+1}{\sum_{r=1}^{m} p_r + m}\|u_j(t,\cdot)\|^2 - \int_{R^n} \prod_{j=1}^{m} |u_j(t,x)|^{p_j+1} dx. \tag{2.3}$$

The main result of this article is stated in the following theorem.

Theorem 2.1. *Let conditions* (2.1) *and* (2.2) *be satisfied. Assume* $u_{k0} \in H^1$ *and* $u_{k1} \in L_2(R^n)$, $k = 1, 2, \dots, m$, *and*

$$E(0) > 0, \tag{2.4}$$

$$I(u_{10}, u_{20}, \dots, u_{m0}) < 0, \tag{2.5}$$

$$\sum_{k=1}^{m} \langle u_{k0}, u_{k1}\rangle \ge 0, \tag{2.6}$$

$$\sum_{j=1}^{m} \frac{p_j+1}{2}|u_{j0}|^2 > \frac{\sum_{j=1}^{m} p_j + m}{\sum_{j=1}^{m} p_j} E(0). \tag{2.7}$$

Then the solution of the Cauchy problem (1.1), (1.2) *blows up in finite time.*

Note that, in the case of $m = 2$, this result was obtained in [1], and in the case $m = 2$, $p_1 = p_2 \ge 1$, it was obtained in [29].

3. Auxiliary assertions

In the Hilbert space $\tilde{H} = L_2(R^n) \times L_2(R^n) \times \cdots \times L_2(R^n)$ we write problem (1.1), (1.2) as the Cauchy problem

$$w'' + Bw' + Aw = F(w), \tag{3.1}$$

$$w(0) = w_0, \quad w'(0) = w_1, \tag{3.2}$$

where

$$w = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_m \end{bmatrix}, \quad w_0 = \begin{bmatrix} u_{10}(x) \\ u_{20}(x) \\ \dots \\ u_{m0}(x) \end{bmatrix}, \quad w_1 = \begin{bmatrix} u_{11}(x) \\ u_{21}(x) \\ \dots \\ u_{m1}(x) \end{bmatrix}$$

Here A and B are linear operators in $\tilde{H}$ defined by

$$A = \begin{pmatrix} -\Delta + 1 & 0 & \dots & 0 \\ 0 & -\Delta + 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\Delta + 1 \end{pmatrix},$$

$$D(A) = \tilde{H}_2 = H^2 \times H^2 \times \dots \times H^2,$$

$$B = \begin{pmatrix} \gamma & 0 & \dots & 0 \\ 0 & \gamma & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \gamma \end{pmatrix},$$

$$D(B) = L_2(R_n) \times L_2(R_n) \times \dots \times L_2(R_n),$$

$$F(w) = \begin{pmatrix} f_1(u_1, u_2, \dots, u_m) \\ f_2(u_1, u_2, \dots, u_m) \\ \dots \\ f_m(u_1, u_2, \dots, u_m) \end{pmatrix}.$$

Note that A is a self-adjoint positive definite operator, B is a linear bounded operator acting in $\tilde{H}$ and conditions (2.1), (2.2) imply that $F(w)$ is a nonlinear operator acting from $\tilde{H}_1 = H^1 \times H^1 \times \dots \times H^1$ to $\tilde{H}$.

Lemma 3.1. *Let $n = 1, 2$, $p_j \geq 1$, $j = 1, 2, \dots, n$, $m = 2, 3, \dots$ or $n = 3$, $m = 2$, $p_1 = p_2 = 1$. Then the nonlinear operator $w \to F(w) : \tilde{H}_1 \to \tilde{H}$ satisfies the local Lipchitz condition, that is for any w^1, $w^2 \in \tilde{H}_1$ we have*

$$\|F(w^1) - F(w^2)\|_{\tilde{H}} \leq c(r)\|w^1 - w^2\|_{\tilde{H}_1}, \tag{3.3}$$

where $c(\cdot) \in C(R_+)$, $c(r) \geq 0$, $r = \sum_{i=1}^{2} \|w^i\|_{\tilde{H}_1}$.

