Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 218, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SOLVABILITY OF SOME NEUMANN-TYPE BOUNDARY VALUE PROBLEMS FOR BIHARMONIC EQUATIONS

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Communicated by Mokhtar Kirane

Abstract. We study some boundary-value problems for inhomogeneous biharmonic equation with periodic boundary conditions. These problems are generalization to periodic data of the Neumann-type boundary-value problems considered before by the authors. We obtain existence and uniqueness of solutions for the problems under consideration.

1. Introduction

Many stationary processes occurring in physics and mechanics are described by equations of elliptic type. One of the important particular cases of fourth-order elliptic equations is the biharmonic equation. Solution of the plane deformation problems in elasticity theory in many cases can be reduced to the integration of biharmonic equations under corresponding boundary conditions. In addition, many problems of continuous media mechanics can be reduced to the solution of harmonic and biharmonic equations. However, convenient analytic expressions for the solutions of these problems are obtained only for domains of particular forms.

Applications of biharmonic problems in mechanics and physics are described in numerous investigations (see, for example, [1, 7, 28]). Applications of boundary value problems for biharmonic equations in mechanics and physics stimulate the study of various boundary value problems for biharmonic equations. One of the well known boundary value problems for biharmonic equations is the Dirichlet problem [3, 6, 14, 15, 16, 17, 36]. Recently other types of boundary value problems for biharmonic equation such as Riquier problem [8, 18, 29], Neumann problem [5, 9, 10, 19, 20, 21, 22, 24, 30, 31], spectral Steklov problem [11], Robin problem [13], generalized Robin problem [23], as well as fractional analogous of Neumann problem [4, 32, 33, 34] are begun to investigate intensively.

The theory of polyharmonic (biharmonic) equations and various boundary value problems for them was described in great detail in [12]. Conditions for the solvability of boundary value problems for elliptic equations and systems of equations contain the so-called complementing conditions. It was established that all problems of the given type are Fredholm-type problems. Therefore, the solvability of

2010 *Mathematics Subject Classification.* 35J40, 31B30.
Key words and phrases. Neumann-type boundary value problem; biharmonic equation; periodic boundary conditions.

Submitted July 28, 2017. Published September 13, 2017.

these problems for homogeneous boundary conditions is guaranteed by the orthogonality of the right-hand sides of the equation to all solutions of the corresponding homogeneous conjugate equation. In the considered below particular case (biharmonic equation) of the common problem more detailed results can be obtained. In the present paper a new class of boundary value problems for inhomogeneous biharmonic equation $\Delta^2 u(x) = f(x)$ in the unit ball with periodic boundary conditions is studied.

This article is organized as follows. In Section 2 the statement of the main problem (2.1)-(2.4) is given. Some preliminary results are cited in Section 3. The necessary and sufficient conditions for solvability of the Neumann-type boundary value problems (3.1)-(3.3) are given in Theorems 3.1–3.3. Auxiliary integral equalities are derived in Lemmas 4.1–4.7 In Section 5 uniqueness conditions for the main problem (2.1)-(2.4) are given: in case of $k = 1$ in Theorem 5.1 and in case of $k = 2$ in Theorem 5.2. The necessary and sufficient existence conditions for the problem (2.1)-(2.4) are obtained in Section 6: in case of $k = 1$ in Theorem 6.1 and in case of $k = 2$ in Theorem 6.2.

2. Statement of the problem

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball, $n \geq 2$, and $\partial\Omega = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere. Denote $\partial\Omega_+ = \{x \in \partial\Omega : x_n \geq 0\}$, $\partial\Omega_- = \{x \in \partial\Omega : x_n \leq 0\}$ and $I = \cap\{x \in \partial\Omega : x_n = 0\}$. To each point $x \in \Omega$ we associate the"opposite" point $x^* = \bar{\alpha}x$, where $\bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_n = -1$ and the other α_j, $j = 1, 2, \dots, n-1$ take one of the values ± 1. Obviously, if $x \in \partial\Omega_+$ then $x^* \in \partial\Omega_-$. Further, let ν be the unit normal to $\partial\Omega$ and $D_\nu^m = \frac{\partial^m}{\partial \nu^m}$ $(m \geq 1)$ be the normal derivative of order m.

In the domain Ω for $k = 1, 2$ consider the following boundary value problems:

$$\Delta^2 u(x) = f(x),\ x \in \Omega, \tag{2.1}$$

$$D_\nu^m u(x) = g(x),\ x \in \partial\Omega, \tag{2.2}$$

$$D_\nu^{\ell_1} u(x) - (-1)^k D_\nu^{\ell_1} u(x^*) = g_1(x),\ x \in \partial\Omega_+, \tag{2.3}$$

$$D_\nu^{\ell_2} u(x) + (-1)^k D_\nu^{\ell_2} u(x^*) = g_2(x),\ x \in \partial\Omega_+, \tag{2.4}$$

where $1 \leq m \leq 3$, $1 \leq \ell_1 < \ell_2 \leq 3$, $\ell_1 \neq m$, $\ell_2 \neq m$.

By a solution of the problem (2.1)-(2.4) we mean a function $u(x) \in C^4(\Omega) \cap C^3(\bar{\Omega})$, which satisfies the conditions (2.1)-(2.4) in the classical sense.

Let $\partial^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}$, where $\beta = (\beta_1, \dots, \beta_n)$ is the multi-index, $|\beta| = \beta_1 + \dots + \beta_n$ and $\partial^0 = I$ is the unit operator.

It is obvious that the necessary condition for existence of the solution to the problem (2.1)-(2.4) belonging to the class $C^3(\bar{\Omega})$ are the following compatibility conditions:

$$\partial^\beta g_1(\tilde{x}, 0) + (-1)^k \partial^\beta g_1(\tilde{\alpha}\tilde{x}, 0) = 0, \quad |\beta| \leq p, \tag{2.5}$$

$$\partial^\beta g_2(\tilde{x}, 0) - (-1)^k \partial^\beta g_2(\tilde{\alpha}\tilde{x}, 0) = 0, \quad |\beta| \leq q, \tag{2.6}$$

where $\tilde{x} = (x_1, \dots, x_{n-1})$, $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{n-1})$, p and q take the values $0, 1, 2, 3$ depending on the order of the boundary operators $D_\nu^{\ell_1}$ and $D_\nu^{\ell_2}$. Note that analogous problems for the Poisson equation were investigated in [25, 26, 27, 34].

3. Preliminary results

In this section we consider the following Neumann-type problems:

$$\Delta^2 u(x) = f(x),\ x \in \Omega, \tag{3.1}$$

$$D_\nu^{m_1} u(x) = g_1(x),\ x \in \partial\Omega, \tag{3.2}$$

$$D_\nu^{m_2} u(x) = g_2(x),\ x \in \partial\Omega, \tag{3.3}$$

where $1 \le m_1 < m_2 \le 3$.

