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POSITIVE SOLUTIONS OF DISCRETE BOUNDARY VALUE PROBLEMS WITH THE (p, q)-LAPLACIAN OPERATOR

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Abstract. We consider a discrete Dirichlet boundary value problem of equations with the (p, q)-Laplacian operator in the principal part and prove the existence of at least two positive solutions. The assumptions on the reaction term ensure that the Euler-Lagrange functional, corresponding to the problem, satisfies an abstract two critical points result.

1. Introduction

Let N be a positive integer and denote by $[1, N]$ the discrete set $\{1, \dots, N\}$. We study the discrete boundary value problem

$$\begin{gathered} -\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) = \lambda g(z, u(z)), \\ \text{for all } z \in [1, N], \\ u(0) = u(N+1) = 0, \end{gathered} \tag{1.1}$$

where $\Delta_r u(z-1) := \Delta(\phi_r(\Delta u(z-1))) = \phi_r(\Delta u(z)) - \phi_r(\Delta u(z-1))$ is the discrete r-Laplacian, $\phi_r : \mathbb{R} \to \mathbb{R}$ is the homomorphism given as $\phi_r(u) = |u|^{r-2}u$ with $u \in \mathbb{R}$ ($z \in [1, N]$), $\Delta u(z-1) = u(z) - u(z-1)$ is the forward difference operator, $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(N+1, t) = 0$ for all $t \in \mathbb{R}$, $\alpha, \beta : [1, N+1] \to \mathbb{R}$, $1 < q < p < +\infty$ and $\lambda \in]0, +\infty[$.

Here, we consider the following hypotheses:

- (H1) $g(z, 0) \geq 0$ for all $z \in [1, N]$, and $g(z, t) = g(z, 0)$ for all $t < 0$ and for all $z \in [1, N]$;
- (H2) $\alpha(z), \beta(z) \geq 0$ for all $z \in [1, N]$.

The solvability of general differential problems, with various boundary value conditions, was a maximum interest field of research over the last decades. It attracts pure and applied mathematicians and has its strong motivation in the possibility to model the dynamical behaviour of real phenomena in physics, economics, engineering and so on (see, for example, Diening-Harjulehto-Hästö-Růžicka [5]). There is a rich literature on this subject, which collects and explains the principal techniques of calculus of variations, critical and fixed points theories, Lyapunov-Schmidt reduction method, critical groups (Morse theory), and many others. We refer, for

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example, to the book of Motreanu-Motreanu-Papageorgiou [9] (for problems with the p-Laplacian operator). So, one can solve other abstract types of boundary value problems, by combining and extending these approaches. For example, the existence and multiplicity of solutions for problems with the (p,q)-Laplacian operator are considered in Marano-Mosconi-Papageorgiou [8], Mugnai-Papageorgiou [11] (equations), and Motreanu-Vetro-Vetro [10] (systems of equations). On the other hand, the increasing of computer performance has allowed researchers (in applied mathematics) to focus on the solution of difference equations, and, in particular, of the discrete version of various continuous differential problems. We refer to the books of Agarwal [1], Kelly-Peterson [7] (difference equations), and the articles of Cabada-Iannizzotto-Tersian [3], D'Aguì-Mawhin-Sciammetta [4], Jiang-Zhou [6] (discrete p-Laplacian operator).

Here, we use the critical point theory to prove the existence of two positive solutions for discrete (p,q)-Laplacian equations subjected to Dirichlet type boundary conditions. Indeed, the idea is to reduce the problem of existence of solutions in variational form, which means to consider the problem of finding critical points for the Euler-Lagrange functional corresponding to problem (1.1). The assumptions on the reaction term ensure that the involved Euler-Lagrange functional satisfies an abstract two critical points result of Bonanno-D'Aguì [2].

2. Mathematical background

We fix the notation as follows. By X and X^* we mean a Banach space and its topological dual, respectively. Here, we consider the N-dimensional Banach space

$$X_d = \{u : [0, N+1] \to \mathbb{R} \text{ such that } u(0) = u(N+1) = 0\},$$

and define the norm

$$\|u\|_{r,h} := \Big(\sum_{z=1}^{N+1} \left[|\Delta u(z-1)|^r + h(z)|u(z)|^r \right] \Big)^{1/r},$$

where $h : [1, N+1] \to \mathbb{R}$, with $h(z) \ge 0$ for all $z \in [1, N]$, and $r \in]1, +\infty[$. By $\|u\|_\infty := \max_{z\in[1,N]} |u(z)|$ we denote the usual sup-norm so that we consider the inequality

$$\|u\|_\infty \le \frac{(N+1)^{\frac{r-1}{r}}}{2} \|u\|_{r,h} \quad \text{for all } u \in X_d \tag{2.1}$$

(see [4] and [6, Lemma 2.2]).

