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SIGN-CHANGING SOLUTIONS FOR ELLIPTIC EQUATIONS WITH FAST INCREASING WEIGHT AND CONCAVE-CONVEX NONLINEARITIES

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Abstract. In this article, we study the problem

$$-\operatorname{div}(K(x)\nabla u) = a(x)K(x)|u|^{q-2}u + b(x)K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $1 < q < 2$, $K(x) = \exp(|x|^\alpha/4)$ with $\alpha \geq 2$. Under some assumptions on the potentials $a(x)$ and $b(x)$, we obtain a pair of sign-changing solutions of the problem via variational methods and certain estimates.

1. Introduction

In this article, we consider the existence of sign-changing solutions for the problem

$$-\operatorname{div}(K(x)\nabla u) = a(x)K(x)|u|^{q-2}u + b(x)K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $1 < q < 2$, $K(x) = \exp(|x|^\alpha/4)$ with $\alpha \geq 2$.

Our motivations of studying the equation (1.1) relies on the fact that, for $\alpha = q = 2$, $a(x) = (N-2)/(N+2)$ and $b(x) \equiv 1$, equation (1.1) occurs when one tries to find self-similar solutions of the form

$$w(t,x) = t^{\frac{2-N}{N+2}} u(xt^{-1/2})$$

for the evolution equation

$$w_t - \Delta w = |w|^{4/(N-2)} w \quad \text{on } (0,\infty) \times \mathbb{R}^N.$$

See [8, 11] for a detailed description.

Equation (1.1) with $q = 2$, $a(x) \equiv \lambda$ and $b(x) \equiv 1$, has been studied in [12, 13, 14, 15]. We also refer to the paper of Catrina et al. [3] where the authors considered the case $q = 2$, $a(x) = \lambda|x|^{\alpha-2}$ and $b(x) \equiv 1$, and showed that the value of α affects the critical dimension of the problem. Later on, Furtado et al. [9] studied the equation

$$-\operatorname{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^\beta |u|^{q-2}u + K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\beta = (\alpha-2)\frac{(2^*-q)}{(2^*-2)}$. In that paper, by using Mountain Pass Theorem, the authors obtained a positive solution if $2 < q < 2^*$. Furthermore, they applied

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Linking Theorem to show that problem (1.2) when $q = 2$ has a nontrivial solution for any $\lambda \geq \lambda_1$, where λ_1 is the first positive eigenvalue of the linear problem

$$-\operatorname{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^{\alpha-2}u, \quad x \in \mathbb{R}^N.$$

With the help of the result of [3], namely there is no positive solution of (1.2) for $q = 2$ and $\lambda \geq \lambda_1$, then they can conclude that this nontrivial solution indeed is a sign-changing solution. Recently, Furtado et al. [10] obtained two nonnegative nontrivial solutions for (1.1) when the potential $a(x)$ has small norm in a suitable weighted Lebesgue space.

On the other hand, for similar problems in bounded domain, Ambrosetti et al. [2] studied the semilinear problem

$$-\Delta u = \lambda u^{q-1} + u^{p-1} \quad \text{in } \Omega,\ u \in H_0^1(\Omega)$$

where $\Omega \subset \mathbb{R}^N$ is bounded, $N \geq 3$, $\lambda > 0$, $1 < q < 2 < p \leq 2^*$. They proved the existence of at least two positive solutions if $\lambda \in (0, \lambda_0)$ for some positive λ_0. We also refer the interested readers to [1, 4, 6, 20] where equations with concave and convex nonlinearity on bounded domains were considered.

