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EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR SUBLINEAR EQUATIONS ON EXTERIOR DOMAINS

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Abstract. In this article we study radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R > 0$, B_R, centered at the origin in $\mathbb{R}^N$ with $u = 0$ on ∂B_R where f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞), $f(u) \sim u^p$ with $0 < p < 1$ for large u and $K(r) \sim r^{-\alpha}$ for large r. We prove that if $N > 2$ and $K(r) \sim r^{-\alpha}$ with $2 < \alpha < 2(N-1)$ then there are no solutions with $\lim_{r\to\infty} u(r) = 0$ for sufficiently large $R > 0$. On the other hand, if $2 < N - p(N-2) < \alpha < 2(N-1)$ and k, n are nonnegative integers with $0 \le k \le n$ then there exist solutions, u_k, with k zeros on (R, ∞) and $\lim_{r\to\infty} u_k(r) = 0$ if $R > 0$ is sufficiently small.

1. Introduction

In this article we study radial solutions of

$$\Delta u + K(r)f(u) = 0 \quad \text{in } \mathbb{R}^N \backslash B_R, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial B_R, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where B_R is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ and $K(r) > 0$. We assume:

- (H1) f is odd and locally Lipschitz, $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞), and $f'(0) < 0$.
- (H2) There exists p with $0 < p < 1$ such that $f(u) = |u|^{p-1}u + g(u)$ where $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$.

We let $F(u) = \int_0^u f(s)\, ds$. Since f is odd it follows that F is even and from (H1) it follows that F is bounded below by $-F_0 < 0$, F has a unique positive zero, γ, with $0 < \beta < \gamma$, and

- (H3) $-F_0 < F < 0$ on $(0, \gamma)$, $F > 0$ on (γ, ∞).

When f grows superlinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, $\Omega = \mathbb{R}^N$, and $K(r) \equiv 1$ then the problem (1.1), (1.3) has been extensively studied [1]-[3], [10, 12, 14].

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Interest in the topic for this paper comes from recent papers [5, 11, 13] about solutions of differential equations on exterior domains. In [7]-[9] we studied (1.1)-(1.3) with $K(r) \sim r^{-\alpha}$, f superlinear, and $\Omega = \mathbb{R}^N \backslash B_R$ with various values for α. In those papers we proved existence of an infinite number of solutions - one with exactly n zeros for each nonnegative integer n such that $u \to 0$ as $|x| \to \infty$ for all $R > 0$. In [6] we studied (1.1)-(1.3) with $K(r) \sim r^{-\alpha}$, f bounded, and $\Omega = \mathbb{R}^N \backslash B_R$. In this paper we consider the case where f grows sublinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u^p} = c_0 > 0$ with $0 < p < 1$.

Since we are interested in radial solutions of (1.1)-(1.3) we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that u solves

$$u''(r) + \frac{N-1}{r} u'(r) + K(r) f(u(r)) = 0 \quad \text{on } (R, \infty) \text{ where } R > 0, \tag{1.4}$$

$$u(R) = 0, u'(R) = b \in \mathbb{R}. \tag{1.5}$$

We will also assume that

(H4) there exist constants $k_1 > 0$, $k_2 > 0$, and α with $0 < \alpha < 2(N-1)$ such that

$$k_1 r^{-\alpha} \le K(r) \le k_2 r^{-\alpha} \quad \text{on } [R, \infty). \tag{1.6}$$

(H5) K is differentiable, on $[R, \infty)$, $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$, and $\frac{rK'}{K} + 2(N-1) > 0$.

Note that (H5) implies $r^{2(N-1)} K(r)$ is increasing. In this article we prove the following result.

Theorem 1.1. *Let $N > 2$, $0 < p < 1$, and $2 < N - p(N-2) < \alpha < 2(N-1)$. Assuming* (H1)–(H5) *then given nonnegative integers k, n with $0 \le k \le n$ then there exist solutions, u_k, of* (1.4)-(1.5) *with k zeros on (R, ∞) and $\lim_{r\to\infty} u_k(r) = 0$ if $R > 0$ is sufficiently small.*

In addition we also prove:

Theorem 1.2. *Let $N > 2$, $0 < p < 1$ and $2 < \alpha < 2(N-1)$. Assuming* (H1)–(H5)*, there are no solutions of* (1.4)-(1.5) *such that $\lim_{r\to\infty} u(r) = 0$ if $R > 0$ is sufficiently large.*

Note that for the superlinear problems studied in [7]-[9] we were able to prove existence for *any* $R > 0$ whereas in the sublinear case and in [6] we only get solutions if R is sufficiently small.

2. Preliminaries and proof of Theorem 1.2

From the standard existence-uniqueness theorem for ordinary differential equations [4] it follows there is a unique solution of (1.4)-(1.5) on $[R, R+\epsilon)$ for some $\epsilon > 0$. We then define

$$E = \frac{1}{2} \frac{u'^2}{K} + F(u). \tag{2.1}$$

Using (H5) we see that

$$E' = -\frac{u'^2}{2rK} \Big(2(N-1) + \frac{rK'}{K} \Big) \le 0 \quad \text{for } 0 < \alpha < 2(N-1). \tag{2.2}$$

Thus E is nonincreasing. Hence it follows that

$$\frac{1}{2} \frac{u'^2}{K} + F(u) = E(r) \le E(R) = \frac{1}{2} \frac{b^2}{K(R)} \text{ for } r \ge R \tag{2.3}$$

and so we see from (H2)–(H4) that u and u' are uniformly bounded wherever they are defined from which it follows that the solution of (1.4)-(1.5) is defined on $[R, \infty)$.