Proposition 2.1. *Let $h = \sum_{z=1}^{N} h(z)$. The following inequalities hold*

$$\frac{2}{N+1}\|u\|_\infty \le \|u\|_{r,h} \le (2^r N + h)^{1/r}\|u\|_\infty.$$

Proof. The left inequality follows by (2.1). On the other hand, since

$$\begin{aligned} \|u\|_{r,h}^r &= \sum_{z=1}^{N+1} \left[|\Delta u(z-1)|^r + h(z)|u(z)|^r \right] \\ &\le 2\|u\|_\infty^r + \sum_{z=2}^{N} 2^r \|u\|_\infty^r + \|u\|_\infty^r \sum_{z=1}^{N} h(z) \\ &= [2^r(N-1) + 2 + h]\|u\|_\infty^r \le [2^r N + h]\|u\|_\infty^r, \end{aligned}$$

we deduce easily the right inequality. $\square$

Now, let X_d be endowed with the norm

$$\|u\| = \|u\|_{p,\alpha} + \|u\|_{q,\beta},$$

where α and β (satisfying (H2)) are the coefficients of ϕ_p and ϕ_q in (1.1), respectively.

We consider the function $G : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ given as

$$G(z,t) = \int_0^t g(z,\xi)d\xi, \quad \text{for all } t \in \mathbb{R},\ z \in [1, N+1],$$

and the functional $B : X_d \to \mathbb{R}$ given as

$$B(u) = \sum_{z=1}^{N+1} G(z, u(z)), \quad \text{for all } u \in X_d.$$

It is clear that $B \in C^1(X_d, \mathbb{R})$ and

$$\langle B'(u), v\rangle = \sum_{z=1}^{N+1} g(z, u(z))v(z), \quad \text{for all } u, v \in X_d.$$

Also, define the functionals $A_1, A_2 : X_d \to \mathbb{R}$ by

$$A_1(u) = \frac{1}{p}\|u\|_{p,\alpha}^p \quad \text{and} \quad A_2(u) = \frac{1}{q}\|u\|_{q,\beta}^q, \quad \text{for all } u \in X_d.$$

Clearly, $A_1, A_2 \in C^1(X_d, \mathbb{R})$ and (by the summation by parts formula) we have the following Gâteaux derivatives at the point $u \in X_d$:

$$\langle A_1'(u), v\rangle = \sum_{z=1}^{N+1} \phi_p(\Delta u(z-1))\Delta v(z-1) + \alpha(z)\phi_p(u(z))v(z),$$
$$\langle A_2'(u), v\rangle = \sum_{z=1}^{N+1} \phi_q(\Delta u(z-1))\Delta v(z-1) + \beta(z)\phi_q(u(z))v(z),$$

for all $u, v \in X_d$. Now, for $r \in]1, +\infty[$, we obtain

$$\begin{aligned}
&\sum_{z=1}^{N+1} \phi_r(\Delta u(z-1))\Delta v(z-1)\\
&= \sum_{z=1}^{N+1} [\phi_r(\Delta u(z-1))v(z) - \phi_r(\Delta u(z-1))v(z-1)]\\
&= \sum_{z=1}^{N} \phi_r(\Delta u(z-1))v(z) - \sum_{z=1}^{N} \phi_r(\Delta u(z))v(z)\\
&= -\sum_{z=1}^{N+1} \Delta\phi_r(\Delta u(z-1))v(z).
\end{aligned}$$

Then, we have

$$\langle A_1'(u), v\rangle = \sum_{z=1}^{N+1} [-\Delta\phi_p(\Delta u(z-1)) + \alpha(z)\phi_p(u(z))]v(z),$$
$$\langle A_2'(u), v\rangle = \sum_{z=1}^{N+1} [-\Delta\phi_q(\Delta u(z-1)) + \beta(z)\phi_q(u(z))]v(z),$$

for all $u, v \in X_d$. Next, we consider the functional $I_\lambda : X_d \to \mathbb{R}$ given as

$$I_\lambda(u) = A_1(u) + A_2(u) - \lambda B(u), \quad \text{for all } u \in X_d.$$

We point out that $I_\lambda(0) = 0$. Also, we have

$$\begin{aligned}\langle I'_\lambda(u), v\rangle = \sum_{z=1}^{N+1} &[-\Delta\phi_p(\Delta u(z-1)) - \Delta\phi_q(\Delta u(z-1)) \\ &+ \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) - \lambda g(z, u(z))]v(z),\end{aligned}$$

for all $u, v \in X_d$. Thus, $u \in X_d$ is a solution of problem (1.1) if and only if u is a critical point of I_λ.

We recall the following notion.

Definition 2.2. Let X be a real Banach space and X^* its topological dual. Then, $I_\lambda : X \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $\{u_n\}$ such that

(i) $\{I_\lambda(u_n)\}$ is bounded;
(ii) $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Our first result is the following auxiliary lemma, which characterizes the functional I_λ.