Motivated by the works we described above, in present paper, we try to seek more solutions of (1.1). Special concern is the existence of sign-changing solutions of (1.1). This kind of problem is variational in nature. Indeed, let us denote by H the Hilbert space obtained as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx\Big)^{1/2}.$$

We also define the weighted Lebesgue spaces

$$L_K^s(\mathbb{R}^N) = \big\{u \text{ measurable in } \mathbb{R}^N : \|u\|_s^s = \int_{\mathbb{R}^N} K(x)|u|^s dx < \infty\big\}.$$

It is proven in [9] that the embedding $H \hookrightarrow L_K^r(\mathbb{R}^N)$ is continuous for $2 \leq r \leq 2^*$, and compact for $2 \leq r < 2^*$. For any $r > 1$, we denote by r' its conjugated exponent, that is, the unique $r' > 1$ so that $1/r + 1/r' = 1$. Throughout this paper, we always use the following assumptions:

(A1) $a(x) > 0$ and $a(x) \in L_K^{\sigma_q}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ for some $(2/q) \leq \sigma_q' < (2^*/q)$;
(A2) $b(x) > 0$ and $b(x) \in L^\infty(\mathbb{R}^N)$;
(A3) the set $\Omega_b^+ := \{x \in \mathbb{R}^N : b(x) > 0\}$ has an interior point;
(A4) there are $x_0 \in \mathbb{R}^N$ and $\delta > 0$ such that $B_\delta(x_0) \subset \Omega_b^+$ and

$$|b(x)|_\infty - b(x) \leq M|x - x_0|^\gamma,$$

for a.e. $x \in B_\delta(x_0)$, with $M > 0$ and $\gamma > N$.

On H, we define the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)b(x)|u|^{2^*} dx.$$

By (A1), (A2) and the above embedding, we conclude that I is well defined and $I \in C^1(H, \mathbb{R})$. Now, it is well known that there exists a one to one correspondence between the critical points and the weak solutions of (1.1). Here, we say $u \in H$ is a weak solution of (1.1), if for any $\phi \in H$, there holds

$$\int_{\mathbb{R}^N} K(x)\big[\nabla u \nabla\phi - a(x)|u|^{q-2}u\phi - b(x)|u|^{2^*-2}u\phi\big]dx = 0.$$

Our main result is stated below.

Theorem 1.1. *Assume that (A1)–(A4). If $N \geq 7$ and $(3N-2)/(2N-4) < q < 2$, then there exists $M_2 > 0$, such that* (1.1) *has at least two nonnegative nontrivial solutions and a pair of sign-changing solutions in H for $\|a(x)\|_{\sigma_q} < M_2$ and $\alpha > (N-2)/2$.*

Since Furtado et al. [10] showed that (1.1) has at least two nonnegative nontrivial solutions in H for $\|a\|_{\sigma_q} < M_1$ with some $M_1 > 0$, we will focus our attentions to find out sign-changing solutions of (1.1). To this end, there are some difficulties. Firstly, since the embedding $H \hookrightarrow L_K^{2^*}(\mathbb{R}^N)$ is not compact, the functional I satisfies (PS) condition only locally. We prove that the energy level belongs to the range where (PS) condition hold by choosing a suitable test function as in [3, 10]. Secondly, as pointed in [5], the Mountain Pass Theorem which was used in [9, 10] is usually unable to prove the existence of sign-changing solutions. Moreover, the Linking Theorem used in [9] can not be applied here becasue to $1 < q < 2$. Instead of the two above Theorems, we shall employ the separation argument for Nehari-type set of the problem, which has been used in [5, 17, 18]. Thirdly, the potentials $a(x)$ and $b(x)$ bring much difficulty to the above separation argument. To overcome this difficulty, inspired by [16], we impose conditions (A1) and (A2) on the potentials $a(x)$ and $b(x)$ respectively, which are stronger than the corresponding ones in [10].

This article is organized as follows. In the next section, we give some notation and preliminaries. Then we prove Theorem 1.1.

2. Preliminaries

Throughout this paper, E^{-1} denotes the dual space of a Banach space E. We denote by $|\cdot|_t$, the norm of the standard Sobolev space $L^t(\mathbb{R}^N)$. $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm of $\int_{\mathbb{R}^N} |\nabla \cdot|^2 dx$. $B_r(x)$ is a ball centered at x with radius r. $\to$ denotes strong convergence. $\rightharpoonup$ denotes weak convergence. d, d_i will denote various positive constants whose exact values are not important. Finally, we write $\int u$, $\|a\|_{\sigma_q}$ and $|b|_\infty$ instead of $\int_{\mathbb{R}^N} u(x)dx$, $\|a(x)\|_{\sigma_q}$ and $|b(x)|_\infty$, respectively.