Proof. Let us take $w^j = (u_1^j, u_2^j, \dots, u_m^j) \in \tilde{H}^1$, $j = 1, 2$. Then, by the mean value theorem we have

$$\begin{aligned} \|F(w^1) - F(w^2)\|_{\tilde{H}}^2 \leq c \sum_{k=1}^{m} \sum_{j=1}^{m} \int_{R_n} (|u_j^1|^{2(\rho_{jk}-1)} + |u_j^2|^{2(\rho_{jk}-1)}) \\ \times \prod_{i=1, i\neq j}^{m} (|u_j^1|^{2\rho_{jk}} + |u_j^2|^{2\rho_{jk}})|u_j^1 - u_j^2|^2 dx. \end{aligned} \tag{3.4}$$

Let $n \geq 2$. By Holder inequality with exponents,

$$\alpha_k^i = \frac{\sum_{r=1}^{m} p_r + m}{\rho_{ki}} \quad \text{if } i \neq j,\ i = 1, \dots, m,$$

$$\alpha_k^j = \frac{\sum_{r=1}^{m} p_r + m}{\rho_{kj} - 1}, \quad \alpha_k^0 = \sum_{r=1}^{m} p_r + m$$

and using interpolation inequalities of Gagliardo and Nirenberg in the case $n = 2$ or Sobolev inequality in case $n = 3$ we have

$$\|F(w^1)-F(w^2)\|^2_{\widetilde{H}} \le c\Big(\|w^1\|^{\sum_{r=1}^m p_r+m-1}_{\widetilde{H}_1}+\|w^2\|^{\sum_{r=1}^m p_r+m-1}_{\widetilde{H}_1}\Big)\|w^1-w^2\|_{\widetilde{H}_1}. \quad (3.5)$$

In case $n = 1$, from (3.4) using embedding theorem we again obtain (3.1). □

By the theorem of solvability of the Cauchy problem for the evolution equation [3], we have the following local solvability theorem for problem (2.3), (2.4).

Theorem 3.2. *Let $n = 1, 2$, $p_j \ge 1$, $j = 1, 2, \dots, m$, $m = 2, 3, \dots$ or $n = 3$, $m = 2$, $p_1 = p_2 = 1$. Then for arbitrary $w_0 \in \tilde{H}_1$, $w_1 \in \tilde{\text{H}}$, there exists $T' > 0$ such that problem (3.1), (3.2) has a unique solution $w(\cdot) \in C([0, T^*]; \tilde{H}_1) \cap C^1([0, T^*]; \tilde{H})$. If $T_{\max} = \sup T^*$, i.e., $T_{\max}$ is the length of the maximal existence interval of the solution $w(\cdot) \in C([0, T_{\max}); \tilde{H}_1) \cap C^1([0, T_{\max}); \tilde{H})$, then either*

(i) *$T_{\max} = +\infty$, or*
(ii) *$\limsup_{t\to T_{\max}-0}[\|w(\cdot)\|_{\tilde{H}_1} + \|\dot{w}(\cdot)\|_{\tilde{H}}] = +\infty$.*

Theorem 3.3. *Let conditions (2.1) and (2.2) be satisfied. Then for arbitrary $w_0 \in \tilde{H}_1$ and $w_1 \in \tilde{H}$ there exists $T' > 0$ such that problem (3.1), (3.2) has a solution $w(\cdot) \in C([0, T']; \widetilde{H}_1) \cap C^1([0, T']; \widetilde{H})$ and $w(t)$ is either global or blow-up in a finite time.*

Proof. We carry out the proof by Galerkin's method, using some considerations from the work [18]. Let $\{w_1, w_2, \dots, w_r \dots\}$ be the basis of the space $\widetilde{H}_1$ and $w_r(t, \cdot) = \sum_{j=1}^r g_{rj}(t)w_j$, $r = 1, 2, \dots$ be defined as a solution of the system

$$(w_r''(t), \omega_j)_{\widetilde{H}} + (Bw_r'(t), w_j)_{\widetilde{H}} + (w_r(t), \omega_j)_{\widetilde{H}_1} = (F(w_r), \omega_j)_{\widetilde{H}} \quad (3.6)$$

with initial data

$$w_r(0, \cdot) = w_{0r}, \quad w_r'(0, \cdot) = w_{1r}, \quad (3.7)$$

where w_{0r} and w_{1r} belongs to the subspace $[\omega_1, \omega_2, \dots, \omega_r]$ generated by the r first vectors of the basis $\{\omega_j\}$, and

$$w_{0r} \to w_0 \text{ in } \widetilde{H}_1 \quad \text{and} \quad w_{1r} \to w_1 \text{ in } \widetilde{H} \text{ if } r \to \infty. \quad (3.8)$$

By multiplying the equation (3.6) by $g_{rj}'(t)$ and summing by k, where k takes the values from 1 to r, we get that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\|w_{rt}(t, \cdot)\|^2_{\widetilde{H}} + \|w_r(t, \cdot)\|^2_{\widetilde{H}_1}] + (Bw_{rt}(t, \cdot), w_{rt}(t, \cdot))_{\widetilde{H}} \\ &= (F(w(t, \cdot)), w'(t, \cdot))_{\widetilde{H}}. \end{aligned} \quad (3.9)$$

Then using Holder's inequalities and (3.5), for

$$y_r(t) = \|w_{rt}(t, \cdot)\|^2_{\widetilde{H}} + \|w_r(t, \cdot)\|^2_{\widetilde{H}_1} \quad (3.10)$$

from (3.9) we get $y_r'(t) \le c(y_r(t))^{\sum_{r=1}^m p_r+m}$.