Lemma 2.3. *Let $m_\infty(z) := \liminf_{t\to+\infty} \frac{G(z,t)}{t^p}$ and $m_\infty := \min_{z\in[1,N]} m_\infty(z)$. If $m_\infty > 0$, and (H1)-(H2) hold, then I_λ satisfies the (PS)-condition and it is unbounded from below for all $\lambda \in \Lambda :=]\frac{(2^p+2^q)N+\alpha+\beta}{q\, m_\infty}, +\infty[$, where $\alpha = \sum_{z=1}^N \alpha(z)$ and $\beta = \sum_{z=1}^N \beta(z)$.*

Proof. As $m_\infty > 0$, let $\lambda > \frac{(2^p+2^q)N+\alpha+\beta}{q\, m_\infty}$ and $m \in \mathbb{R}$ such that $m_\infty > m > \frac{(2^p+2^q)N+\alpha+\beta}{q\lambda}$. We consider a sequence $\{u_n\} \subset X_d$ such that $I_\lambda(u_n) \to c \in \mathbb{R}$ and $I'_\lambda(u_n) \to 0$ in X_d^*, as $n \to +\infty$. Let $u_n^+ = \max\{u_n, 0\}$ and $u_n^- = \max\{-u_n, 0\}$ for all $n \in \mathbb{N}$. We show that the sequence $\{u_n^-\}$ is bounded. So, we have

$$\begin{aligned}|\Delta u_n^-(z-1)|^p &\le |\Delta u_n^-(z-1)|^{p-2}\Delta u_n^-(z-1)\Delta u_n^-(z-1) \\ &\le -|\Delta u_n(z-1)|^{p-2}\Delta u_n(z-1)\Delta u_n^-(z-1) \\ &= -\phi_p(\Delta u_n(z-1))\Delta u_n^-(z-1),\end{aligned}$$

for all $z \in [1, N+1]$. Also, we obtain

$$\alpha(z)|u_n^-(z)|^p = -\alpha(z)|u_n(z)|^{p-2}u_n(z)u_n^-(z) = -\alpha(z)\phi_p(u_n(z))u_n^-(z),$$

for all $z \in [1, N+1]$. Consequently, we have

$$\begin{aligned}\|u_n^-\|_{p,\alpha}^p &= \sum_{z=1}^{N+1} [|\Delta u_n^-(z-1)|^p + \alpha(z)|u_n^-(z)|^p] \\ &\le -\sum_{z=1}^{N+1} [\phi_p(\Delta u_n(z-1))\Delta u_n^-(z-1) + \alpha(z)\phi_p(u_n(z))u_n^-(z)] \\ &= -\langle A'_1(u_n), u_n^-\rangle.\end{aligned}$$

In a similar fashion, one has $\|u_n^-\|_{q,\beta}^q \le -\langle A_2'(u_n), u_n^-\rangle$. On the other hand, we obtain

$$\langle B'(u_n), u_n^-\rangle = \sum_{z=1}^{N+1} g(z, u_n(z))u_n^-(z) \ge 0 \quad \text{(by (H1))}.$$

So, we obtain

$$\begin{aligned}\|u_n^-\|_{p,\alpha}^p &\le \|u_n^-\|_{p,\alpha}^p + \|u_n^-\|_{q,\beta}^q \\ &\le -\langle A_1'(u_n), u_n^-\rangle - \langle A_2'(u_n), u_n^-\rangle + \lambda\langle B'(u_n), u_n^-\rangle \\ &= -\langle I_\lambda'(u_n), u_n^-\rangle,\end{aligned}$$

for all $n \in \mathbb{N}$, which leads to $\|u_n^-\|_{p,\alpha}^{p-1} \to 0$ as $n \to +\infty$. Similarly, we deduce that $\|u_n^-\|_{q,\beta}^{q-1} \to 0$ as $n \to +\infty$, and hence $\|u_n^-\| \to 0$ as $n \to +\infty$. It follows that there exists $\rho > 0$ such that

$$\|u_n^-\| \le \rho \;\Rightarrow\; \|u_n^-\|_\infty \le \frac{\rho + \rho N}{2} := \gamma, \quad \text{for all } n \in \mathbb{N}.$$

Next, we suppose that the sequence $\{u_n\}$ is unbounded, which means that $\{u_n^+\}$ is unbounded. We may suppose without any loss of generality (passing to a subsequence if necessary) that $\|u_n\| \to +\infty$ as $n \to +\infty$. By the assumption on m_∞ at the beginning of the proof, we deduce that there is $\delta_z \ge \max\{\gamma, 1\}$ such that $G(z,t) > mt^p$ for all $t > \delta_z$. Now, for all $z \in [1, N]$, as $G(z,\cdot)$ is a continuous function, there exists a constant $C(z) \ge 0$ such that $G(z,t) \ge m|t|^p - C(z)$ for all $t \in [-\gamma, \delta_z]$. This implies that $G(z,t) \ge m|t|^p - C(z)$ for all $t \ge -\gamma$, all $z \in [1, N]$. It follows easily that