Problems (3.1)-(3.3) for different values of m_1 and m_2 are studied in [5, 9, 10, 19, 20, 21, 22, 24, 30, 31]. The case $f(x) = 0,\ m_1 = 1,\ m_2 = 2$ was considered by Bitsadze in [5]. It was established that the necessary and sufficient condition for solvability of the problem (3.1)-(3.3) have the form

$$\int_{\partial\Omega} [g_2(x) - g_1(x)] dS_x = 0.$$

Further in [19] the following statement is established.

Theorem 3.1. *Let $m_1 = 1,\ m_2 = 2,\ f(x) \in C(\bar\Omega),\ g_1(x) \in C^1(\partial\Omega),\ g_2(x) \in C(\partial\Omega)$. Then for solvability of the problem (3.1)-(3.3) it is necessary and sufficient that the following condition be fulfilled*

$$\int_{\partial\Omega} [f_2(x) - f_1(x)]\, dS_x = \frac{1}{2}\int_\Omega \left(1 - |x|^2\right) f(x) dx. \tag{3.4}$$

If a solution of the problem exists then it is unique up to a constant term.

The case $m_1 = 2,\ m_2 = 3$ is investigated in [30]. The following statement is proved.

Theorem 3.2. *Let $m_1 = 2,\ m_2 = 3,\ f(x) \in C^{\lambda+1}(\bar\Omega),\ g_1(x) \in C^{\lambda+2}(\partial\Omega),\ g_2(x) \in C^{\lambda+1}(\partial\Omega)$. Then for solvability of problem (3.1)-(3.3) it is necessary and sufficient that the following conditions be fulfilled*

$$\int_{\partial\Omega} g_2(x)\, dS_x = \frac{n-1}{2}\int_\Omega |x|^2 f(x)\, dx - \frac{n-3}{2}\int_\Omega f(x)\, dx, \tag{3.5}$$

$$\int_{\partial\Omega} x_j [g_2(x) - g_1(x)]\, dS_x = \frac{n-1}{2}\int_\Omega x_j |x|^2 f(x)\, dx - \frac{n-3}{2}\int_\Omega x_j f(x)\, dx \tag{3.6}$$

for $j = 1, \dots, n$. If solution of the problem exists, then it is unique up to the first order polynomials.

In [21] the following statement is obtained.

Theorem 3.3. *Let $m_1 = 1$, $m_2 = 3$, $f(x) = C(\bar\Omega)$, $g_1(x) \in C(\partial\Omega)$, $g_2(x) \in C(\partial\Omega)$. Then for solvability of problem (3.1)-(3.3) it is necessary and sufficient that the following condition be fulfilled*

$$\int_{\partial\Omega} g_2(x) dS_x = \frac{n-1}{2}\int_\Omega |x|^2 f(x)\, dx - \frac{n-3}{2}\int_\Omega f(x)\, dx. \tag{3.7}$$

If a solution of the problem exists, then it is unique up to a constant term.

4. Auxiliary integral equalities

In what follows we need some integral equalities. Let $f(x) \in C(\bar\Omega)$, $g_1(x) \in C(\partial\Omega)$, $g_2(x) \in C(\partial\Omega)$. Denote

$$\begin{gathered} f^{\pm}(x) = \frac{f(x) \pm f(x^*)}{2},\ g^{\pm}(x) = \frac{g(x) \pm g(x^*)}{2}, \\ \tilde g^{\pm}(x) = \frac{1}{2}\begin{cases} g(x), & x \in \partial\Omega_+, \\ \pm g(x^*), & x \in \partial\Omega_-\,. \end{cases} \end{gathered} \tag{4.1}$$

Lemma 4.1. *Let $f(x) = C(\bar\Omega)$ and $g(x) \in C(\partial\Omega)$. Then*

$$\int_\Omega f(x^*)\,dx = \int_\Omega f(x)\,dx, \tag{4.2}$$

$$\int_{\partial\Omega} g(x^*)dx = \int_{\partial\Omega} g(x)dx. \tag{4.3}$$

Proof. Let $\bar\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_n = -1$ and the other α_j, $j = \overline{1, n-1}$ take one of the values ±1. Consider the matrix

$$P = \begin{pmatrix} \alpha_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix}.$$

It is obvious that $P^T = P$ and $P \cdot P^T = E$. Consequently, P is an orthogonal matrix. It is known (see e.g. [2]), that if P is an orthogonal matrix, then

$$\int_\Omega f(Px)dx = \int_\Omega f(x)dx, \quad \int_{\partial\Omega} g(Px)dS_x = \int_{\partial\Omega} g(x)dS_x.$$

Since $x^* = Px$ then we obtain (4.2) and (4.3). □

Corollary 4.2. *Let $f(x) \in C(\bar\Omega)$ and $g(x) \in C(\partial\Omega)$. Then the following equalities hold:*

$$\int_\Omega f^+(x)dx = \int_\Omega f(x)dx, \quad \int_\Omega |x|^2 f^+(x)dx = \int_\Omega |x|^2 f(x)dx, \tag{4.4}$$

$$\int_\Omega f^-(x)dx = 0, \int_\Omega |x|^2 f^-(x)dx = 0, \tag{4.5}$$

$$\int_{\partial\Omega} g^+(x)dS_x = \int_{\partial\Omega} g(x)dS_x, \tag{4.6}$$

$$\int_{\partial\Omega} g^-(x)dS_x = 0. \tag{4.7}$$

Proof. By the definition of the functions $f^{\pm}(x)$ we obtain

$$\int_\Omega f^{\pm}(x)\,dx = \frac12\int_\Omega f(x)\,dx \pm \frac12\int_\Omega f(x^*)\,dx = \frac12\int_\Omega f(x)dx \pm \frac12\int_\Omega f(x)\,dx.$$

Further, since $|x| = |x^*|$ then (4.2) implies equalities (4.4) and (4.5). Equalities (4.6) and (4.7) can be proved similarly. □

Lemma 4.3. *Let $g(x) \in C(\partial\Omega)$. Then the following equality holds*

$$\int_{\partial\Omega_-} g(x^*)\,dS_x = \int_{\partial\Omega_+} g(x)\,dS_x. \tag{4.8}$$

Proof. To prove (4.8) we pass to the spherical coordinate system:

$$x_1 = \cos\theta_1, \quad x_2 = \sin\theta_1\cos\theta_2, \dots,\ x_{n-1} = \sin\theta_1 \dots \sin\theta_2 \cos\theta_{n-1},$$
$$x_n = \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2}\sin\theta_{n-1},$$

where

$$0 \le \theta_j \le \pi, \quad j = 1,2,\dots,\ n-2,\ 0 \le \theta_{n-1} \le 2\pi.$$