Lemma 2.1. *Let $N > 2$, $0 < p < 1$, and $0 < \alpha < 2(N-1)$. Assume* (H1)–(H5) *and suppose u satisfies* (1.4)-(1.5) *with $b > 0$. If u has a zero, z_b, with $u > 0$ on (R, z_b) or if $u > 0$ for $r > R$ and $\lim_{r\to\infty} u = 0$ then u has a local maximum, M_b, with $R < M_b$, $u' > 0$ on (R, M_b), $M_b \to \infty$ as $b \to \infty$, and $u(M_b) \to \infty$ as $b \to \infty$.*

Proof. Since $u(R) = 0$ and $u'(R) = b > 0$ we see that u gets positive for $r > R$ and if u has a zero, z_b, or if $u > 0$ and $\lim_{r\to\infty} u(r) = 0$ then u has a critical point, M_b, such that $u' > 0$ on (R, M_b). Then $u'(M_b) = 0$ and $u''(M_b) \le 0$. By uniqueness of solutions of initial value problems it follows that $u''(M_b) < 0$ and thus M_b is a local maximum. Next suppose there exists $M_0 > R$ such that $M_b \le M_0$ for all $b > 0$. Letting $v_b(r) = \frac{u(r)}{b}$ then from (1.5) we have $v_b(R) = 0$, $v_b'(R) = 1$ and

$$v_b''(r) + \frac{N-1}{r} v_b'(r) + K(r)\frac{f(bv_b(r))}{b} = 0 \quad \text{for } r \ge R. \tag{2.4}$$

It follows from (2.1)-(2.2) that

$$\Big(\frac{1}{2}\frac{v_b'^2}{K} + \frac{F(bv_b)}{b^2}\Big)' \le 0 \quad \text{for } r \ge R$$

and thus

$$\frac{1}{2}\frac{v_b'^2}{K} + \frac{F(bv_b)}{b^2} \le \frac{1}{2K(R)} \quad \text{for } r \ge R. \tag{2.5}$$

It then follows from (2.5) and (H2)–(H4) that $|v_b'|$ is uniformly bounded for large $b > 0$ on $[R, \infty)$. So there is a constant $C_1 > 0$ such that

$$|v_b'| \le C_1 \text{ for large } b > 0 \quad \text{and all } r \ge R. \tag{2.6}$$

We now fix a compact set $[R, R_0]$. Then on $[R, R_0]$ we have by (2.6)

$$|v_b| = |(r-R) + \int_R^r v_b'(t)\,dt| \le (1 + C_1)(R_0 - R)$$

so we see that $|v_b|$ is uniformly bounded for large b on $[R, R_0]$.

In addition from (H1)–(H2) it follows there is a constant $C_2 > 0$ such that

$$|f(u)| \le C_2 |u|^p \quad \text{for all } u \tag{2.7}$$

and therefore since the v_b are uniformly bounded on $[R, R_0]$ and $0 < p < 1$ it follows that

$$|\frac{f(bv_b)}{b}| \le \frac{C_2|v_b|^p}{b^{1-p}} \to 0 \quad \text{as } b \to \infty. \tag{2.8}$$

Then from (2.4) and (2.8) we see that $|v_b''|$ is uniformly bounded on $[R, R_0]$. So by the Arzela-Ascoli theorem there is a subsequence of v_b (still denoted v_b) such that $v_b \to v_0$ and $v_b' \to v_0'$ uniformly on $[R, R_0]$ as $b \to \infty$. It then follows from (2.4) that v_b'' converges uniformly to v_0'' on $[R, R_0]$ and $v_0'' + \frac{N-1}{r}v_0' = 0$. Since R_0 is arbitrary we see that $v_0'' + \frac{N-1}{r}v_0' = 0$ on $[R, \infty)$. Thus, $r^{N-1}v_0' = R^{N-1}$ and $v_0 = \frac{R^{N-1}[R^{2-N} - r^{2-N}]}{N-2}$. Now since $M_b \le M_0$ for all $b > 0$ then a subsequence of M_b converges to some M and since $v_b'(M_b) = 0$ it follows that $v_0'(M) = 0$. However this contradicts that $v_0' = \frac{R^{N-1}}{r^{N-1}} > 0$. Therefore our assumption that the M_b are bounded is false and so we see $M_b \to \infty$ as $b \to \infty$.

Next we see that since $M_b \to \infty$ then $M_b > 2R$ if b is sufficiently large and since u is increasing on $[R, M_b]$ then $\frac{u(M_b)}{b} \ge \frac{u(2R)}{b} = v_b(2R) \to v_0(2R) > 0$ for sufficiently large b. Thus $u(M_b) > \frac{v_0(2R)}{2} b$ for sufficiently large b and so we see that $u(M_b) \to \infty$ as $b \to \infty$. This completes the proof. □

Lemma 2.2. *Let $N > 2$, $0 < p < 1$, $2 < \alpha < 2(N-1)$, and assume* (H1)–(H5)*. If $u(z_b) = 0$ with $u > 0$ on (R, z_b) or $u > 0$ on (R, ∞) with $\lim_{r\to\infty} u = 0$ then*

$$[u(M_b)]^{\frac{1-p}{2}} M_b^{\frac{\alpha}{2}-1} \le \frac{k_2}{\frac{\alpha}{2}-1} \sqrt{\frac{1}{p+1} + \frac{F_0}{\gamma^{p+1}}}. \tag{2.9}$$