$$B(u_n) = \sum_{z=1}^{N+1} G(z, u_n(z)) \ge \sum_{z=1}^{N} m|u_n(z)|^p - C \ge m\|u_n\|_\infty^p - C, \quad \text{for all } n \in \mathbb{N},$$

where $C = \sum_{z=1}^N C(z)$. For all u_n such that $\|u_n\|_\infty \ge 1$, we conclude that

$$\begin{aligned}I_\lambda(u_n) &= A_1(u_n) + A_2(u_n) - \lambda B(u_n) \\ &= \frac{1}{p}\|u_n\|_{p,\alpha}^p + \frac{1}{q}\|u_n\|_{q,\beta}^q - \lambda B(u_n) \\ &\le \Big(\frac{2^pN + \alpha}{p} + \frac{2^qN + \beta}{q}\Big)\|u_n\|_\infty^p - \lambda m\|u_n\|_\infty^p + \lambda C \\ &\le \Big[\frac{(2^p + 2^q)N + \alpha + \beta}{q} - \lambda m\Big]\|u_n\|_\infty^p + \lambda C\,.\end{aligned}$$

So, since $\frac{(2^p+2^q)N+\alpha+\beta}{q} - \lambda m < 0$, we obtain $I_\lambda(u_n) \to -\infty$ as $n \to +\infty$ ($\|u_n\| \to +\infty \Rightarrow \|u_n\|_\infty \to +\infty$). This is an absurd sentence and so $\{u_n\}$ is a bounded sequence. This ensures that I_λ satisfies the (PS)-condition.

On the other hand, again reasoning on a sequence $\{u_n\} \subset X_d$ such that $\{u_n^-\}$ is bounded and $\|u_n\| \to +\infty$ as $n \to +\infty$, we deduce easily that $I_\lambda(u_n) \to -\infty$ as $n \to +\infty$ and hence I_λ is unbounded from below. □

As in D'aguì-Mawhin-Sciammetta [4], our key-theorem is the following two non-zero critical points result of Bonanno-D'Aguì [2], which we arrange according to our notation and further use.

Theorem 2.4. *Let $X_d = \{u : [0, N+1] \to \mathbb{R}$ such that $u(0) = u(N+1) = 0\}$ and $A_1, A_2, B \in C^1(X_d, \mathbb{R})$ three functionals such that $\inf_{u \in X_d}(A_1(u) + A_2(u)) = A_1(0) + A_2(0) = B(0) = 0$. Assume that*

(i) *there are $s \in \mathbb{R}$ and $\widehat{u} \in X_d$, with $0 < A_1(\widehat{u}) + A_2(\widehat{u}) < s$, such that*
$$\frac{B(\widehat{u})}{A_1(\widehat{u}) + A_2(\widehat{u})} > \frac{\sup_{u \in (A_1+A_2)^{-1}(]-\infty, s])} B(u)}{s};$$
(ii) *the functional $I_\lambda : X_d \to \mathbb{R}$ given as $I_\lambda(u) = A_1(u) + A_2(u) - \lambda B(u)$ for all $u \in X_d$ satisfies the (PS)-condition and it is unbounded from below for all $\lambda \in \overline{\Lambda} := \big] \frac{A_1(\widehat{u})+A_2(\widehat{u})}{B(\widehat{u})}, \frac{s}{\sup_{u \in (A_1+A_2)^{-1}(]-\infty,s])} B(u)} \big[$.*

Then I_λ admits two non-zero critical points $u_{\lambda,1}, u_{\lambda,2} \in X_d$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$, for all $\lambda \in \overline{\Lambda}$.

3. Main results

We start this section with an useful observation. Let $h : [0, N+1] \to [0, +\infty[$ and $r \in]1, +\infty[$. If $-\Delta(\phi_r(\Delta u(z-1))) + h(z)\phi_r(u(z)) \geq 0$ and $u(z) \leq 0$, then
$$\Delta u(z) \begin{cases} \leq 0 & \text{if } \Delta u(z-1) \leq 0; \\ < 0 & \text{if } \Delta u(z-1) < 0. \end{cases} \tag{3.1}$$
Indeed, if $u(z) \leq 0$ then $\phi_r(u(z)) \leq 0$ and hence $-\Delta(\phi_r(\Delta u(z-1))) \geq 0$. So, we have $\phi_r(\Delta u(z)) \leq \phi_r(\Delta u(z-1))$, which implies that (3.1) holds.

We denote by $C_+ := \{u \in X_d : u(z) > 0 \text{ for all } z \in [1, N]\}$. A solution u of problem (1.1) is positive if $u \in C_+$. Now, we are ready to establish the following strong maximum principle result type.