For each $r \in [2, 2^*]$, the existence of the embedding $H \hookrightarrow L_K^r(\mathbb{R}^N)$ enables us to define

$$S_r = \inf\Big\{\int K(x)|\nabla u|^2 : u \in H, \int K(x)|u|^r = 1\Big\}. \tag{2.1}$$

In particular, when $r = 2^*$, we only write $S := S_{2^*}$. It is worth pointing out that this constant is equal to the best constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, see [3].

By the condition (A4), we can choose $\eta > 0$ small enough such that $B_{2\eta}(x_0) \subset B_\delta(x_0)$ with $x_0 \in \operatorname{int}(\Omega_b^+)$ and $\delta > 0$. Define a cutoff function $\psi(x)$ satisfying $\psi(x) \equiv 1$ in $B_\eta(x_0)$, $\psi(x) \equiv 0$ outside $B_{2\eta}(x_0)$ and $0 \leq \psi \leq 1$. Inspired by [3, 10], we consider the function

$$u_\varepsilon(x) = K(x)^{-1/2}\psi(x)\Big(\frac{1}{\varepsilon + |x-x_0|^2}\Big)^{(N-2)/2},$$

and set

$$U_\varepsilon(x) = K(x)^{-1/2}\Big(\frac{1}{\varepsilon + |x-x_0|^2}\Big)^{(N-2)/2}, \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{2^*}}.$$

Without loss of generality, we assume that $x_0 = 0$ from now on. To prove Theorem 1.1, we first give the next three Lemmas which will be useful later.

Lemma 2.1. *For $\varepsilon > 0$ small,*

$$\int u_\varepsilon^\mu = O(1) \quad \text{if } 0 < \mu < \frac{N}{N-2}, \tag{2.2}$$

$$\int u_\varepsilon^\mu = O\big(\varepsilon^{\frac{N}{2}-\frac{N-2}{2}\mu}|\ln\varepsilon|\big) \quad \text{if } \mu = \frac{N}{N-2}, \tag{2.3}$$

$$\int u_\varepsilon^\mu = O\big(\varepsilon^{\frac{N}{2}-\frac{N-2}{2}\mu}\big) \quad \text{if } \frac{N}{N-2} < \mu < 2^*. \tag{2.4}$$

Proof. Note that

$$\begin{aligned}\int u_\varepsilon^\mu &\le d\int_{B_{2\eta(0)}} \frac{dx}{(\varepsilon+|x|^2)^{(N-2)\mu/2}}\\ &\le d_1\int_0^{2\eta/\sqrt{\varepsilon}} \frac{\varepsilon^{N/2}\rho^{N-1}d\rho}{\varepsilon^{(N-2)\mu/2}(1+|\rho|^2)^{(N-2)\mu/2}}\,.\end{aligned}$$

Since $N-1-(N-2)\mu > -1$, when $0 < \mu < N/(N-2)$, we have

$$\int u_\varepsilon^\mu \le d_2\int_{B_{2\eta(0)}} \frac{1}{|x|^{(N-2)\mu}} = O(1).$$

Thus, (2.2) holds. The proofs of (2.3) and (2.4) are similar. □

Lemma 2.2. *For $\varepsilon > 0$ small, we have*

$$\begin{aligned}\int v_\varepsilon^\mu &= \frac{\int u_\varepsilon^\mu}{\|u_\varepsilon\|_{2^*}^\mu}\\ &= O\big(\varepsilon^{(N-2)\mu/4}\big) \quad \text{if } 0<\mu<\frac{N}{N-2}, &(2.5)\\ &= O\big(\varepsilon^{\frac{N}{2}-\frac{N-2}{4}\mu}|\ln\varepsilon|\big) \quad \text{if } \mu=\frac{N}{N-2}, &(2.6)\\ &= O\big(\varepsilon^{\frac{N}{2}-\frac{N-2}{4}\mu}\big) \quad \text{if } \frac{N}{N-2}<\mu<2^*. &(2.7)\end{aligned}$$