Proof. We first show that if $u(z_b) = 0$ with $u > 0$ on (M_b, z_b) then $u' < 0$ on (M_b, z_b) and if $u > 0$ on (M_b, ∞) with $\lim_{r\to\infty} u(r) = 0$ then $u' < 0$ on (M_b, ∞). In the first case, if u has a positive local minimum, m_b, with $M_b < m_b < z_b$ then $u'(m_b) = 0$, $u''(m_b) \le 0$, so $f(u(m_b)) \ge 0$ which implies $0 < u(m_b) \le \beta$. On the other hand, since E is nonincreasing $0 > F(u(m_b)) = E(m_b) \ge E(z_b) = \frac{1}{2}\frac{u'^2(z_b)}{K(z_b)} \ge 0$ which is impossible. Secondly, suppose $u > 0$ on (R, ∞) and $\lim_{r\to\infty} u(r) = 0$. Since E is nonincreasing it follows that $\lim_{r\to\infty} E(r)$ exists and since $\frac{1}{2}\frac{u'^2}{K} \ge 0$ and $F(u(r)) \to 0$ as $r \to \infty$ we see that $\lim_{r\to\infty} E(r) \ge 0$. Thus $E(r) \ge 0$ for all $r \ge R$. On the other hand, if u has a positive local minimum, m_b, then $0 < u(m_b) \le \beta$ and $E(m_b) = F(u(m_b)) < 0$ again yielding a contradiction.

Next, it follows from (2.1)-(2.2) that $E(t) \le E(M_b)$ for $t \ge M_b$. Rewriting this inequality we obtain

$$\frac{|u'(t)|}{\sqrt{2}\sqrt{F(u(M_b)) - F(u(t))}} \le \sqrt{K} \text{ for } t \ge M_b. \tag{2.10}$$

If $u(z_b) = 0$ then integrating (2.10) on (M_b, z_b) and using that $u' < 0$ on (M_b, z_b) gives

$$\begin{aligned} \int_0^{u(M_b)} \frac{dt}{\sqrt{F(u(M_b)) - F(t)}} &= \int_{M_b}^{z_b} \frac{-u'(t)}{\sqrt{2}\sqrt{F(u(M_b)) - F(u(t))}}\, dt \\ &\le \int_{M_b}^{z_b} \sqrt{K}\, dt \\ &\le \frac{k_2}{\frac{\alpha}{2}-1}(M_b^{1-\frac{\alpha}{2}} - z_b^{1-\frac{\alpha}{2}}) \\ &\le \frac{k_2}{\frac{\alpha}{2}-1} M_b^{1-\frac{\alpha}{2}}. \end{aligned} \tag{2.11}$$

Similarly if $u(r) > 0$ and $\lim_{r\to\infty} u = 0$ then integrating (2.10) on (M_b, ∞) and using that $u' < 0$ on (M_b, ∞) we again obtain

$$\int_0^{u(M_b)} \frac{dt}{\sqrt{F(u(M_b)) - F(t)}} \le \frac{k_2}{\frac{\alpha}{2}-1} M_b^{1-\frac{\alpha}{2}}.$$

Next from (H2), (H3) and (2.7) it follows that $-F_0 \le F(u) \le \frac{C_2|u|^{p+1}}{p+1}$ for all u. Therefore estimating the left-hand side of (2.11) gives

$$\int_0^{u(M_b)} \frac{dt}{\sqrt{F(u(M_b)) - F(t)}} \ge \frac{u(M_b)}{\sqrt{\frac{C_2[u(M_b)]^{p+1}}{p+1} + F_0}} = \frac{[u(M_b)]^{\frac{1-p}{2}}}{\sqrt{\frac{C_2}{p+1} + \frac{F_0}{[u(M_b)]^{p+1}}}}. \tag{2.12}$$

Also from (2.1)-(2.2) if $u(z_b) = 0$ then we have $F(u(M_b)) = E(M_b) \ge E(z_b) = \frac{1}{2}\frac{u'^2(z_b)}{K(z_b)} \ge 0$ and so $u(M_b) \ge \gamma$. On the other hand, if $u > 0$ and $\lim_{r\to\infty} u = 0$

then as we saw earlier $E(r) \ge 0$ for all $r \ge R$. Thus $F(u(M_b)) = E(M_b) \ge 0$ and again we see $u(M_b) \ge \gamma$. Now using (2.12) in (2.11) and rewriting gives

$$\begin{aligned} \gamma^{\frac{1-p}{2}} M_b^{\frac{\alpha}{2}-1} &\le \frac{k_2}{\frac{\alpha}{2}-1}\sqrt{\frac{C_2}{p+1} + \frac{F_0}{[u(M_b)]^{p+1}}} \\ &\le \frac{k_2}{\frac{\alpha}{2}-1}\sqrt{\frac{C_2}{p+1} + \frac{F_0}{\gamma^{p+1}}}. \end{aligned} \tag{2.13}$$