The Jacobian of this mapping has the form

$$J(\theta) = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \dots \sin\theta_{n-1}.$$

Furthermore, we use the following elementary equalities:

$$\cos(\pi \pm \theta) = -\cos\theta, \quad \sin(\pi \pm \theta) = \pm \sin\theta.$$

Since $\partial\Omega_- = \partial\Omega \cap \{x \in \mathbb{R}^n : x_n \le 0\}$ if and only if $\pi \le \theta_{n-1} \le 2\pi$, $0 \le \theta_j \le \pi$, $j = 1, 2, \dots, n-2$, then

$$\begin{aligned}\int_{\partial\Omega_-} g(x^*) dS_x = \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{n-2} \int_\pi^{2\pi} g(\alpha_1\cos\theta_1, \alpha_2 \sin\theta_1\cos\theta_2, \dots,\\ -\sin\theta_1 \dots \sin\theta_{n-1}) J(\theta) d\theta_{n-1}.\end{aligned}$$

Let us make the change of variables in the last integral,

$$\theta_j = \begin{cases} \pi - \xi_j, & \alpha_j = -1 \\ \xi_j, & \alpha_j = 1,\ j = 1,2,\dots,n-2, \end{cases},$$
$$\theta_{n-1} = \pi + \xi_{n-1}.$$

Note that under this change of variables we obtain the equality (if $\alpha_j = -1$, $j = 1,2,\dots,n-2$)

$$\begin{aligned} J(\xi) &= \sin^{n-2}(\pi - \xi_1)\sin^{n-3}(\pi - \xi_2)\dots\sin(\pi - \xi_{n-2}) \\ &= \sin^{n-2}\xi_1 \sin^{n-3}\xi_2 \dots \sin\xi_{n-2}, \end{aligned}$$

i.e. the Jacobian's sign is not changed. Further, since

$$-\cos\theta_1 = -\cos(\pi - \xi_1) = \cos\xi_1,$$
$$-\sin\theta_j = -\sin(\pi - \xi_j) = -\cos\pi\sin\xi_j = \sin\xi_j,$$

then after the change of variables we have

$$\begin{aligned} \int_{\partial\Omega_-} g(x^*)\, dS_x &= \int_0^\pi d\xi_1 \dots \int_0^\pi d\xi_{n-2} \int_0^\pi g(\cos\xi_1, \sin\xi_1\cos\xi_2, \dots, \\ &\quad \sin\xi_1 \dots \sin\xi_{n-1}) J(\xi)\, d\xi_{n-1} \\ &= \int_{\partial\Omega_+} g(x)\, dS_x. \end{aligned}$$

□

Corollary 4.4. *Let $g(x) \in C(\partial\Omega)$. Then*

$$\int_{\partial\Omega} \tilde{g}^+(x) dS_x = \int_{\partial\Omega_+} g(x) dS_x, \tag{4.9}$$

$$\int_{\partial\Omega} \tilde{g}^-(x) dS_x = 0. \tag{4.10}$$

Proof. Using definition of the function $\tilde{g}^+(x)$ and the equality (4.8), we have

$$\begin{aligned}\int_{\partial\Omega}\tilde{g}^+(x)dS_x &= \frac{1}{2}\int_{\partial\Omega_+}g(x)dS_x+\frac{1}{2}\int_{\partial\Omega_-}g(x^*)dS_x\\ &= \frac{1}{2}\int_{\partial\Omega_+}g(x)dS_x+\frac{1}{2}\int_{\partial\Omega_+}g(x)dS_x\\ &= \int_{\partial\Omega_+}g(x)dS_x.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_{\partial\Omega}\tilde{g}^-(x)dS_x &= \frac{1}{2}\int_{\partial\Omega_+}g(x)dS_x-\frac{1}{2}\int_{\partial\Omega_-}g(x^*)dS_x\\ &= \frac{1}{2}\int_{\partial\Omega_+}g(x)dS_x-\frac{1}{2}\int_{\partial\Omega_+}g(x)dS_x=0.\end{aligned}$$

□

Lemma 4.5. *Let $f(x)\in C(\partial\bar{\Omega})$. Then*

$$\int_\Omega x_jf(x^*)dx=\alpha_j\int_\Omega x_jf(x)dx,\quad j=1,2,\dots,n \tag{4.11}$$

Proof. Since $x^*=(\alpha_1x_1,\alpha_2x_2,\dots,\alpha_{n-1}x_{n-1},-x_n)$, it follows that

$$\begin{aligned}\int_\Omega x_jf(x^*)dx=\int_{-1}^{1}\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\dots\int_{-\sqrt{1-x_1^2-\dots-x_{n-1}^2}}^{\sqrt{1-x_1^2-\dots-x_{n-1}^2}}x_jf(\alpha_1x_1,\alpha_2x_2,\dots,\\ \alpha_{n-1}x_{n-1},-x_n)dx_n\dots dx_1.\end{aligned}$$

In the above integral we make the change of variables $y_k=\alpha_kx_k$, $k=1,2,\dots,n$, where $\alpha_n=-1$. Then

$$\begin{aligned}&\int_\Omega x_jf(x^*)dx\\ &=\int_{-\alpha_1}^{\alpha_1}\int_{-\alpha_2\sqrt{1-y_1^2}}^{\alpha_2\sqrt{1-y_1^2}}\dots\int_{-\alpha_n\sqrt{1-y_1^2-\dots-y_{n-1}^2}}^{\alpha_n\sqrt{1-y_1^2-\dots-y_{n-1}^2}}\frac{y_j}{\alpha_j}f(y_1,\dots,y_{n-1},y_n)\frac{dy_n}{\alpha_n}\dots\frac{dy_1}{\alpha_1}\\ &=\alpha_j\int_{-1}^{1}\int_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}}\dots\int_{-\sqrt{1-y_1^2-\dots-y_{n-1}^2}}^{\sqrt{1-y_1^2-\dots-y_{n-1}^2}}y_jf(y_1,\dots,y_{n-1},y_n)dy_n\dots dy_1\\ &=\alpha_j\int_\Omega x_jf(x)dx.\end{aligned}$$

□

Corollary 4.6. *If $f(x)\in C(\bar{\Omega})$, then for all $j=1,2,\dots,n$ we have*

$$\int_\Omega x_jf^+(x)dx=\frac{1+\alpha_j}{2}\int_\Omega x_jf(x)dx, \tag{4.12}$$

$$\int_\Omega x_jf^-(x)dx=\frac{1-\alpha_j}{2}\int_\Omega x_jf(x)dx. \tag{4.13}$$