Proof. According to [3],

$$\|u_\varepsilon\|_{2^*}^{2^*} = \int K(x)|u_\varepsilon|^{2^*} = \varepsilon^{-N/2}A_0 + O(1), \quad \text{if } N>2,$$

with

$$A_0 = \int \frac{1}{(1+|x|^2)^N}, \quad \text{if } N>2,$$

from which it follows that

$$\begin{aligned}\|u_\varepsilon\|_{2^*}^\mu &= \big(\varepsilon^{-N/2}A_0 + O(1)\big)^{\mu/2^*}\\ &= d\varepsilon^{-(N-2)\mu/4} + O\big(\varepsilon^{-\frac{N}{2}(\frac{\mu}{2^*}-1)}\big).\end{aligned}$$

This and (2.2) imply that for $0 < \mu < N/(N-2)$ and ε small enough

$$\begin{aligned}\int v_\varepsilon^\mu &= \frac{\int u_\varepsilon^\mu}{\|u_\varepsilon\|_{2^*}^\mu}\\ &= \frac{O(1)}{d\varepsilon^{-(N-2)\mu/4} + O\big(\varepsilon^{\frac{-N}{2}(\frac{\mu}{2^*}-1)}\big)} = O\big(\varepsilon^{(N-2)\mu/4}\big).\end{aligned}$$

Thus, (2.5) follows. Similar arguments arrive at (2.6) and (2.7). □

Lemma 2.3. *Let w_1 be a nonnegative nontrivial solution of* (1.1)*. For $1 < q < 2$ and $\varepsilon > 0$ small, then we have*

$$\int K(x)a(x)v_\varepsilon^q \geq d\varepsilon^{\frac{N}{2}-\frac{N-2}{4}q} + O\big(\varepsilon^{\frac{(N-2)q}{4}}\big) \quad \text{if } \frac{N}{N-2} < q < 2, \tag{2.8}$$

$$\int K(x)a(x)|w_1|v_\varepsilon^{q-1} = O\big(\varepsilon^{\frac{(N-2)(q-1)}{4}}\big), \tag{2.9}$$

$$\int K(x)a(x)|w_1|^{q-1}v_\varepsilon = O\big(\varepsilon^{\frac{N-2}{4}}\big), \tag{2.10}$$

$$\int K(x)a(x)|w_1|^{q-2}w_1v_\varepsilon = O\big(\varepsilon^{\frac{N-2}{4}}\big), \tag{2.11}$$

$$\int K(x)b(x)|w_1|^{2^*-1}v_\varepsilon = O\big(\varepsilon^{\frac{N-2}{4}}\big), \tag{2.12}$$

$$\int K(x)b(x)|w_1|^{2^*-2}w_1v_\varepsilon = O\big(\varepsilon^{\frac{N-2}{4}}\big), \tag{2.13}$$

$$\int K(x)b(x)|w_1|v_\varepsilon^{2^*-1} = O\big(\varepsilon^{\frac{N-2}{4}}\big). \tag{2.14}$$

Proof. We only prove part(i). The rest parts of the Lemma can be proved by a similar argument. Using (A1), one has