Theorem 3.1. *Let $u \in X_d$ be fixed so that one of the following inequalities holds for each $z \in [1, N]$:*

(a) *$u(z) > 0$;*
(b) *$-\Delta(\phi_p(\Delta u(z-1))) + \alpha(z)\phi_p(u(z)) \geq 0$;*
(c) *$-\Delta(\phi_q(\Delta u(z-1))) + \beta(z)\phi_q(u(z)) \geq 0$.*

Then, either $u \in C_+$ or $u \equiv 0$, provided that (H2) holds too.

Proof. Let $u \in X_d \setminus \{0\}$ and $J = \{z \in [1, N] : u(z) \leq 0\}$. If $J = \emptyset$, then $u \in C_+$. Proceeding by absurd, we assume that $J \neq \emptyset$. Now, if $\min J = 1$, then from (3.1) we deduce that $\Delta u(1) \leq 0$, which implies $u(2) \leq 0$. By iterating this argument, we obtain easily
$$0 = u(N+1) \leq u(N) \leq \dots \leq u(2) \leq u(1) \leq 0,$$
which leads to contradiction (i.e., $u \equiv 0$). On the other hand, if $\min J = j \in [2, N]$, then $\Delta u(j-1) = u(j) - u(j-1) < 0$ (note that $u(j-1) > 0$). By (3.1), we obtain
$$\Delta u(j) < 0 \ \Rightarrow\ u(j+1) < u(j) \leq 0.$$
By iterating this argument, we obtain easily
$$u(N+1) < u(N) < \dots < u(j+1) < u(j) \leq 0,$$
which leads to a contradiction (i.e., $u(N+1) < 0$). Then, $J = \emptyset$ and hence $u \in C_+$. $\square$

In the sequel, let $\xi^+ = \max\{0, \xi\}$ and we denote with $g_+ : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ the function defined by $g_+(z, \xi) = g(z, \xi^+)$ for all $z \in [1, N]$, all $\xi \in \mathbb{R}$.

Remark 3.2. If the function $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ is such that $g(z, 0) \geq 0$ for all $z \in [1, N]$, then g_+ satisfies the condition (H1).

Now, consider the function $G^+ : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ given as

$$G^+(z,t) = \int_0^t g_+(z,\xi)d\xi, \quad \text{for all } t \in \mathbb{R},\ z \in [1, N+1],$$

and the functional $B^+ : X_d \to \mathbb{R}$ defined by

$$B^+(u) = \sum_{z=1}^{N+1} G^+(z, u(z)), \quad \text{for all } u \in X_d.$$

It is clear that $B^+ \in C^1(X_d, \mathbb{R})$. Also, the functional $I_\lambda^+ : X_d \to \mathbb{R}$ given as

$$I_\lambda^+(u) = A_1(u) + A_2(u) - \lambda B^+(u), \quad \text{for all } u \in X_d,$$

has as critical points the solutions of the problem

$$\begin{aligned} &-\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) \\ &= \lambda g_+(z, u(z)), \quad \text{for all } z \in [1, N], \\ &\qquad\qquad u(0) = u(N+1) = 0. \end{aligned} \tag{3.2}$$

Remark 3.3. It is immediate to check that Lemma 2.3 holds for the functional I_λ^+, if we assume that $g(z, 0) \geq 0$ for all $z \in [1, N]$. In fact, this ensures that (H1) holds for g_+ (by Remark 3.2).

The proof of the following proposition is an immediate consequence of Theorem 3.1 (see also [4]).

Proposition 3.4. *If the function $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ is such that $g(z, 0) \geq 0$ for all $z \in [1, N]$, then each non-zero critical point of I_λ^+ is a positive solution of* (1.1)*, provided that* (H2) *holds.*

Proof. We note that each positive solution $u \in X_d$ of (3.2) is a positive solution of (1.1), since $g_+(z, u(z)) = g(z, u(z))$ for all $z \in [1, N]$. So, we prove that the non-zero solutions of (3.2) are positive. Assume that $u \in X_d \setminus \{0\}$ is a solution of (3.2). If for some $z \in [1, N]$ we have $u(z) \leq 0$, then

$$\begin{aligned} &-\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) \\ &= \lambda g(z, u^+(z)) = \lambda g(z, 0) \geq 0. \end{aligned}$$

This ensures that either (b) or (c) holds for each $z \in [1, N]$ such that $u(z) \leq 0$. So, by an application of Theorem 3.1, we conclude that $u \in C_+$. It follows that the non-zero solutions of (3.2) are positive and hence are positive solutions of (1.1). □

Invoking Theorem 2.4, we have the following result concerning problem (1.1). We establish it with respect to the functional I_λ^+.