Proof. Using (4.10) and according to definition (4.1) of function $f^+(x)$ we obtain

$$\int_\Omega x_jf^+(x)dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_\Omega x_j f(x)\,dx + \frac{1}{2}\int_\Omega x_j f(x^*)\,dx \\
&= \frac{1}{2}\int_\Omega x_j f(x)\,dx + \frac{\alpha_j}{2}\int_\Omega x_j f(x)\,dx \\
&= \frac{1+\alpha_j}{2}\int_\Omega x_j f(x)\,dx.
\end{aligned}
$$

Similarly we can obtain

$$
\int_\Omega x_j f^-(x)\,dx = \frac{1}{2}\int_\Omega x_j f(x)\,dx - \frac{1}{2}\int_\Omega x_j f(x^*)\,dx = \frac{1-\alpha_j}{2}\int_\Omega x_j f(x)\,dx.
$$

□

Lemma 4.7. *Let $g(x) \in C(\partial\Omega)$. Then for $j = 1, 2, \dots, n$,*

$$
\int_{\partial\Omega} x_j g(x^*)\,dS_x = \alpha_j \int_{\partial\Omega} x_j g(x)\,dS_x, \tag{4.14}
$$

$$
\int_{\partial\Omega_-} x_j g(x^*)\,dS_x = \alpha_j \int_{\partial\Omega_+} x_j g(x)\,dS_x. \tag{4.15}
$$

Proof. To prove this statement we pass to the spherical coordinate system (see Lemma 4.3). Then we obtain

$$
\begin{aligned}
\int_{\partial\Omega} x_j g(x^*)\,dS_x &= \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{n-2} \int_0^\pi \sin\theta_1 \dots \sin\theta_{j-1}\cos\theta_j \\
&\quad \times g(\alpha_1\cos\theta_1, \dots, -\sin\theta_1 \dots \sin\theta_{n-1}) J(\theta) d\theta_{n-1} \\
&\quad + \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{n-2} \int_\pi^{2\pi} \sin\theta_1 \dots \sin\theta_{j-1}\cos\theta_j \\
&\quad \times g(\alpha_1\cos\theta_1, \dots, -\sin\theta_1 \dots \sin\theta_{n-1}) J(\theta) d\theta_{n-1}.
\end{aligned}
$$

For the first integral we make the change of variables

$$
\begin{gathered}
\theta_j = \begin{cases} \pi - \xi_j, & \alpha_j = -1 \\ \xi_j, \alpha_j = 1, & j = 1, 2, \dots, n-2, \end{cases} \\
\theta_{n-1} = \xi_{n-1} - \pi,
\end{gathered}
$$

and use the equality $\theta_{n-1} = \xi_{n-1} + \pi$ for the second integral. Note that under these changes of variables we have

$$
\begin{gathered}
\sin\theta_k = \sin(\pi - \xi_k) = \sin\pi\cos\xi_k - \cos\pi\sin\xi_k = \sin\xi_k,\ k \le n-2, \\
\sin\theta_{n-1} = \sin(\xi_{n-1} - \pi) = \sin\xi_{n-1}\cos\pi - \cos\xi_{n-1}\sin\pi = -\sin\xi_{n-1},
\end{gathered}
$$

or

$$
\begin{gathered}
\sin\theta_{n-1} = \sin(\xi_{n-1} + \pi) = \sin\xi_{n-1}\cos\pi + \cos\xi_{n-1}\sin\pi = -\sin\xi_{n-1}, \\
\cos\theta_k = \cos(\pi - \xi_k) = \cos\pi\cos\xi_k - \sin\pi\sin\xi_k = -\cos\xi_k.
\end{gathered}
$$

Consequently for the monomial x_j we obtain:

(a) if $\alpha_j = -1$, then

$$
x_j \to \sin\theta_1 \dots \sin\theta_{j-1}\cos\theta_j \to \sin\xi_1 \dots \sin\xi_{j-1}(-\cos\xi_j) \to \alpha_j x_j;
$$

(b) if $\alpha_j = +1$, then

$$x_j \to \alpha_j \sin\xi_1 \dots \sin\xi_{j-1}\cos\xi_j \to \alpha_j x_j.$$

Thus we have the equality

$$\begin{aligned}
&\int_{\partial\Omega} x_j g(x^*)\, dS_x \\
&= \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{n-2} \int_0^\pi \sin\theta_1 \dots \sin\theta_{j-1}\cos\theta_j \\
&\quad \times g(\alpha_1\cos\theta_1, \dots, -\sin\theta_1 \dots \sin\theta_{n-1}) J(\theta) d\theta_{n-1} \\
&\quad + \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{n-2} \int_\pi^{2\pi} \sin\theta_1 \dots \sin\theta_{j-1}\cos\theta_j \\
&\quad \times g(\alpha_1\cos\theta_1, \dots, -\sin\theta_1 \dots \sin\theta_{n-1}) J(\theta) d\theta_{n-1} \\
&= \int_0^\pi d\xi_1 \dots \int_0^\pi d\xi_{n-2} \int_\pi^{2\pi} \sin\xi_1 \dots \sin\xi_{j-1}\cdot\cos\xi_j \\
&\quad \times g(\cos\xi_1, \dots, \sin\xi_1 \dots \sin\xi_{n-1}) J(\xi) d\xi_{n-1} \\
&\quad + \alpha_j \int_0^\pi d\xi_1 \dots \int_0^\pi d\xi_{n-2} \int_0^\pi \sin\xi_1 \dots \sin\xi_{j-1}\cdot\cos\xi_j g(\cos\xi_1, \dots, \\
&\quad \sin\xi_1 \dots \sin\xi_{n-1}) J(\xi) d\xi_{n-1} \\
&= \alpha_j \int_{\partial\Omega} x_j g(x)\, dS_x.
\end{aligned}$$

Thus the equality (4.14) is proved. Consider the equality (4.15). In this case we have

$$\begin{aligned}
&\int_{\partial\Omega_-} x_j g(x^*)\, dS_x \\
&= \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{n-2} \int_\pi^{2\pi} \sin\theta_1 \dots \sin\theta_{j-1}\cos\theta_j \\
&\quad \times g(\alpha_1\cos\theta_1, \dots, -\sin\theta_1 \dots \sin\theta_{n-1}) J(\theta) d\theta_{n-1} \\
&= \alpha_j \int_0^\pi d\xi_1 \dots \int_0^\pi d\xi_{n-2} \int_\pi^{2\pi} \sin\xi_1 \dots \sin\xi_{j-1}\cos\xi_j \\
&\quad \times g(\alpha_1\cos\xi_1, \dots, -\sin\xi_1 \dots \sin\xi_{n-1}) J(\xi) d\xi_{n-1} = \alpha_j \int_{\partial\Omega_+} x_j g(x)\, dS_x.
\end{aligned}$$