This completes the proof. □

Proof of Theorem 1.2. If u has a zero, z_b, with $u > 0$ on (R, z_b) or $u >$ on (R, ∞) with $\lim_{r\to\infty} u(r) = 0$ then by Lemmas 2.1 and 2.2 we know that u has a local maximum, M_b, with $R < M_b$ and $u' > 0$ on (R, M_b). In addition, from the proof of Lemma 2.2 we have $u(M_b) \ge \gamma$. Combining this with (2.13) and the fact that $\alpha > 2$ and $0 < p < 1$ we obtain

$$\gamma^{\frac{1-p}{2}} R^{\frac{\alpha}{2}-1} \le [u(M_b)]^{\frac{1-p}{2}} M_b^{\frac{\alpha}{2}-1} \le \frac{k_2}{\frac{\alpha}{2}-1}\sqrt{\frac{1}{p+1} + \frac{F_0}{\gamma^{p+1}}}. \tag{2.14}$$

Thus we see that if R is sufficiently large then (2.14) is violated and so we obtain a contradiction. This completes the proof of Theorem 1.2. □

3. Proof of Theorem 1.1

We now turn to the proof of existence for $N > 2$, $0 < p < 1$, $2 < N - p(N-2) < \alpha < 2(N-1)$ and $R > 0$ sufficiently small. First we make the change of variables:

$$u(r) = u_1(r^{2-N}).$$

Using (1.4) we see that u_1 satisfies

$$u_1'' + h(t) f(u_1) = 0 \tag{3.1}$$

where it follows from (H4)–(H5) that:

$$0 < h(t) = \frac{t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}})}{(N-2)^2} \quad \text{and} \quad h'(t) < 0 \text{ for } t > 0, \tag{3.2}$$

$$u_1(R^{2-N}) = 0 \quad \text{and} \quad u_1'(R^{2-N}) = -\frac{bR^{N-1}}{N-2} < 0. \tag{3.3}$$

In addition, from (H4) we have

$$\frac{k_1}{(N-2)^2 t^q} \le h(t) \le \frac{k_2}{(N-2)^2 t^q} \quad \text{for all } t > 0, \quad \text{where } q = \frac{2(N-1)-\alpha}{N-2}. \tag{3.4}$$

Note: Since $2 < \alpha < 2(N-1)$, $N > 2$, and $q = \frac{2(N-1)-\alpha}{N-2}$ it follows that $0 < q < 2$.

Now instead of considering (3.1) with (3.3) we consider (3.1) with

$$u_1(0) = 0, \quad u_1'(0) = b_1 > 0. \tag{3.5}$$

Integrating (3.1) twice on $(0, t)$ and using (3.5) we see that a solution of (3.1), (3.5) is equivalent to a solution of:

$$u_1 = b_1 t - \int_0^t \int_0^s h(x) f(u_1)\, dx\, ds. \tag{3.6}$$

Letting $u_1 = tv_1$ we see that a solution of (3.6) is equivalent to a solution of

$$v_1 = b_1 - \frac{1}{t}\int_0^t \int_0^s h(x) f(xv_1)\,dx\,ds. \tag{3.7}$$

Now we define

$$Tv_1 = b_1 - \frac{1}{t}\int_0^t \int_0^s h(x) f(xv_1)\,dx\,ds. \tag{3.8}$$

Let $0 < \epsilon < 1$. Denoting $\|w\| = \sup_{[0,\epsilon]} |w(x)|$ we let

$$B = \{v \in C[0,\epsilon] \mid \|v - b_1\| \le 1\}$$

where $C[0,\epsilon]$ is the set of continuous functions on $[0,\epsilon]$. It follows from (H1)–(H2) that there exists $L > 0$ such that

$$|f(u)| \le L|u| \quad \text{for all } u. \tag{3.9}$$

Then by (3.4), (3.8)-(3.9), and since $q < 2$ as well as $|v_1| \le 1 + b_1$:

$$\begin{aligned}
|Tv_1 - b_1| &\le \frac{Lk_2}{(N-2)^2 t}\int_0^t \int_0^s x^{-q} x |v_1|\,dx\,ds \\
&\le \frac{Lk_2(1+b_1)t^{2-q}}{(2-q)(3-q)(N-2)^2} \\
&\le \frac{Lk_2(1+b_1)\epsilon^{2-q}}{(2-q)(3-q)(N-2)^2}.
\end{aligned}$$

Thus for sufficiently small $\epsilon > 0$ we have $T : B \to B$. Next we see by the mean value theorem, (3.4), and (3.9) that we have

$$\begin{aligned}
|Tv_1 - Tv_2| &= \Big|\frac{1}{t}\int_0^t \int_0^s h(x)[f(xv_1) - f(xv_2)]\,dx\,ds\Big| \\
&\le \frac{L}{t}\int_0^t \int_0^s xh(x)|v_1 - v_2|\,dx\,ds \\
&\le \frac{Lk_2}{(N-2)^2}\|v_1 - v_2\| \frac{1}{t}\int_0^t \int_0^s x x^{-q}\,dx\,ds \\
&\le \frac{Lk_2 \epsilon^{2-q}}{(2-q)(3-q)(N-2)^2}\|v_1 - v_2\|.
\end{aligned}$$

Thus for small enough $\epsilon > 0$ we see that T is a contraction for any $b_1 > 0$ and so by the contraction mapping principle there is a solution of (3.7) and hence of (3.1), (3.5) on $[0,\epsilon]$ for some $\epsilon > 0$.