$$\begin{aligned}
&\int K(x)a(x)|u_\varepsilon|^q \\
&= \int_{B_{2\eta}(0)} \frac{K(x)a(x)K(x)^{-q/2}\psi^q(x)}{(\varepsilon+|x|^2)^{q(N-2)/2}} \\
&\geq d_1 \int_{B_{2\eta}(0)} \frac{\psi^q(x)}{(\varepsilon+|x|^2)^{q(N-2)/2}} \\
&= d_1\Big(\int_{B_{2\eta}(0)} \frac{1}{(\varepsilon+|x|^2)^{q(N-2)/2}} + \int_{B_{2\eta}(0)} \frac{\psi^q(x)-1}{(\varepsilon+|x|^2)^{q(N-2)/2}}\Big) \\
&= d_1\Big(\varepsilon^{\frac{N}{2}-\frac{(N-2)q}{2}} \int_{B_{2\eta/\sqrt{\varepsilon}}(0)} \frac{1}{(1+|x|^2)^{q(N-2)/2}} + \int_{B_{2\eta}(0)} \frac{\psi^q(x)-1}{(\varepsilon+|x|^2)^{q(N-2)/2}}\Big) \\
&= d_2\varepsilon^{\frac{N}{2}-\frac{(N-2)q}{2}} + O(1)
\end{aligned}$$

whenever $q > N/(N-2)$. Therefore,

$$\begin{aligned}
\int K(x)a(x)|v_\varepsilon|^q &= \frac{\int K(x)a(x)|u_\varepsilon|^q}{\|u_\varepsilon\|_{2^*}^q} \\
&\geq \frac{d_2\varepsilon^{\frac{N}{2}-\frac{(N-2)q}{2}} + O(1)}{d_3\varepsilon^{-(N-2)q/4} + O\big(\varepsilon^{\frac{-N}{2}(\frac{q}{2^*}-1)}\big)} \\
&= d\varepsilon^{\frac{N}{2}-\frac{N-2}{4}q} + O\big(\varepsilon^{\frac{(N-2)q}{4}}\big).
\end{aligned}$$

Hence, we obtain (2.8) holds. □

3. Existence of sign-changing solutions

Following Tarantello [18] and Chen [5], we first decompose the Nehari-type set of the considered problem, then consider minimization problems of I on its proper subset. Set

$$\Lambda = \{u \in H : \langle I'(u), u\rangle = 0\}.$$

Consider the following three subsets of Λ:

$$\Lambda_0 = \{u \in \Lambda : (2-q)\|u\|^2 - (2^* - q)\int K(x)b(x)|u|^{2^*} = 0\},$$

$$\Lambda^+ = \{u \in \Lambda : (2-q)\|u\|^2 - (2^* - q)\int K(x)b(x)|u|^{2^*} > 0\},$$

$$\Lambda^- = \{u \in \Lambda : (2-q)\|u\|^2 - (2^* - q)\int K(x)b(x)|u|^{2^*} < 0\}.$$

Furthermore, if we denote

$$\overline{M} = \Big(\frac{2-q}{2^*-q}\Big)^{\frac{2-q}{2^*-2}}\Big(\frac{2^*-2}{2^*-q}\Big)S^{\frac{N}{2}-\frac{N}{4}q}S_{q\sigma_q'}^{q/2}|b|_\infty^{\frac{q-2}{2^*-2}},$$

we indeed get that for $\|a\|_{\sigma_q} < \overline{M}$ the following minimization problems:

$$c_0 = \inf_{u\in\Lambda^+} I(u) \quad \text{and} \quad c_1 = \inf_{u\in\Lambda^-} I(u)$$

attain their infimum at u_0 and u_1, respectively. Additionally, u_0 and u_1 are non-negative nontrivial solutions of (1.1). Next, we start establishing the existence of sign-changing solutions of (1.1).

3.1. **Some lemmas.** For every $u \in H$ and $u \neq 0$, we set

$$t_{\max} = \Big[\frac{(2-q)\|u\|^2}{(2^*-q)\int K(x)b(x)|u|^{2^*}}\Big]^{\frac{1}{2^*-2}}.$$

Then we have the following result.