Integrating this inequality and taking into account the inequality (3.4), we find that there exists $T' > 0$ and r_0 such that

$$y_r(t) \le c, \quad t \in [0, T'],\ r \ge r_0. \quad (3.11)$$

From (3.10), (3.11) it follows that there exists a subsequence still denoted by the same symbols, such that

$$w_r \to w \quad \text{weak star in } L_\infty(0, T'; \widetilde{H}_1),$$

$$w_r' \to w' \quad \text{weak star in } L_\infty(0,T';\widetilde{H}), F(w_r) \to \chi \quad \text{weak star in } L_\infty(0,T';\widetilde{H}).$$

Further, using the method given in [16], we obtain that $\chi = F(w)$.

Passing to the limit is carried out by the standard method (for example, see [[16, 18]). Thus, problem (3.1), (3.2) has the solution $w \in L_\infty(0,T';\widetilde{H}_1)$, such that $w' \in L_\infty(0,T';\widetilde{H})$ and $F(w) \in L_\infty(0,T';\widetilde{H})$.

Further applying the linear theory of the hyperbolic equations, considering equation (3.1) as a linear equation with a given right-hand side of $\chi(t) = F(w) \in L_\infty(0,T';\widetilde{H})$, we find that $w \in C([0,T']\widetilde{H}_1) \cap C^1([0,T']\widetilde{H})$ (see [17]). □

Remark 3.4. *labelrmk3.1* If $w_0 \in \tilde{H}_2$ and $w_1 \in \tilde{H}_1$, then $w(\cdot) \in C([0,T_{\max});\tilde{H}_2) \cap C^1([0,T_{\max});\tilde{H}_1)$.

Lemma 3.5. *Let conditions* (2.1), (2.2) *and* (2.4)-(2.7) *be satisfied. Then*

$$I(u_1(t,.),u_2(t,.),\dots,u_m(t,.)) < 0, \quad t \in [0,T_{\max}).$$

Proof. By (2.5) there exists $T_1 > 0$, such that

$$I(u_1(t,\cdot),u_2(t,\cdot),\dots,u_m(t,\cdot)) < 0, \quad t \in [0,T_1). \tag{3.12}$$

We shall prove that $T_1 = T_{\max}$. Assume that $T_1 < T_{\max}$. Then by the continuity of $I(u_1(t,\cdot),u_2(t,\cdot),\dots,u_m(t,\cdot))$ we have

$$I(u_1(T_1,\cdot),u_2(T_1,\cdot),\dots,u_m(T_1,\cdot)) = 0. \tag{3.13}$$

We introduce the functional $F(t) = \sum_{j=1}^m (p_j+1)|u_j(t,\cdot)|^2$. Taking into account Remark 2.1 and using (1.1), (1.2) we obtain:

$$F'(t) = 2\sum_{j=1}^m (p_j+1)\langle u_j(t,\cdot),\dot{u}_j(t,\cdot)\rangle,$$

and

$$\begin{aligned} F''(t) &= 2\sum_{j=1}^m (p_j+1)|u_j'(t,\cdot)|^2 - 2\sum_{j=1}^m (p_j+1)\big[\|u_j(t,\cdot)\|^2 + \gamma\langle u_j(t,\cdot),\dot{u}_j(t,\cdot)\rangle\big] \\ &\quad + 2\Big(\sum_{k=1}^m p_k + m\Big)\int_{R^n}\prod_{j=1}^m |u_j(t,x)|^{p_j+1}dx. \end{aligned}$$

Therefore,

$$F''(t) + \gamma F'(t) = \varphi(t), \quad t \in [0,T_1), \tag{3.14}$$

where

$$\varphi(t) = 2\sum_{j=1}^m (p_j+1)|u_j'(t,\cdot)|^2 - 2\Big(\sum_{k=1}^m p_k + m\Big) I(u_1(t,\cdot),u_2(t,\cdot),\dots,u_m(t,\cdot)).$$