Next from (3.7) and (3.9) we have

$$\begin{aligned}
\Big|\frac{u_1}{t}\Big| = |v_1| &\le b_1 + \frac{L}{t}\int_0^t \int_0^s xh(x)|v_1(x)|\,dx\,ds \qquad (3.10)\\
&\le b_1 + \frac{Lk_2}{(N-2)^2 t}\int_0^t \int_0^s x^{1-q}|v_1(x)|\,dx\,ds \\
&\le b_1 + \frac{k_2 L}{(N-2)^2}\int_0^t x^{1-q}|v_1(x)|\,dx. \qquad (3.11)
\end{aligned}$$

Now let $w_1 = \int_0^t s^{1-q}|v_1(s)|\,ds$. Then

$$w_1' = t^{1-q}|v_1(t)| = t^{-q}|u_1(t)| \tag{3.12}$$

and from (3.10)-(3.12) we obtain

$$w_1' - \frac{k_2 L}{(N-2)^2} t^{1-q} w_1 \le b_1 t^{1-q}. \tag{3.13}$$

Multiplying (3.13) by $\mu(t) = e^{-\frac{k_2 L t^{2-q}}{(2-q)(N-2)^2}} \le 1$, integrating on $[0,t]$, and rewriting gives

$$w_1 \le \frac{b_1}{\mu(t)} \int_0^t s^{1-q} \mu(s)\, ds \le \frac{b_1}{(2-q)} \frac{t^{2-q}}{\mu(t)}. \tag{3.14}$$

Then from (3.12)-(3.14) we obtain

$$u_1 \le \Big(\frac{k_2 L}{(2-q)(N-2)^2} \Big) \frac{b_1 t^{3-q}}{\mu(t)} + b_1 t = b_1 \left(t + B(t) t^{3-q} \right) \tag{3.15}$$

where

$$B(t) = \Big(\frac{k_2 L}{(2-q)(N-2)^2} \Big) \frac{1}{\mu(t)}. \tag{3.16}$$

Note that $\mu(t)$ is decreasing and continuous hence $B(t)$ is increasing and continuous.

Next it follows from (3.6) that

$$u_1' = b_1 - \int_0^t h(x) f(u_1)\, dx \tag{3.17}$$

and thus from (3.4), (3.15), (3.17), and since $B(t)$ is increasing:

$$\begin{aligned} |u_1'| &\le b_1 + \frac{k_2 L}{(N-2)^2} \int_0^t x^{-q} b_1 \left(x + B(x) x^{3-q} \right)\, dx \\ &\le b_1 + \frac{k_2 L b_1}{2(N-2)^2 (2-q)} \left(2t^{2-q} + B(t) t^{4-2q} \right). \end{aligned} \tag{3.18}$$

Thus from (3.15) and (3.18) we see that u_1 and u_1' are bounded on $[0,t]$ and so it follows that the solution of (3.1), (3.5) exists on $[0,t]$. Since t is arbitrary it follows that the solution of (3.1), (3.5) exists on $[0,\infty)$.

Lemma 3.1. *Let $N > 2$, $0 < p < 1$, and $2 < \alpha < 2(N-1)$. Assuming* (H1)–(H5) *and that u_1 solves* (3.1), (3.5) *then there exists $t_{b_1} > 0$ such that $u_1(t_{b_1}) = \beta$ and $0 < u_1 < \beta$ on $(0, t_{b_1})$. In addition, $u_1'(t) > 0$ on $[0, t_{b_1}]$.*

Proof. Since $u_1'(0) = b_1 > 0$ we see that u_1 is initially increasing, positive, and less than β. On this set $f(u_1) < 0$ and so by (3.1) we have $u_1'' > 0$. Thus by (3.5) we have $u_1' > b_1 > 0$ when $0 < u_1 < \beta$ and so on this set we have $u_1 > b_1 t$. Since $b_1 t$ exceeds β for sufficiently large t we see then that there exists $t_{b_1} > 0$ such that $u_1(t_{b_1}) = \beta$ and $0 < u_1 < \beta$ on $(0, t_{b_1})$. This completes the proof. □

Lemma 3.2. *Let $N > 2$, $0 < p < 1$, and $2 < \alpha < 2(N-1)$. Assuming* (H1)–(H5) *and that u_1 solves* (3.1), (3.5) *then $t_{b_1} \to \infty$ as $b_1 \to 0^+$.*

Proof. Evaluating (3.15) at $t = t_{b_1}$ gives:

$$\beta = u_1(t_{b_1}) \le b_1 (t_{b_1} + B(t_{b_1}) t_{b_1}^{3-q}). \tag{3.19}$$

Since $2 < \alpha < 2(N-1)$ it then follows from the note after (3.4) that $0 < q < 2$. Now if t_{b_1} is bounded as $b_1 \to 0^+$ then the right-hand side of (3.19) goes to 0 as $b_1 \to 0^+$ which violates (3.19). Thus we obtain a contradiction and so we see that $t_{b_1} \to \infty$ as $b_1 \to 0^+$. This completes the proof. □

Lemma 3.3. *Let $N > 2$, $0 < p < 1$, and $N - p(N-2) < \alpha < 2(N-1)$. Assuming* (H1)–(H5) *and that u_1 solves* (3.1), (3.5) *then u_1 has a local maximum, M_{b_1}, on* $(0, \infty)$.