Lemma 3.1. *Let $\|a\|_{\sigma_q} < \overline{M}$. For every $u \in H$ and $u \neq 0$, we have*

(i) *there exists a unique $t^+ = t^+(u) > t_{\max} > 0$ such that $t^+u \in \Lambda^-$ and $I(t^+u) = \max_{t\ge t_{\max}} I(tu)$.*
(ii) *there exists a unique $0 < t^- = t^-(u) < t_{\max}$ such that $t^-u \in \Lambda^+$ and $I(t^-u) = \min_{0\le t\le t^+} I(tu)$.*

Proof. From direct computations, we have

$$\frac{\partial I}{\partial t}(tu) = t^{q-1}\Big(t^{2-q}\|u\|^2 - t^{2^*-q}\int K(x)b(x)|u|^{2^*} - \int K(x)a(x)|u|^q\Big).$$

Let

$$\varphi(t) = t^{2-q}\|u\|^2 - t^{2^*-q}\int K(x)b(x)|u|^{2^*} - \int K(x)a(x)|u|^q.$$

By (A1), (A2) and easy calculations show that $\lim_{t\to 0^+}\varphi(t) = -\int K(x)a(x)|u|^q < 0$ and $\lim_{t\to+\infty}\varphi(t) = -\infty$. In addition, $\varphi(t)$ is concave and attains its maximum at the point $t_{\max}$. Also

$$\varphi(t_{\max}) = \Big(\frac{2-q}{2^*-q}\Big)^{\frac{2-q}{2^*-2}}\Big(\frac{2^*-2}{2^*-q}\Big)\Big[\frac{\|u\|^{2(2^*-q)}}{(\int K(x)b(x)|u|^{2^*})^{(2-q)}}\Big]^{\frac{N-2}{4}} - \int K(x)a(x)|u|^q.$$

From (A1), (A2) and (2.1), it is easily verified that

$$\varphi(t_{\max}) \ge \Big(\frac{2-q}{2^*-q}\Big)^{\frac{2-q}{2^*-2}}\Big(\frac{2^*-2}{2^*-q}\Big)S^{N(2-q)/4}\|u\|^q|b|_\infty^{\frac{q-2}{2^*-2}} - \int K(x)a(x)|u|^q$$

$$\geq \Big[\Big(\frac{2-q}{2^*-q}\Big)^{\frac{2-q}{2^*-2}}\Big(\frac{2^*-2}{2^*-q}\Big)S^{N(2-q)/4}|b|_\infty^{\frac{q-2}{2^*-2}} - \|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\Big]\|u\|^q.$$

Thus, for $\|a\|_{\sigma_q} < \overline{M}$, we have $\varphi(t_{\max}) > 0$. It then follows that $\varphi(t)$ has exactly two points $0 < t^- < t_{\max} < t^+$ such that

$$\varphi(t^+) = 0 = \varphi(t^-) \quad \text{and} \quad \varphi'(t^+) < 0 < \varphi'(t^-).$$

Equivalently, we obtain $t^+u \in \Lambda^-$ and $t^-u \in \Lambda^+$. Also $I(t^+u) \geq I(tu)$, for any $t \geq t_{max}$ and $I(t^-u) \leq I(tu)$, for any $t \in [0, t^+]$. □

Lemma 3.2. *Let $\|a\|_{\sigma_q} < \overline{M}$, then $\Lambda_0 = \{0\}$.*

Proof. Suppose to the contrary, there exists $w \in \Lambda_0$, $w \neq 0$ such that $(2-q)\|w\|^2 - (2^*-q)\int K(x)b(x)|w|^{2^*} = 0$. Combining this with (2.1), we can obtain that $\|w\| \geq \big(\frac{2-q}{2^*-q}\big)^{(N-2)/4}|b|_\infty^{(2-N)/4}S^{N/4}$. On the other hand, we infer from $w \in \Lambda$ that

$$\begin{aligned}
0 &= \|w\|^2 - \int K(x)a(x)|w|^q - \int K(x)b(x)|w|^{2^*}\\
&\geq \Big(\frac{2^*-2}{2^*-q}\Big)\|w\|^2 - \|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\|w\|^q\\
&\geq \|w\|^q\Big[\frac{2^*-2}{2^*-q}\Big(\frac{2-q}{2^*-q}\Big)^{(N-2)(2-q)/4}|b|_\infty^{(q-2)/(2^*-2)}S^{N(2-q)/4} - \|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\Big] > 0,
\end{aligned}$$

which is a contradiction. This completes the proof. □

Lemma 3.3. *Let $\|a\|_{\sigma_q} < \overline{M}$. Given $u \in \Lambda^-$, there are $\rho_u > 0$ and a differential function $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ defined for $w \in H$, $w \in B_{\rho_u}(0)$ such that*