□

Corollary 4.8. *Let $g(x) \in C(\partial\Omega)$. Then for $j = 1, 2, \dots, n$ the following equalities hold:*

$$\int_{\partial\Omega} x_j g^{\pm}(x) dS_x = \frac{1\pm\alpha_j}{2}\int_{\partial\Omega} x_j g(x)\, dS_x, \tag{4.16}$$

$$\int_{\partial\Omega} x_j \tilde{g}^{\pm}(x) dS_x = \frac{1\pm\alpha_j}{2}\int_{\partial\Omega_+} x_j g(x)\, dS_x. \tag{4.17}$$

Proof. According to Lemma 4.7 we have

$$\int_{\partial\Omega} x_j g(x^*)\, dS_x = \alpha_j \int_{\partial\Omega} x_j g(x)\, dS_x.$$

Therefore, using (4.1) we obtain

$$\begin{aligned}\int_{\partial\Omega} x_j g^{\pm}(x)\,dS_x &= \frac{1}{2}\Big[\int_{\partial\Omega} x_j g(x)\,dS_x \pm \int_{\partial\Omega} x_j g(x^*)\,dS_x\Big]\\ &= \frac{1}{2}\Big[\int_{\partial\Omega} x_j g(x)\,dS_x \pm \alpha_j \int_{\partial\Omega} x_j g(x)\,dS_x\Big]\\ &= \frac{1\pm\alpha_j}{2}\int_{\partial\Omega} x_j g(x)\,dS_x.\end{aligned}$$

Similarly we can get

$$\begin{aligned}\int_{\partial\Omega} x_j \tilde{g}^{\pm}(x)\,dS_x &= \frac{1}{2}\Big[\int_{\partial\Omega_+} x_j g(x)\,dS_x \pm \int_{\partial\Omega_-} x_j g(x^*)\,dS_x\Big]\\ &= \frac{1}{2}\Big[\int_{\partial\Omega_+} x_j g(x)\,dS_x \pm \alpha_j \int_{\partial\Omega_+} x_j g(x)\,dS_x\Big]\\ &= \frac{1\pm\alpha_j}{2}\int_{\partial\Omega_+} x_j g(x)\,dS_x.\end{aligned}$$

□

Remark 4.9. Since $\alpha_n = -1$, ti follows that (4.16) and (4.17) imply

$$\begin{gathered}\int_{\partial\Omega} x_n g^{+}(x) dS_x = 0, \quad \int_{\partial\Omega} x_n g^{-}(x) dS_x = \int_{\partial\Omega} x_n g(x) dS_x,\\ \int_{\partial\Omega} x_n \tilde{g}^{+}(x) dS_x = 0, \quad \int_{\partial\Omega} x_n \tilde{g}^{-}(x) dS_x = \int_{\partial\Omega_+} x_n g(x) dS_x.\end{gathered}$$

5. Uniqueness conditions

In this section we study uniqueness of solutions of the problems (2.1)-(2.4).

Theorem 5.1. *Let $k = 1$ and solution of problem (2.1)-(2.4) exist. Then*

(1) in the case $m = 1$, $\ell_1 = 2$, $\ell_2 = 3$, the solution of problem (2.1)-(2.4) is unique up to constant term;

(2) if $m = 2$, $\ell_1 = 1$, $\ell_2 = 3$, or $m = 3$, $\ell_1 = 1$, $\ell_2 = 2$, then the following cases are possible:

(a) *if for all $1 \le j \le 1$, $\alpha_j = -1$, then the solution of homogeneous problem (2.1)-(2.4) is a function of the form*

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j;$$

(b) *if for some $j_0 \in \{1, 2, \dots, n-1\}$ the equality $\alpha_{j_0} = 1$ holds, then solution of the homogeneous problem (2.1)-(2.4) is a function of the form*

$$u(x) = c_0 + \sum_{j=1, j\neq j_0}^{n} c_j x_j.$$

In particular, if $\alpha_j = 1$, $1 \le j \le n-1$ then $u(x) = c_0 + c_n x_n$.

Proof. Let $k = 1$ and function $u(x)$ is a solution of the homogeneous problem (2.1)-(2.4). Then $u(x)$ is a biharmonic function that satisfies boundary conditions:

$$D_\nu^m u(x) = 0,\ x \in \partial\Omega, \tag{5.1}$$

$$D_\nu^{\ell_1} u(x) = -D_\nu^{\ell_1} u(x^*),\ x \in \partial\Omega_+, \tag{5.2}$$

$$D_\nu^{\ell_2} u(x) = D_\nu^{\ell_2} u(x^*),\ x \in \partial\Omega_+. \tag{5.3}$$

If $x \in \partial\Omega_-$ then $x^* \in \partial\Omega_+$ and therefore from the conditions (5.2) and (5.3) it follows that

$$D_\nu^{\ell_1} u(x^*) = -D_\nu^{\ell_1} u(x),\ x \in \partial\Omega_-, \quad D_\nu^{\ell_2} u(x^*) = D_\nu^{\ell_2} u(x),\ x \in \partial\Omega_-.$$

Then for all $x \in \partial\Omega$ the following equalities hold

$$D_\nu^{\ell_1} u(x) = -D_\nu^{\ell_1} u(x^*), \quad x \in \partial\Omega, \tag{5.4}$$

$$D_\nu^{\ell_2} u(x) = D_\nu^{\ell_2} u(x^*),\ x \in \partial\Omega. \tag{5.5}$$

On the other hand differentiating (5.4) along the normal ν give us

$$D_\nu^{\ell_2} u(x) = -D_\nu^{\ell_2} u(x^*), \quad x \in \partial\Omega. \tag{5.6}$$

Then from the equalities (5.5) and (5.6) it follows that

$$D_\nu^{\ell_2} u(x) = 0, x \in \partial\Omega.$$

Thereby the function $u(x)$ is a solution of the problem

$$\Delta^2 u(x) = 0, \quad x \in \Omega, \tag{5.7}$$

$$D_\nu^m u(x)\big|_{\partial\Omega} = 0, \quad D_\nu^{\ell_2} u(x)\big|_{\partial\Omega} = 0. \tag{5.8}$$

Furthermore when we use the results of Section 3. The following cases are possible:

(1) if $m = 1$, $\ell_2 = 3$, then by Theorem 3.3, the function $u(x) = c_0 \equiv const$ is a unique solution of the problem (5.7)-(5.8). Obviously, this function satisfies all conditions of the homogeneous problem (2.1)-(2.4) for $k = 1$. Consequently, if $m = 1, \ell_1 = 2, \ell_2 = 3$ then solution of the homogeneous problem (2.1)-(2.4) is a function $u(x) = c_0$.