Proof. From Lemma 3.1 it follows that there exists $t_{b_1} > 0$ such that $u_1(t_{b_1}) = \beta$ and $u_1' > 0$ on $[0, t_{b_1}]$. Now if u_1 does not have a local maximum then $u_1' \geq 0$ for $t > t_{b_1}$ and so $u_1 \geq u_1(t_{b_1} + \delta) > \beta > 0$ for $t > t_{b_1} + \delta$ and some $\delta > 0$. Then from (H2) we see that there is a $C_3 > 0$ such that $f(u_1) \geq C_3$ on $[t_{b_1} + \delta, \infty)$. Thus

$$-u_1'' = h(t)f(u_1) \geq C_3 h(t) \text{ for } t > t_{b_1} + \delta. \tag{3.20}$$

We now divide the rest of the proof into 3 cases.

Case 1: $N < \alpha < 2(N-1)$ In this case we see from (3.4) that $0 < q < 1$ so integrating (3.20) on $(t_{b_1} + \delta, t)$ and using (3.4) gives

$$u_1' \leq u_1'(t_{b_1} + \delta) - \frac{k_1 C_3}{(1-q)(N-2)^2}\left(t^{1-q} - (t_{b_1} + \delta)^{1-q}\right) \to -\infty \quad \text{as } t \to \infty.$$

Thus u_1' gets negative which contradicts that $u_1' \geq 0$ for $t > 0$ and so u_1 must have a local maximum.

Case 2: $\alpha = N$ In this case we have $q = 1$ by (3.4) and so again integrating (3.20) on $(t_{b_1} + \delta, t)$ we obtain

$$u_1' \leq u_1'(t_{b_1} + \delta) - \frac{k_1 C_3}{(N-2)^2}\left(\ln(t) - \ln(t_{b_1} + \delta)\right) \to -\infty \text{ as } t \to \infty$$

which again contradicts that $u_1' \geq 0$ for $t > 0$. Thus u_1 must have a local maximum.

Case 3: $N - p(N-2) < \alpha < N$ We denote

$$E_1 = \frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1) \tag{3.21}$$

and observe from (3.1)-(3.2) that

$$E_1' = \Big(\frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1)\Big)' = -\frac{u_1'^2 h'}{2h^2} \geq 0. \tag{3.22}$$

In addition we see from (3.4) that $E_1(0) = 0$ and so $E_1(t) \geq 0$ for $t \geq 0$.

We suppose now that u_1 is increasing for $t > t_{b_1}$. We first show that there exists $t_{b_2} > t_{b_1}$ such that $u(t_{b_2}) = \gamma$. So we suppose by the way of contradiction that $0 < u_1 < \gamma$ and $u_1' \geq 0$ for $t > t_{b_1}$.

Then from (3.1)-(3.2) and (H3) we have

$$\Big(\frac{1}{2}u_1'^2 + h(t)F(u_1)\Big)' = h'(t)F(u_1) \geq 0 \quad \text{when } 0 \leq u_1 \leq \gamma. \tag{3.23}$$

Now we recall from (H1) that $\lim_{u_1 \to 0} \frac{F(u_1)}{u_1^2} = \frac{f'(0)}{2}$. Also since $u_1(0) = 0$ and $u_1'(0) = b_1$ then $\lim_{t \to 0^+} \frac{u_1}{t} = b_1$. Therefore for small positive t and (3.4) we have

$$0 \leq h(t)|F(u_1)| = t^2 h(t)\frac{|F(u_1)|}{u_1^2}\frac{u_1^2}{t^2} \leq \frac{|f'(0)|\, k_2\, b_1^2\, t^{2-q}}{(N-2)^2} \to 0 \tag{3.24}$$

as $t \to 0^+$ since $q < 2$. Therefore, integrating (3.23) on $(0, t)$ and using (3.24) we obtain

$$\frac{1}{2}u_1'^2 + h(t)F(u_1) \geq \frac{1}{2}b_1^2 \quad \text{when } 0 \leq u_1 \leq \gamma. \tag{3.25}$$

In addition, since $0 \le u_1 \le \gamma$ it follows that $h(t)F(u_1) \le 0$ and thus from (3.25),

$$u_1' \ge b_1 \quad \text{when } 0 \le u_1 \le \gamma. \tag{3.26}$$

Integrating on $(0,t)$ we obtain

$$u_1 \ge b_1 t \to \infty \text{ as } t \to \infty$$

- a contradiction since we assumed $u_1 < \gamma$. Thus there exists $t_{b_2} > t_{b_1}$ such that $u(t_{b_2}) = \gamma$ and $u_1' \ge b_1 > 0$ on $[0, t_{b_2}]$ by (3.26).

We show now that $u_1(t) \to \infty$ as $t \to \infty$. If not then u_1 is bounded from above and so there exists $Q > \gamma$ such that $\lim_{t\to\infty} u_1(t) = Q$. Returning to (3.1) we see that this implies:

$$\lim_{t\to\infty} \frac{u_1''}{h(t)} = -f(Q) < 0. \tag{3.27}$$

In particular, $u_1'' < 0$ for large t and so u_1' is decreasing for large t. Since $u_1' > 0$ for large t it follows that $\lim_{t\to\infty} u_1'$ exists. This limit must be zero otherwise this would imply $u_1 \to \infty$ as $t \to \infty$ contradicting the assumption that u_1 is bounded. Thus $\lim_{t\to\infty} u_1' = 0$. Next denoting $H(t) = \int_t^\infty h(s)\,ds$ we see that since $N - p(N-2) < \alpha < N$ and $q = \frac{2(N-1)-\alpha}{N-2}$ this implies:

$$1 < q < 1 + p < 2. \tag{3.28}$$

Therefore by (3.4) we see that $h(t)$ is integrable at infinity so $H(t)$ is defined. Then by (3.27) and L'Hôpital's rule we see that

$$\lim_{t\to\infty} \frac{u_1'}{H(t)} = \lim_{t\to\infty} -\frac{u_1''}{h(t)} = f(Q) > 0. \tag{3.29}$$