Taking into account inequality (3.12), we obtain

$$\varphi(t) > 0, \quad t \in [0,T_1). \tag{3.15}$$

It follows from condition (2.6) and relations (3.14) and (3.15) that

$$F'(t) > 0, \quad t \in [0,T_1).$$

Therefore, the function $F(t)$ is monotone increasing on $[0,T_1)$. Consequently,

$$F(t) > F(0) = \sum_{j=1}^m (p_j+1)|u_{j0}|^2. \tag{3.16}$$

By taking into account the continuity of the function $F(t)$, from condition (2.7) and inequalities (3.16), we obtain

$$F(T_1) > \frac{2\big[\sum_{j=1}^m p_j + m\big]}{\sum_{j=1}^m p_{j_3}} E(0). \tag{3.17}$$

On the other hand it follows from (1.1) and (1.2) that

$$E(t) = E(0) \tag{3.18}$$

for every $t \in [0, T_{\max})$. From (3.13) and (3.18) we obtain the inequality

$$\Big(1 - \frac{2}{\sum_{j=1}^m p_j + m}\Big) \sum_{j=1}^m \frac{p_j+1}{2} \|u_j(T_1, \cdot)\|^2 \le E(0).$$

It follows that

$$F(T_1) \le \frac{2\big[\sum_{j=1}^m p_j + m\big]}{\sum_{j=1}^m p_j + m - 2} E(0). \tag{3.19}$$

The resulting contradiction (3.17) with (3.19) shows that our assumption fails. Therefore $T_1 = T_{\max}$.

Let $T_2 > 0$, $T_3 > 0$ and $k > 0$ be some numbers. Consider the functional

$$\begin{aligned} R(t) = \sum_{j=1}^m \frac{p_j+1}{2} \Big[|u_j(t, \cdot)|^2 + \gamma \int_0^t |u_j(s, \cdot)|^2 ds + \gamma |u_{j0}|^2 (T_2 - t) \Big] \\ + k(T_3 + t)^2. \end{aligned} \tag{3.20}$$

□

Lemma 3.6. *Let (2.4)–(2.7) be satisfied. Then $\ddot{R}(t) > 0$ for $t \in [0, T_{\max})$.*

Proof. A simple computation gives us

$$R'(t) = \sum_{j=1}^m \frac{p_j+1}{2} \big[2\langle u_j(t, \cdot), u_j'(t, \cdot) \rangle + \gamma |u_j(t, \cdot)|^2 - \gamma |u_{j0}|^2 \big] + 2k(t + T_3). \tag{3.21}$$

Next, from (3.18), (3.21) by using relations (1.1) and (1.2), we obtain

$$\begin{aligned} R''(t) &= \sum_{j=1}^m (p_j+1)[|u_j'(t, \cdot)|^2 - \|u_j(t, \cdot)\|^2] \\ &\quad + \Big[\sum_{j=1}^m p_j + m \Big] \int_{R^n} \prod_{j=1}^m |u_j(t,x)|^{p_j+1} dx + 2k. \end{aligned} \tag{3.22}$$

It follows from (2.3) and (3.22) that

$$R''(t) \ge -\Big[\sum_{j=1}^m p_j + m \Big] I(u_1(t, .), \dots, u_3(t, .)) + 2k, \quad t \in [0, T_{\max}).$$

By Lemma 3.5, for sufficiently small k it holds

$$R''(t) > 0, \quad t \in [0, T_{\max}). \tag{3.23}$$

□

4. Proof of main result

We first assume that $u_{i0} \in H^2$, $u_{i1} \in H^1$, $i = 1, 2, \dots, m$. We shall prove that under conditions (2.1), (2.2) and (2.4)-(2.7), $T_{\max} < +\infty$. Suppose the contrary: $T_{\max} = +\infty$. It follows from (1.1) and (1.2) that

$$\begin{aligned}&\int_{R^n} \prod_{j=1}^m |u_j(t,x)|^{p_j+1} dx \\ &= -E(0) + \sum_{j=1}^m \frac{p_j+1}{2} [|u_j'(t,\cdot)|^2 + \|u_j(t,\cdot)\|^2 + 2\gamma \int_0^t |u_j'(s,\cdot)|^2 ds].\end{aligned}$$