(2) if $m = 2, \ell_2 = 3$ then according to Theorem 3.2, the unique solution of the homogeneous problem (5.7)-(5.8) is a function of the form:

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j,$$

where c_j are constants, $j = 0, 1, \dots, n$. In this case $\ell_1 = 1$ and for all $x \in \partial\Omega$,

$$D_\nu^1 u(x)\big|_{\partial\Omega} = r\frac{\partial u(x)}{\partial r}\big|_{\partial\Omega} = \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x).$$

Since $u(x^*) = C_0 + \sum_{j=1}^n c_j \alpha_j x_j$, it follows that

$$0 = D_\nu^1 u(x) + D_\nu^1 u(x^*)|_{\partial\Omega} = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} c_j \alpha_j x_j = \sum_{j=1}^{n} (1+\alpha_j) c_j x_j.$$

Further, if for all $1 \le j \le n$: $\alpha_j = -1$, then c_j are arbitrary numbers and if for some $j_0 \in \{1, 2, \dots, n-1\}$, $\alpha_{j_0} = 1$ then for the equality

$$D_\nu^1 u(x) + D_\nu^1 u(x^*) = 0$$

to be correct it is necessary that $c_{j_0} = 0$. Hence, if $\alpha_j = -1,\ 1 \le j \le n$ then the function

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j$$

is a solution of the homogeneous problem (2.1)-(2.4). If for some $j_0 \in \{1, 2, \dots, n-1\}$, $\alpha_{j_0} = 1$, then the solution of the homogeneous problem (2.1)-(2.4) is a function of the form

$$u(x) = c_0 + \sum_{j=1, j\neq j_0}^{n} c_j x_j.$$

In particular, if $\alpha_j = 1$ for all $j = 1, 2, \dots, n-1$, then

$$u(x) = c_0 + c_n x_n.$$

(3) Let $m = 3$, $\ell_1 = 1$, $\ell_2 = 2$. Then as in the case (2) the function of the form

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j$$

is a solution of the homogeneous problem (5.7)-(5.8). Making the same arguments as in the case $m = 2$, $\ell_1 = 1$, $\ell_2 = 3$ we obtain: If $\alpha_j = -1,\ 1 \le j \le n$ then solution of the homogeneous problem (2.1)-(2.4) is the function

$$u(x) = c_0 + \sum_{j=1}^{n} c_j x_j.$$

If for some $j_0 \in \{1, 2, \dots, n-1\}$, $\alpha_{j_0} = 1$, then solution of the homogeneous problem (2.1)-(2.4) is a function of the form

$$u(x) = c_0 + \sum_{j=1, j\neq j_0}^{n} c_j x_j.$$

In particular, if $\alpha_j = 1$ for all $1 \le j \le n-1$ then the solution has the form $u(x) = c_0 + c_n x_n$. $\square$

The following statement can be proved similarly.

Theorem 5.2. *Let $k = 2$ and a solution of problem (2.1)-(2.4) exist. Then*

(1) in the case $m = 1, \ell_1 = 2, \ell_2 = 3$ solution of the problem (2.1)-(2.4) is unique up to constant term;

(2) if $m = 2, \ell_1 = 1, \ell_2 = 3$ or $m = 3$, $\ell_1 = 1$, $\ell_2 = 2$, then the following cases are possible:

(a) *if for all $1 \le j \le n-1$, $\alpha_j = 1$ then the solution of the homogeneous problem (2.1)-(2.4) is function of the form:*

$$u(x) = c_0 + \sum_{j=1}^{n-1} c_j x_j;$$

(b) *if for some $j_0 \in \{1, 2, \dots, n\}$, $\alpha_{j_0} = -1$ then solution of the homogeneous problem (2.1)-(2.4) is a function of the form*

$$u(x) = c_0 + \sum_{j=1, j\neq j_0}^{n} c_j x_j.$$

In particular, if $\alpha_j=-1,\ 1\le j\le n$, then $u(x)=c_0$.

6. Existence conditions

In this section we present results on existence of a solution of problem (2.1)-(2.4).

Theorem 6.1. *Let $k=1$ and functions $f(x),\ g(x),\ g_j(x)\ j=1,2$ be smooth enough and compatibility conditions (2.5), (2.6) be fulfilled. Then the necessary and sufficient conditions for solvability of problem (2.1)-(2.4) have the following form:*

(1) if $m=1,\ \ell_1=2,\ \ell_2=3$, then

$$\frac{1}{2}\int_\Omega (1-|x|^2)f(x)dx=\int_{\partial\Omega_+} g_1(x)dS_x-\int_{\partial\Omega} g(x)dS_x; \tag{6.1}$$

(2) if $m=2,\ \ell_1=1,\ \ell_2=3$, then

$$\begin{aligned}&\frac{1}{2}\int_\Omega (1-|x|^2)f(x)dx\\ &=\int_{\partial\Omega} g(x)dS_x-\int_{\partial\Omega_+} g_1(x)dS_x,,\end{aligned} \tag{6.2}$$

$$\begin{aligned}&\frac{n-1}{2}\int_\Omega x_j|x|^2f(x)\,dx-\frac{n-3}{2}\int_\Omega x_jf(x)\,dx\\ &=\int_{\partial\Omega_+} x_jg_2(x)\,dS_x-\int_{\partial\Omega} x_jg(x)\,dS_x\end{aligned} \tag{6.3}$$

for all j such that $\alpha_j=-1$;

(3) if $m=3,\ \ell_1=1,\ \ell_2=2$, then

$$\frac{n-1}{2}\int_\Omega |x|^2f(x)\,dx-\frac{n-3}{2}\int_\Omega f(x)\,dx=\int_{\partial\Omega} g(x)\,dS_x, \tag{6.4}$$

$$\begin{aligned}&\frac{n-1}{2}\int_\Omega x_j|x|^2f(x)\,dx-\frac{n-3}{2}\int_\Omega x_jf(x)\,dx\\ &=\int_{\partial\Omega} x_jg(x)\,dS_x-\int_{\partial\Omega_+} x_jg_2(x)\,dS_x,\end{aligned} \tag{6.5}$$

for all $j\in\{1,2,\dots,n\}$ such that $\alpha_j=-1$.