Then from (3.4) and (3.28)-(3.29) we see

$$u_1' \ge \frac{f(Q)}{2} H(t) \ge \frac{k_1 f(Q)}{2(q-1)(N-2)^2} t^{1-q} \quad \text{for large } t. \tag{3.30}$$

Now integrating (3.30) on (t_0, t) where t_0 and t are sufficiently large gives

$$u_1 \ge u_1(t_0) + \frac{k_1 f(Q)}{2(q-1)} \frac{t^{2-q}}{(2-q)(N-2)^2} \to \infty \quad \text{as } t \to \infty \text{ since } q < 2$$

- a contradiction since we assumed u_1 was bounded. Thus if $u_1' > 0$ for $t > 0$ then it must be that $u_1 \to \infty$ as $t \to \infty$.

Next recalling (3.23) we have

$$\Big(\frac{1}{2} u_1'^2 + h(t)F(u_1)\Big)' = h'(t)F(u_1) < 0 \quad \text{when } u_1 > \gamma. \tag{3.31}$$

Integrating this on (t_{b_2}, t) gives

$$\frac{1}{2} u_1'^2 + h(t)F(u_1) \le \frac{1}{2} u_1'^2(t_{b_2}) \quad \text{for } t > t_{b_2}. \tag{3.32}$$

On (t_{b_2}, t) we have $h(t)F(u_1) > 0$ and thus from (3.32):

$$|u_1'| < |u_1'(t_{b_2})| \text{ for } t > t_{b_2}. \tag{3.33}$$

We claim now that

$$\lim_{t\to\infty} \frac{t^2 h(t) f(u_1)}{u_1} = \infty. \tag{3.34}$$

Integrating (3.33) on (t_{b_2}, t) gives

$$u_1 < \gamma + (t - t_{b_2})|u_1'(t_{b_2})| \le C_4 t \quad \text{for some } C_4 > 0 \text{ for large } t. \tag{3.35}$$

Next from (H2) we have

$$\frac{f(u_1)}{u_1^p} \geq 1 - \epsilon \text{ for large } u_1.$$

Thus by (3.35),

$$\frac{f(u_1)}{u_1} \geq \frac{(1-\epsilon)u_1^p}{u_1} = \frac{(1-\epsilon)}{u_1^{1-p}} \geq \frac{(1-\epsilon)}{C_4^{1-p} t^{1-p}} \quad \text{for large } t. \tag{3.36}$$

Therefore by (3.4), (3.28), and (3.36):

$$\frac{t^2 h(t) f(u_1)}{u_1} \geq \frac{k_1(1-\epsilon)}{C_4^{1-p}(N-2)^2} \frac{t^{2-q}}{t^{1-p}} = \frac{k_1(1-\epsilon)}{C_4^{1-p}(N-2)^2} t^{1+p-q} \to \infty,$$

since $1 + p > q$. This establishes (3.34).

Next we rewrite (3.1) as

$$u_1'' + \frac{t^2 h(t) f(u_1)}{u_1} \frac{u_1}{t^2} = 0. \tag{3.37}$$

Now it follows from (3.34) that we may choose t_0 sufficiently large so that

$$\frac{t^2 h(t) f(u_1)}{u_1} \geq A > \frac{1}{4} \quad \text{on } [t_0, \infty).$$

Next let y_1 be the solution of

$$y_1'' + A\frac{y_1}{t^2} = 0 \tag{3.38}$$

with $y_1(t_0) = u_1(t_0) = \gamma$ and $y_1'(t_0) = u_1'(t_0) > 0$. It follows then for some constants $d_1 \neq 0$ and d_2 that

$$y_1 = d_1\sqrt{t}\Big(\sin\big(\ln\big(t\sqrt{A - \frac{1}{4}}\big) + d_2\big)\Big)$$

and so clearly y_1 has an infinite number of local extrema on $[t_0, \infty)$. Consider now the interval $[t_0, M]$ such that $y_1 > 0$, $y_1' > 0$ on $[t_0, M]$ and $y_1'(M) = 0$. We claim now that u_1' must get negative on $[t_0, M]$. So suppose not. Then $u_1' \geq 0$ on $[t_0, M]$. Then multiplying (3.37) by y_1, multiplying (3.38) by u_1, and subtracting we obtain

$$(y_1 u_1' - y_1' u_1)' + \Big(\frac{t^2 h(t) f(u_1)}{u_1} - A\Big)\frac{y_1 u_1}{t^2} = 0.$$

Integrating this on $[t_0, M]$ gives

$$y_1(M)u_1'(M) + \int_{t_0}^{M} \Big(\frac{t^2 h(t) f(u_1)}{u_1} - A\Big)\frac{y_1 u_1}{t^2}\, dt = 0. \tag{3.39}$$

The integral term in (3.39) is positive by (3.34) and also $y_1(M)u_1'(M) \geq 0$ yielding a contradiction. Therefore we see that u_1 must have a maximum, $M_{b_1} > 0$, and $u_1' > 0$ on $[0, M_{b_1})$. This completes the proof. □