Taking into account this relation in (3.22), we obtain

$$\begin{aligned}R''(t) &= \frac{\sum_{j=1}^m p_j + m + 2}{2} \sum_{j=1}^m (p_j+1)|u_j'(t,\cdot)|^2 \\ &\quad + \frac{\sum_{j=1}^m p_j + m - 2}{2} \sum_{j=1}^m (p_j+1)\|u_j(t,\cdot)\|^2 + \gamma \Big[\sum_{j=1}^m p_j + m\Big] \\ &\quad \times \sum_{j=1}^3 (p_j+1) \int_0^t |u_j'(s,\cdot)|^2 ds - \Big[\sum_{j=1}^m p_j + m\Big] E(0) + 2k.\end{aligned} \tag{4.1}$$

By (3.9) we have

$$\begin{aligned}R'^2(t) &\le \Big[\sum_{j=1}^m (p_j+1)(|u_j(t,\cdot)|^2 + \gamma \int_0^t |u_j(s,\cdot)|^2 ds) + k(t+T_3)^2\Big] \\ &\quad \times \Big[\sum_{j=1}^m (p_j+1)(|u_j'(t,\cdot)|^2 + \gamma \int_0^t |u_j'(s,\cdot)|^2 ds) + k\Big].\end{aligned} \tag{4.2}$$

By choosing a sufficiently large T_3, from Lemma 3.5 and relations (3.19), (4.1), and (4.2), we obtain

$$\begin{aligned}&R(t)R''(t) - \frac{\sum_{j=1}^m p_j + m + 2}{4} (R'(t))^2 \\ &\ge R(t)\cdot R''(t) - \frac{\sum_{j=1}^m p_j + m + 2}{4} \Big[2R(t) - (T_1 - t)\sum_{j=1}^m (p_j+1)\cdot |u_{j0}|^2\Big] \\ &\quad \times \Big[\sum_{j=1}^m (p_j+1)(|u_j'(t,\cdot)|^2 + \gamma \int_0^t |u_j'(s,\cdot)|^2 ds) + k] \\ &\ge R(t)\Big\{\frac{\sum_{j=1}^m p_j + m}{2} \sum_{j=1}^m (p_j+1)|u_j'(t,\cdot)|^2 \\ &\quad + \frac{\sum_{j=1}^m p_j + m - 2}{2} \sum_{j=1}^m (p_j+1)\|u_j(t,\cdot)\|^2 \\ &\quad + [\sum_{j=1}^m p_j + m] \sum_{j=1}^m (p_j+1) \int_0^t |u_j'(s,\cdot)|^2 ds - \Big[\sum_{j=1}^m p_j + m\Big] E(0) + 2k\end{aligned}$$

$$
\begin{aligned}
&- \frac{\sum_{j=1}^m p_j + m + 2}{2} \sum_{j=1}^m (p_j+1)\Big(|u_j'(t,\cdot)|^2 + \int_0^t |u_j'(s,\cdot)|^2 ds \Big) + k \Big\} \\
&= R(t)y(t) + \frac{\sum_{j=1}^m p_j + m - 2}{2} \sum_{j=1}^m \int_0^t |u_j'(s,\cdot)|^2 ds, \qquad (4.3)
\end{aligned}
$$

where

$$
\begin{aligned}
y(t) = &\frac{\sum_{j=1}^m p_j + m - 2}{2} \sum_{j=1}^m (p_j+1)\|u_j(t,\cdot)\|^2 \\
&- \Big[\sum_{j=1}^m p_j + m \Big] E(0) - \frac{p_1+p_2+p_3+1}{2} k.
\end{aligned}
$$

Having in mind Lemma 3.5, and choosing a sufficiently small $k > 0$, we obtain that $y(t) \geq 0$. Thus, for sufficiently large $T_2 > 0$, $T_3 > 0$, and for sufficiently small $k > 0$ we ahve

$$
R(t) \cdot R''(t) - \frac{\sum_{j=1}^m p_j + m + 2}{4} R'^2(t) \geq 0\,. \qquad (4.4)
$$

On the other hand,

$$
R'(0) = \sum_{j=1}^m (p_j+1)\langle u_{j0}, u_{j1}\rangle + 2kT_2\,.
$$

Therefore, $R'(0) > 0$. Using this inequality and (4.4) bya standard procedure, we obtain that there exists $0 < T^* < +\infty$ such that $\lim_{t\to T^*-0} R(t) = +\infty$. We obtain a contradiction, which shows that $T_{\max} < +\infty$.

If $u_{i0} \in H^1$ and $u_{i1} \in L_2(R^n)$, $i = 1, 2, \dots, m$, then the justification can be carried out in a standard way, by approximation of the initial data by functions from H^2 and H^1, respectively.

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