Proof. We introduce two auxiliary functions

$$v(x)=\frac{1}{2}[u(x)+u(x^*)],\quad w(x)=\frac{1}{2}[u(x)-u(x^*)].$$

It is obvious that $u(x)=v(x)+w(x)$. It is easy to see that the functions $v(x)$ and $w(x)$ are solutions of the following Neumann-type problems:

$$\Delta^2v(x)=f^+(x),\ x\in\Omega,\ \ D_\nu^m v(x)\big|_{\partial\Omega}=g^+(x),\ D_\nu^{\ell_1}v(x)\big|_{\partial\Omega}=\tilde g_1^+(x), \tag{6.6}$$

$$\Delta^2w(x)=f^-(x),\ x\in\Omega,\ D_\nu^m w(x)\big|_{\partial\Omega}=g^-(x),\ D_\nu^{\ell_2}w(x)\big|_{\partial\Omega}=\tilde g_2^-(x). \tag{6.7}$$

Indeed applying the biharmonic operator Δ^2 to the function $v(x)$, we obtain

$$\Delta^2v(x)=\frac{1}{2}[\Delta^2u(x)+\Delta^2u(x^*)]=\frac{1}{2}[f(x)+f(x^*)]=f^+(x),\ x\in\Omega.$$

Further, taking the boundary conditions (2.2), (2.3) into account, we have

$$D_\nu^m v(x)=\frac{1}{2}[D_\nu^m u(x)+D_\nu^m u(x^*)]=\frac{1}{2}[g(x)+g(x^*)]=g^+(x),\quad x\in\partial\Omega,$$

$$
\begin{aligned}
D_\nu^{\ell_1} v(x) &= \frac{1}{2}[D_\nu^{l_1} u(x) + D_\nu^{\ell_1} u(x^*)] = \frac{1}{2} g(x), \quad x \in \partial\Omega_+,\\
D_\nu^{\ell_1} v(x) &= \frac{1}{2}[D_\nu^{\ell_1} u(x) + D_\nu^{\ell_1} u(x^*)]\\
&= \frac{1}{2}[D_\nu^{\ell_1} u(x^*) + D_\nu^{\ell_1} u(x)] = \frac{1}{2} g(x^*),\ x \in \partial\Omega_-.
\end{aligned}
$$

Similarly, for function $w(x)$ we obtain

$$
\begin{aligned}
\Delta^2 w(x) &= \frac{1}{2}[\Delta^2 u(x) - \Delta^2 u(x^*)] = \frac{1}{2}[f(x) - f(x^*)] = f^-(x), \quad x \in \Omega,\\
D_\nu^m w(x) &= \frac{1}{2}[D_\nu^m u(x) - D_\nu^m u(x^*)] = \frac{1}{2}[g(x) - g(x^*)] = g^-(x),\ x \in \partial\Omega,\\
D_\nu^{\ell_2} w(x) &= \frac{1}{2}[D_\nu^{\ell_2} u(x) - D_\nu^{\ell_2} u(x^*)] = \frac{1}{2} g_2(x),\ x \in \partial\Omega_+,\\
D_\nu^{\ell_2} w(x) &= \frac{1}{2}[D_\nu^{\ell_2} u(x) - D_\nu^{\ell_2} u(x^*)]\\
&= -\frac{1}{2}[D_\nu^{\ell_2} u(x^*) - D_\nu^{\ell_2} u(x)] = -\frac{1}{2}\tilde g_2^-(x^*),\ x \in \partial\Omega_-.
\end{aligned}
$$

Note that if the function $f(x)$ is a smooth enough function defined on the domain $\bar\Omega$, and function $g(x)$ is defined on the sphere $\partial\Omega$, then it is obvious that the functions $f^\pm(x)$ and $g^\pm(x)$ have the same properties. Moreover, if functions $g_1(x)$ and $g_2(x)$ are smooth on $\partial\Omega_+$, then because of compatibility conditions (2.5), (2.6) the functions $\tilde g_1^\pm(x)$ and $\tilde g_2^\pm(x)$ have the same properties. Further, to study the solvability of the problems (6.6) and (6.7) we use the statements of Theorems 3.1, 3.3.

(1) if $m = 1$, $\ell_1 = 2$, $\ell_2 = 3$ then the necessary and sufficient conditions for solvability of the problems (6.6) and (6.7), respectively, are:

$$
\frac{1}{2}\int_\Omega (1 - |x|^2) f^+(x)\,dx = \int_{\partial\Omega} [\tilde g_1^+(x) - g^+(x)]\,dS_x, \tag{6.8}
$$

$$
\frac{1}{2}\int_\Omega [(n-1)|x|^2 - (n-3)] f^-(x)\,dx = \int_{\partial\Omega} \tilde g_2^-(x)\,dS_x. \tag{6.9}
$$

From equalities (4.4), (4.6) and (4.9) we obtain

$$
\frac{1}{2}\int_\Omega (1 - |x|^2) f^+(x)\,dx = \frac{1}{2}\int_\Omega (1 - |x|^2) f(x)\,dx.
$$

Similarly, from (4.4), (4.6) and (4.9), it follows that

$$
\begin{gathered}
\frac{1}{2}\int_\Omega (1 - |x|^2) f^+(x)\,dx = \frac{1}{2}\int_\Omega (1 - |x|^2) f(x)\,dx,\\
\int_{\partial\Omega} \tilde g_1^+(x)\,dS_x - \int_{\partial\Omega} g^+(x)\,dS_x = \int_{\partial\Omega_+} g_1(x)\,dS_x - \int_{\partial\Omega} g(x)\,dS_x.
\end{gathered}
$$

Consequently, condition (6.9) always holds and condition (6.8) can be rewritten in the form (6.1).

(2) if $m = 2$, $\ell_1 = 1$, $\ell_2 = 3$, then the necessary and sufficient condition for solvability of the problem (6.6) has the form

$$
\frac{1}{2}\int_\Omega (1 - |x|^2) f^+(x)\,dx = \int_{\partial\Omega} [g^+(x) - \tilde g_1^+(x)]\,dS_x,
$$

which can be rewritten in the form

$$\frac{1}{2}\int_{\Omega}(1-|x|^2)f(x)\,dx=\int_{\partial\Omega}g(x)dS_x-\int_{\partial\Omega_+}g_1(x)\,dS_x.$$

For problem (6.7) we obtain the conditions:

$$\frac{1}{2}\int_{\Omega}[(n-1)|x|^2-(n-3)]f^-(x)\,dx=\int_{\partial\Omega}\tilde{g}_2^-(x)\,dS_x, \tag{6.10}$$

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}x_j[(n-1)|x|^2-(n-3)]f^-(x)dx\\&=\int_{\partial\Omega}x_j[\tilde{g}_2^-(x)-g^-(x)]dS_x,\quad j=1,2,\dots,n.\end{aligned} \tag{6.11}$$

From equalities (4.4) and (4.9), the condition (6.10) always holds. Further, using (4.12), (4.15) and (4.16) we have

$$\frac{1}{2}\int_{\Omega}x_j[(n-1)|x|^2-(n-3)]f^-(x)dx=\frac{1-\alpha_j}{2}\int_{\Omega}x_j[(n-1)|x|^2-(n-3)]f(x)dx,$$

$$\int_{\partial\Omega}x_j\tilde{g}_2^-(x)dS_x=\frac{1-\alpha_j}{2}\int_{\partial\Omega_+}x_jg_2(x)dS_x,$$

$$\int_{\partial\Omega}x_jg^-(x)dS_x=\frac{1-\alpha_j}{2}\int_{\partial\Omega}x_jg(x)dS_x.$$