Lemma 3.4. *Let $N > 2$, $0 < p < 1$, and $N - p(N-2) < \alpha < 2(N-1)$. Assuming* (H1)–(H5) *and that u_1 solves* (3.1), (3.5) *then there exists $t_{b_3} > M_{b_1}$ such that $u_1(t_{b_3}) = \frac{\beta+\gamma}{2}$ and $u_1' < 0$ on $(M_{b_1}, t_{b_3}]$.*

Proof. If $u_1 \geq \frac{\beta+\gamma}{2}$ for all $t \geq M_{b_1}$, then $f(u_1) > 0$ for $t \geq M_b$. Then from (3.1) it follows that $u_1'' < 0$ and thus $u_1'(t) \leq u_1'(t_0) < 0$ for $t > t_0 > M_{b_1}$. Integrating this inequality on (t_0, t) gives

$$u_1(t) \leq u_1(t_0) + u_1'(t_0)(t - t_0) \to -\infty \quad \text{as } t \to \infty$$

which gives a contradiction since we assumed $u_1 \geq \frac{\beta+\gamma}{2}$ for all $t \geq M_{b_1}$. Thus there exists $t_{b_3} > M_{b_1}$ such that $u_1(t_{b_3}) = \frac{\beta+\gamma}{2}$, $u_1 > \frac{\beta+\gamma}{2}$, and $u_1' < 0$ on $(M_{b_1}, t_{b_3}]$. □

Lemma 3.5. *Let $N > 2$, $0 < p < 1$, and $N - p(N-2) < \alpha < 2(N-1)$. Assuming* (H1)–(H5) *and that u_1 solves* (3.1), (3.5) *then there exists $z_{1,b_1} > M_{b_1}$ such that $u_1(z_{1,b_1}) = 0$. In fact, u_1 has an infinite number of zeros on* $(0, \infty)$.

Proof. Suppose now by the way of contradiction that $0 < u_1 < \gamma$ and thus $F(u_1) < 0$ for $t > t_{b_3}$. Then from (3.21)-(3.22) we have

$$\frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1) \geq F(u_1(M_{b_1})) > 0 \text{ for } t \geq M_{b_1}. \tag{3.40}$$

Therefore by (3.4) and (3.40) we have

$$u_1'^2 \geq 2h(t)F(u_1(M_{b_1})) \geq \frac{2k_1 F(u_1(M_{b_1}))}{(N-2)^2 t^q}$$

for $t > t_{b_3}$. Thus:

$$-u_1' \geq C_5 t^{-q/2} \quad \text{where } C_5 = \frac{\sqrt{2k_1 F(u_1(M_{b_1}))}}{N-2} > 0 \text{ for } t > t_{b_3}. \tag{3.41}$$

Integrating (3.41) on (t_{b_3}, t) gives

$$u_1 \leq \frac{\beta+\gamma}{2} - C_5\Big(\frac{t^{1-\frac{q}{2}} - t_{b_3}^{1-\frac{q}{2}}}{1-\frac{q}{2}}\Big) \to -\infty \quad \text{as } t \to \infty \text{ since } q < 2.$$

Thus u_1 gets negative contradicting that $u_1 > 0$ on $(0, \infty)$. Hence there exists $z_{1,b_1} > M_{b_1}$ such that $u_1(z_{1,b_1}) = 0$ and $u_1' < 0$ on $(M_{b_1}, z_{1,b_1}]$.

In a similar way to Lemma 3.3 we can show that u_1 has a negative local minimum, $m_{b_1} > z_{1,b_1}$, and similar to Lemma 3.5 we can show that u_1 has a second zero $z_{2,b_1} > m_{b_1}$. It then in fact follows that u_1 has an infinite number of zeros z_{n,b_1}. This completes the proof. □

Proof of Theorem 1.1. By continuous dependence on initial conditions it follows that z_{1,b_1} is a continuous function of b_1. In addition, by Lemma 3.2 it follows that $t_{b_1} \to \infty$ as $b_1 \to 0^+$ and since $z_{1,b_1} > t_{b_1}$ it follows that $z_{1,b_1} \to \infty$ as $b_1 \to 0^+$.

So now let k, n be nonnegative integers with $0 \leq k \leq n$. Choose $R > 0$ sufficiently small so that $z_{1,b_1} < \cdots < z_{n,b_1} < R^{2-N}$. Then by the intermediate value theorem there exists a smallest value of $b_1 > 0$, say $b_{1,k}$, such that $z_{k,b_{1,k}} = R^{2-N}$. Then $u_1(t, b_{1,k})$ is a solution of (3.1) and (3.5) such that $u_1(t, b_{1,k})$ has k zeros on $(0, R^{2-N})$.

Finally defining

$$U_k(r) = (-1)^k u_1(r^{2-N}, b_{1,k})$$

we see that U_k solves (1.4), U_k has k zeros on (R, ∞), and $\lim_{r\to\infty} U_k(r) = 0$. This completes the proof. □

Note: A crucial step in proving Theorem 1.1 is Lemma 3.3 which says that if $N - p(N-2) < \alpha < 2(N-1)$ then every solution of (3.1), (3.5) must have a local maximum. We conjecture that a similar lemma does not hold for $2 < \alpha < N - p(N-2)$ because for an appropriate constant $c > 0$ the function $ct^{\frac{\alpha-2}{(N-2)(1-p)}}$ is a monotonically increasing solution of the model equation

$$u'' + \frac{1}{t^q} u^p = 0$$

with $q = \frac{2(N-1)-\alpha}{N-2}$ and $0 < p < 1$.

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