Theorem 3.5. *Let $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(z, 0) \geq 0$ for all $z \in [1, N]$ and $g(N+1, t) = 0$ for all $t \in \mathbb{R}$. Assume that (H2) holds, and there exist $c, d \in]0, +\infty[$ with $c > d$ such that the following inequality is* *satisfied:*

$$\begin{aligned} &c^{-p}\sum_{z=1}^{N+1}\max_{0\le\xi\le c}G(z,\xi)\\ &<\frac{(N+1)^{1-p}}{p}\min\Big\{\frac{\sum_{z=1}^{N+1}G(z,d)}{d^pp^{-1}(2+\alpha)+d^qq^{-1}(2+\beta)},\frac{q\,m_\infty}{(2^p+2^q)N+\alpha+\beta}\Big\}, \end{aligned}\tag{3.3}$$

where $m_\infty>0$ *is as in Lemma 2.3. Then problem (1.1) has at least two positive solutions, for each* $\lambda\in\Lambda^*$ *with* Λ^* *being the open interval*

$$\begin{aligned} &\Big]\max\Big\{\frac{d^pp^{-1}(2+\alpha)+d^qq^{-1}(2+\beta)}{\sum_{z=1}^{N+1}G(z,d)},\frac{(2^p+2^q)N+\alpha+\beta}{q\,m_\infty}\Big\},\\ &\frac{p^{-1}(N+1)^{1-p}c^p}{\sum_{z=1}^{N+1}\max_{0\le\xi\le c}G(z,\xi)}\Big[. \end{aligned}$$

Proof. We show that there are $s\in\mathbb{R}$ and $\widehat{u}\in X_d$, with $0<A_1(\widehat{u})+A_2(\widehat{u})<s$, such that

$$\frac{B^+(\widehat{u})}{A_1(\widehat{u})+A_2(\widehat{u})}>\frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])}B^+(u)}{s}.$$

Let

$$s:=\frac{c^p}{p(N+1)^{p-1}}.$$

For all $u\in(A_1+A_2)^{-1}(]-\infty,s])$, we have

$$\begin{aligned} &\frac{1}{p}\|u\|_{p,\alpha}^p+\frac{1}{q}\|u\|_{q,\beta}^q\le s,\\ &\Rightarrow\ \frac{1}{p}\|u\|_{p,\alpha}^p\le s,\\ &\Rightarrow\ \|u\|_{p,\alpha}\le(ps)^{\frac{1}{p}},\\ &\Rightarrow\ \|u\|_\infty\le\frac{(N+1)^{\frac{p-1}{p}}}{2}\|u\|_{p,\alpha}\le\frac{(N+1)^{\frac{p-1}{p}}}{2}(ps)^{1/p}<c\quad\text{(by (2.1))}. \end{aligned}$$

Since $G^+(z,t)\le G^+(z,0)=G(z,0)$ for all $t<0$ and $z\in[1,N]$, we have

$$B^+(u)=\sum_{z=1}^{N+1}G^+(z,u(z))\le\sum_{z=1}^{N+1}\max_{0\le\xi\le c}G(z,\xi),$$

for all $u\in X_d$ with $u\in(A_1+A_2)^{-1}(]-\infty,s])$, and hence

$$\frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])}B^+(u)}{s}\le p(N+1)^{p-1}\frac{\sum_{z=1}^{N+1}\max_{0\le\xi\le c}G(z,\xi)}{c^p}.\tag{3.4}$$

Next, let $\widehat{u}\in X_d$ be given as $\widehat{u}(z)=d$ for all $z\in[1,N]$. We have

$$A_1(\widehat{u})+A_2(\widehat{u})=\frac{(2+\alpha)d^p}{p}+\frac{(2+\beta)d^q}{q}=d^pp^{-1}(2+\alpha)+d^qq^{-1}(2+\beta)$$

implies

$$\begin{aligned} \frac{B^+(\widehat{u})}{A_1(\widehat{u})+A_2(\widehat{u})}&=\frac{\sum_{z=1}^{N+1}G(z,d)}{d^pp^{-1}(2+\alpha)+d^qq^{-1}(2+\beta)}\\ &>p(N+1)^{p-1}\frac{\sum_{z=1}^{N+1}\max_{0\le\xi\le c}G(z,\xi)}{c^p}, \end{aligned}$$

which implies

$$\frac{B^+(\widehat{u})}{A_1(\widehat{u})+A_2(\widehat{u})} > \frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])} B^+(u)}{s}$$

by (3.4). We observe that $0<d<c$ implies

$$\sum_{z=1}^{N+1} G(z,d) \le \sum_{z=1}^{N+1} \max_{0\le\xi\le c} G(z,\xi).$$

So, by (3.3), we obtain

$$0 < d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta) < \frac{c^p}{p(N+1)^{p-1}}.$$

Also, we have

$$0 < A_1(\widehat{u}) + A_2(\widehat{u}) = d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta) < \frac{c^p}{p(N+1)^{p-1}} = s.$$

By an application of Theorem 2.4, since the functional I_λ^+ satisfies Lemma 2.3, we conclude that the problem (3.2) has at least two non-zero solutions, for each $\lambda\in\Lambda^*$. Finally, Proposition 3.4 implies that the two solutions are positive and hence they are positive solutions of the problem (1.1). □