Then equality (6.11) can be rewritten in the form

$$\begin{aligned}&\frac{1-\alpha_j}{4}\int_{\Omega}x_j[(n-1)|x|^2-(n-3)]f(x)dx\\&=\frac{1-\alpha_j}{2}[\int_{\partial\Omega_+}x_jg_2(x)dS_x-\int_{\partial\Omega}x_jg(x)dS_x],\ j=1,2,\dots,n.\end{aligned} \tag{6.12}$$

If for all $1\le j\le n-1$, $\alpha_j=1$ then condition (6.12) always holds for these indexes and in this case condition (6.11) for $j=n$ can be rewritten in the form

$$\frac{n-1}{2}\int_{\Omega}x_n|x|^2f(x)dx-\frac{n-3}{2}\int_{\partial\Omega}x_nf(x)dx=\int_{\partial\Omega_+}x_ng_2(x)dS_x-\int_{\partial\Omega}x_ng(x)dS_x.$$

If for some $j_0\in\{1,2,\dots,n\}$, $\alpha_{j_0}=-1$, then for this j_0, condition (6.11) can be rewritten in the form

$$\begin{aligned}&\frac{n-1}{2}\int_{\Omega}x_{j_0}|x|^2f(x)dx-\frac{n-3}{2}\int_{\partial\Omega}x_{j_0}f(x)dx\\&=\int_{\partial\Omega_+}x_{j_0}g_2(x)dS_x-\int_{\partial\Omega}x_{j_0}g(x)dS_x.\end{aligned}$$

(3) if $m=3$, $\ell_1=1$, $\ell_2=2$, then by Theorem 3.3 the problem's solvability condition has the form

$$\frac{1}{2}\int_{\Omega}[(n-1)|x|^2-(n-3)]f^+(x)dx=\int_{\partial\Omega}g^+(x)dS_x.$$

According to (4.3) and (4.5) the last condition can be rewritten in the form

$$\frac{1}{2}\int_{\Omega}[(n-1)|x|^2-(n-3)]f(x)dx=\int_{\partial\Omega}g(x)dS_x.$$

Further, using Theorem 3.2 the solvability condition of problem (6.7) can be rewritten in the form

$$\frac{1}{2}\int_{\Omega}[(n-1)|x|^2-(n-3)]f^-(x)dx=\int_{\partial\Omega}g^-(x)dS_x, \tag{6.13}$$

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}x_j[(n-1)|x|^2-(n-3)]f^-(x)dx\\&=\int_{\partial\Omega}x_j[g^-(x)-\tilde{g}_2^-(x)]dS_x,\quad j=1,2,\dots,n.\end{aligned}\tag{6.14}$$

From (4.4) and (4.6) it follows that condition (6.13) always holds. From (4.12), (4.15) and (6.14) we obtain

$$\begin{gathered}\frac{1}{2}\int_{\Omega}x_j|x|^2f^-(x)dx=\frac{1-\alpha_j}{2}\int_{\Omega}x_j|x|^2f(x)dx,\\ \int_{\partial\Omega}x_jg^-(x)dS_x=\frac{1-\alpha_j}{2}\int_{\partial\Omega}x_jg(x)dS_x,\\ \int_{\partial\Omega}x_j\tilde{g}_2^-(x)dS_x=\frac{1-\alpha_j}{2}\int_{\partial\Omega_+}x_jg_2(x)dS_x.\end{gathered}$$

Then (6.14) can be rewritten as follows

$$\begin{aligned}&\frac{1-\alpha_j}{4}\int_{\Omega}x_j[(n-1)|x|^2-(n-3)]f(x)dx\\&=\frac{1-\alpha_j}{2}\Big[\int_{\partial\Omega}x_jg(x)dS_x-\int_{\partial\Omega_+}x_jg_2(x)dS_x\Big],\quad j=1,2,\dots,n.\end{aligned}\tag{6.15}$$

If $\alpha_j=1$ then (6.15) holds, and if $\alpha_j=-1$ then this condition can be rewritten in the form

$$\begin{aligned}&\frac{n-1}{2}\int_{\Omega}x_j|x|^2f(x)dx-\frac{n-3}{2}\int_{\Omega}x_jf(x)dx\\&=\int_{\partial\Omega}x_jg(x)dS_x-\int_{\partial\Omega_+}x_jg_2(x)dS_x,\quad j=1,2,\dots,n.\end{aligned}$$

Thus equality (6.5) and, consequently, the theorem are proved. □

The following statement can be proved similarly to Theorem 6.1.

Theorem 6.2. *Let $k=2$ and the functions $f(x)$, $g(x)$, $g_j(x)$, $j=1,2$ be smooth enough on the domains $\bar{\Omega}$, $\partial\Omega$ and $\partial\Omega_+$, respectively, and the compatibility conditions (2.5) and (2.6) hold. Then the necessary and sufficient conditions for solvability of problem (2.1)-(2.4) have the form:*

(1) if $m=1$, $\ell_1=2$, $\ell_2=3$, then

$$\frac{n-1}{2}\int_{\Omega}|x|^2f(x)dx-\frac{n-3}{2}\int_{\Omega}f(x)dx=\int_{\partial\Omega_+}g_2(x)dS_x;$$

(2) if $m=2$, $l_1=1$, $l_2=3$, then

$$\frac{n-1}{2}\int_{\Omega}|x|^2f(x)dx-\frac{n-3}{2}\int_{\Omega}f(x)dx=\int_{\partial\Omega_+}g_2(x)dS_x,$$

and

$$\frac{n-1}{2}\int_{\Omega}x_j|x|^2f(x)dx-\frac{n-3}{2}\int_{\Omega}x_jf(x)dx=\int_{\partial\Omega_+}x_jg_2(x)dS_x-\int_{\partial\Omega}x_jg(x)dS_x,$$

for all $j \in \{1, 2, \dots, n-1\}$ *for which* $\alpha_j = 1$*;*

(3) if $m = 3$*,* $l_1 = 1$*,* $l_2 = 3$*, then*

$$\frac{n-1}{2}\int_\Omega |x|^2 f(x)dx - \frac{n-3}{2}\int_\Omega f(x)dx = \int_{\partial\Omega} g(x)dS_x,$$

and

$$\frac{n-1}{2}\int_\Omega x_j|x|^2 f(x)dx - \frac{n-3}{2}\int_\Omega x_j f(x)dx = \int_{\partial\Omega} x_j g(x)dS_x - \int_{\partial\Omega_+} x_j g_2(x)dS_x,$$

for all $j \in \{1, 2, \dots, n-1\}$ *such that* $\alpha_j = 1$*.*

Acknowledgements. This research was supported by the target program 0085 / PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan and by Act 211 Government of the Russian Federation, contract no. 02.A03.21.0011.

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