(i) $g_{\rho_u}(0) = 1, \quad g_{\rho_u}(w)(u+w) \in \Lambda^-$,

(ii)

$$\begin{aligned}
&\langle g'_{\rho_u}(0), \phi\rangle\\
&= \Big(-2\int K(x)\nabla u\nabla\phi + 2^*\int K(x)b(x)|u|^{2^*-2}u\phi\\
&\quad + q\int K(x)a(x)|u|^{q-2}u\phi\Big)\Big/\Big((2-q)\|u\|^2 - (2^*-q)\int K(x)b(x)|u|^{2^*}\Big).
\end{aligned}$$

Proof. Define $F : \mathbb{R}\times H \to \mathbb{R}$ by:

$$F(t,w) = t^{2-q}\|u+w\|^2 - t^{2^*-q}\int K(x)b(x)|u+w|^{2^*} - \int K(x)a(x)|u+w|^q.$$

In view of $u \in \Lambda^- \subset \Lambda$, we obtain $F(1,0) = 0$ and

$$F_t(1,0) = (2-q)\|u\|^2 - (2^*-q)\int K(x)b(x)|u|^{2^*} < 0.$$

Using Implicit function Theorem for F at the point $(1,0)$, we know that there is $\bar\varepsilon > 0$ so that for $w \in H$, $\|w\| < \bar\varepsilon$, the equation $F(t,w) = 0$ has a unique solution $t = g_{\rho_u}(w) > 0$ with $g_{\rho_u}(0) = 1$. Since $F(g_{\rho_u}(w), w) = 0$ for $w \in H$, $\|w\| < \bar\varepsilon$, we have

$$\begin{aligned}
&g_{\rho_u}^{2-q}(w)\|u+w\|^2 - g_{\rho_u}^{2^*-q}(w)\int K(x)b(x)|u+w|^{2^*} - \int K(x)a(x)|u+w|^q\\
&= \Big(\|g_{\rho_u}(w)(u+w)\|^2 - \int K(x)b(x)|g_{\rho_u}(w)(u+w)|^{2^*}
\end{aligned}$$

$$- \int K(x)a(x)|g_{\rho_u}(w)(u+w)|^q\Big)/\Big(g_{\rho_u}^q(w)\Big) = 0,$$

namely, $g_{\rho_u}(w)(u+w) \in \Lambda$ for all $w \in H$ with $\|w\| < \bar{\varepsilon}$. Since $F_t(1,0) < 0$ and

$$\begin{aligned}
&F_t(g_{\rho_u}(w), w) \\
&= (2-q)g_{\rho_u}^{1-q}(w)\|u+w\|^2 - (2^*-q)g_{\rho_u}^{2^*-q-1}(w)\int K(x)b(x)|u+w|^{2^*} \\
&= \frac{(2-q)\|g_{\rho_u}(w)(u+w)\|^2 - (2^*-q)\int K(x)b(x)|g_{\rho_u}(w)(u+w)|^{2^*}}{g_{\rho_u}^{1+q}(w)},
\end{aligned}$$

we can choose $\varepsilon > 0$ small enough ($\varepsilon < \bar{\varepsilon}$) such that for $w \in H$ and $\|w\| < \varepsilon$,

$$(2-q)\|g_{\rho_u}(w)(u+w)\|^2 - (2^*-q)\int K(x)b(x)|g_{\rho_u}(w)(u+w)|^{2^*} < 0,$$

which means that

$$g_{\rho_u}(w)(u+w) \in \Lambda^-, \quad \text{for all } w \in H,\ \|w\| < \varepsilon.$$