Now, we assume that $g:[1,N+1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $g(z,0)\ge 0$ for all $z\in[1,N]$, $g(N+1,t)=0$ for all $t\in\mathbb{R}$, and

$$\limsup_{\xi\to 0^+} \frac{G(z,\xi)}{\xi^p} = +\infty, \quad \lim_{\xi\to+\infty} \frac{G(z,\xi)}{\xi^p} = +\infty \quad \text{for all } z\in[1,N]. \tag{3.5}$$

Note that the second limit in (3.5) ensures that $m_\infty = +\infty$. On the other hand, the first limit in (3.5) ensures that

$$\max_{0\le\xi\le c} G(z,\xi) > 0 \quad \text{for all } z\in[1,N], \text{ all } c>0.$$

So, we put

$$\overline{\lambda} = \frac{1}{p(N+1)^{p-1}} \sup_{c>0} \frac{c^p}{\sum_{z=1}^{N+1} \max_{0\le\xi\le c} G(z,\xi)} > 0.$$

It follows that for all $\lambda<\overline{\lambda}$ there exists $c>0$ such that

$$\lambda < \frac{1}{p(N+1)^{p-1}} \frac{c^p}{\sum_{z=1}^{N+1} \max_{0\le\xi\le c} G(z,\xi)} > 0.$$

By the first limit in (3.5), we obtain that there is $d\in]0,c[$ such that

$$\frac{\sum_{z=1}^{N+1} G(z,d)}{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)} > \frac{1}{\lambda}.$$

Consequently

$$c^{-p} \sum_{z=1}^{N+1} \max_{0\le\xi\le c} G(z,\xi) < \frac{1}{p(N+1)^{p-1}} \frac{\sum_{z=1}^{N+1} G(z,d)}{d^p p^{-1}(2+\alpha) + d^q q^{-1}(2+\beta)}.$$

By using Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Let $g : [1, N+1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(z,0) \geq 0$ for all $z \in [1,N]$ and $g(N+1,t) = 0$ for all $t \in \mathbb{R}$. Also, assume that (H2),* (3.5) *hold. Then problem* (1.1) *has at least two positive solutions, for each $\lambda \in]0, \overline{\lambda}[$.*

Along the lines of Theorem 3.5, in the case $c, d \in]0,1]$, we have the following result.

Corollary 3.7. *Let $g : [1, N+1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function with $g(N+1,t) = 0$ for all $t \in \mathbb{R}$. Assume also that* (H1)–(H2) *hold, and there exist $c, d \in]0,1]$ with $c > d$ such that the following inequality is satisfied:*

$$c^{-q} \sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi) < 2^q (N+1)^{1-q} \min\Big\{ \frac{\sum_{z=1}^{N+1} G(z,d)}{(4+\alpha+\beta)d^q}, \frac{m_\infty}{(2^p+2^q)N+\alpha+\beta} \Big\}, \tag{3.6}$$

where $m_\infty > 0$ is as in Lemma 2.3. Then problem (1.1) *has at least two positive solutions, for each $\lambda \in \overline{\Lambda}^*$ with $\overline{\Lambda}^*$ being the open interval*

$$\Big] \max\Big\{ \frac{4+\alpha+\beta}{q} \frac{d^q}{\sum_{z=1}^{N+1} G(z,d)}, \frac{(2^p+2^q)N+\alpha+\beta}{q m_\infty} \Big\}, \frac{2^q q^{-1}(N+1)^{1-q} c^q}{\sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi)} \Big[.$$

Proof. Arguing as in the proof of Theorem 3.5, we can show that there are $s \in \mathbb{R}$ and $\widehat{u} \in X_d$, with $0 < A_1(\widehat{u}) + A_2(\widehat{u}) < s$, such that

$$\frac{B(\widehat{u})}{A_1(\widehat{u}) + A_2(\widehat{u})} > \frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])} B(u)}{s}.$$

Let

$$s = \frac{2^q c^q}{q(N+1)^{q-1}}.$$

For all $u \in (A_1 + A_2)^{-1}(] - \infty, s])$, we have

$$\begin{aligned}
&\frac{1}{p}\|u\|_{p,\alpha}^p + \frac{1}{q}\|u\|_{q,\beta}^q \leq s,\\
&\Rightarrow \frac{1}{q}\|u\|_{q,\beta}^q \leq s,\\
&\Rightarrow \|u\|_{q,\beta} \leq (qs)^{\frac{1}{q}},\\
&\Rightarrow \|u\|_\infty \leq \frac{(N+1)^{\frac{q-1}{q}}}{2}\|u\|_{q,\beta} \leq \frac{(N+1)^{\frac{q-1}{q}}}{2}(qs)^{\frac{1}{q}} \leq c \quad \text{(by (2.1))}.
\end{aligned}$$

Since $G(z,t) \leq G(z,0)$ for all $t < 0$, we obtain

$$B(u) = \sum_{z=1}^{N+1} G(z,u(z)) \leq \sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi),$$

for all $u \in X_d$ with $u \in (A_1 + A_2)^{-1}(] - \infty, s])$. Then, we have

$$\frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])} B(u)}{s} \leq \frac{q(N+1)^{q-1}}{2^q} \frac{\sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi)}{c^q}. \tag{3.7}$$

Moreover, let $\widehat{u} \in X_d$ be given as $\widehat{u}(z) = d$ for all $z \in [1,N]$. So, we obtain that

$$A_1(\widehat{u}) + A_2(\widehat{u}) = \frac{(2+\alpha)d^p}{p} + \frac{(2+\beta)d^q}{q} < \frac{(4+\alpha+\beta)d^q}{q}$$

implies

$$\frac{B(\widehat{u})}{A_1(\widehat{u})+A_2(\widehat{u})} \geq \frac{q\sum_{z=1}^{N+1} G(z,d)}{(4+\alpha+\beta)d^q} > \frac{q(N+1)^{q-1}}{2^q} \frac{\sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi)}{c^q},$$

which implies

$$\frac{B(\widehat{u})}{A_1(\widehat{u})+A_2(\widehat{u})} > \frac{\sup_{u\in(A_1+A_2)^{-1}(]-\infty,s])} B(u)}{s} \quad \text{(by (3.7))}.$$

Now, $0<d<c$ implies $\sum_{z=1}^{N+1} G(z,d) \leq \sum_{z=1}^{N+1} \max_{0\leq\xi\leq c} G(z,\xi)$, and by (3.6) we obtain

$$0<d<\frac{2c}{[(4+\alpha+\beta)(N+1)^{q-1}]^{\frac{1}{q}}},$$

and hence we deduce that $0 < A_1(\widehat{u})+A_2(\widehat{u}) < s$. By an application of Theorem 2.4 we conclude that problem (1.1) has at least two non-zero solutions, for each $\lambda\in\overline{\Lambda}^*$. Now, the assumption that (H1) holds for the function g, ensures that the solutions of problem (1.1) are also solutions of problem (3.2). By Proposition 3.4, problem (1.1) has at least two positive solutions, for each $\lambda\in\overline{\Lambda}^*$. $\square$

The next result is a particular case of Theorem 3.5. That is, we deal with the problem

$$\begin{gathered} -\Delta_p u(z-1) - \Delta_q u(z-1) + \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) = \lambda\omega(z)f(u(z)), \\ \text{for all } z\in[1,N], \\ u(0)=u(N+1)=0, \end{gathered} \tag{3.8}$$

where $f:\mathbb{R}\to[0,+\infty[$, and $\omega:[1,N+1]\to[0,+\infty[$ with $\omega(z)>0$ for all $z\in[1,N]$, and $w(N+1)=0$. Let $W=\sum_{z=1}^{N}\omega(z)$, $F(t)=\int_0^t f(\xi)d\xi$ for all $t\in\mathbb{R}$ and $m_\infty^* := \min_{z\in[1,N]}\omega(z)\liminf_{t\to+\infty}\frac{F(t)}{t^p} > 0$. Then, we have the following result.

Corollary 3.8. *Let $f:\mathbb{R}\to[0,+\infty[$ be a continuous function. Assume that* (H2) *holds, and that there exist $c,d\in]0,+\infty[$ with $c>d$ such that the following inequality is satisfied:*

$$\begin{aligned} &c^{-p}F(c)W \\ &< \frac{(N+1)^{1-p}}{p}\min\Big\{\frac{F(d)W}{d^p p^{-1}(2+\alpha)+d^q q^{-1}(2+\beta)}, \frac{qm_\infty^*}{(2^p+2^q)N+\alpha+\beta}\Big\}. \end{aligned}$$

Then problem (3.8) *has at least two positive solutions, for each $\lambda\in\Lambda^*$ with Λ^* being the open interval*

$$\Big]\max\Big\{\frac{d^p p^{-1}(2+\alpha)+d^q q^{-1}(2+\beta)}{F(d)W}, \frac{(2^p+2^q)N+\alpha+\beta}{qm_\infty^*}\Big\}, \frac{p^{-1}(N+1)^{1-p}c^p}{F(c)W}\Big[.$$

Proof. Consider the function $g:[1,N+1]\times\mathbb{R}\to\mathbb{R}$ given as

$$g(z,\xi)=\omega(z)f(\xi), \quad \text{for all } z\in[1,N+1], \text{ all } \xi\in\mathbb{R},$$

so that

$$\sum_{z=1}^{N+1}\max_{0\leq\xi\leq c} G(z,\xi) = F(c)W \quad \text{and} \quad \sum_{z=1}^{N+1} G(z,d) = F(d)W$$

Then, all the assumptions of Theorem 3.5 hold and so we conclude that problem (3.8) has at least two positive solutions, for each $\lambda\in\overline{\Lambda}^*$. $\square$

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