Moreover, for any $\phi \in H,\ r > 0$, we have

$$\begin{aligned}
&F(1, 0+r\phi) - F(1,0) \\
&= \int K(x)|\nabla(u+r\phi)|^2 - \int K(x)b(x)|u+r\phi|^{2^*} - \int K(x)a(x)|u+r\phi|^q \\
&\quad - \int K(x)|\nabla u|^2 + \int K(x)b(x)|u|^{2^*} + \int K(x)a(x)|u|^q \\
&= \int K(x)(2r\nabla u\nabla\phi + r^2|\nabla\phi|^2) - \int K(x)b(x)\Big(|u+r\phi|^{2^*} - |u|^{2^*}\Big) \\
&\quad - \int K(x)a(x)\Big(|u+r\phi|^q - |u|^q\Big)
\end{aligned}$$

and so

$$\begin{aligned}
\langle F_w, \phi\rangle|_{t=1,w=0} &= \lim_{r\to 0}\frac{F(1,0+r\phi) - F(1,0)}{r} \\
&= 2\int K(x)\nabla u\nabla\phi - 2^*\int K(x)b(x)|u|^{2^*-2}u\phi \\
&\quad - q\int K(x)a(x)|u|^{q-2}u\phi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle g'_{\rho_u}(0), \phi\rangle \\
&= -\frac{\langle F_w, \phi\rangle}{F_t}\Big|_{t=1,w=0} \\
&= \frac{-2\int K(x)\nabla u\nabla\phi + 2^*\int K(x)b(x)|u|^{2^*-2}u\phi + q\int K(x)a(x)|u|^{q-2}u\phi}{(2-q)\|u\|^2 - (2^*-q)\int K(x)b(x)|u|^{2^*}}.
\end{aligned}$$

This completes the proof. □

3.2. **Existence results.** We are now in a position to prove Theorem 1.1. To this end, we need to make comparisons among some minimization problems. Set

$$\Lambda_1^- = \{u = u^+ - u^- \in \Lambda : u^+ \in \Lambda^-\},$$
$$\Lambda_2^- = \{u = u^+ - u^- \in \Lambda : -u^- \in \Lambda^-\},$$

where $u^+ = \max\{u, 0\}$ and $u^- = u^+ - u$. Define

$$\beta_1 = \inf_{u \in \Lambda_1^-} I(u),$$
$$\beta_2 = \inf_{u \in \Lambda_2^-} I(u).$$

Lemma 3.4. *Let $\|a\|_{\sigma_q} < \overline{M}$, then Λ_1^- and Λ_2^- are closed.*

Proof. Let $\{u_n\}$ be a sequence in Λ_1^- with $u_n \to u_0$. It then follows from $\{u_n\} \subset \Lambda_1^- \subset \Lambda$ that

$$\begin{aligned} \|u_0\|^2 &= \lim_{n\to\infty} \|u_n\|^2 \\ &= \lim_{n\to\infty} \Big[\int K(x)a(x)|u_n|^q + \int K(x)b(x)|u_n|^{2^*} \Big] \\ &= \int K(x)a(x)|u_0|^q + \int K(x)b(x)|u_0|^{2^*} \end{aligned}$$

and

$$\begin{aligned} &(2-q)\|u_0^+\|^2 - (2^* - q)\int K(x)b(x)|u_0^+|^{2^*} \\ &= \lim_{n\to\infty} \Big[(2-q)\|u_n^+\|^2 - (2^* - q)\int K(x)b(x)|u_n^+|^{2^*}\Big] \le 0, \end{aligned}$$

namely, $u_0 \in \Lambda$ and $u_0^+ \in \Lambda^- \cup \Lambda_0$.

Since there exists a positive d_1 such that $\|u^+\| \ge d_1 > 0$ for all $u \in \Lambda_1^-$, we know $u_0^+ \neq 0$. Combining this with Lemma 3.2, for $\|a\|_{\sigma_q} < \overline{M}$, we have $u_0^+ \notin \Lambda_0$. In turn, $u_0^+ \in \Lambda^-$ and hence, $u_0 \in \Lambda_1^-$. Thus, Λ_1^- is closed for $\|a\|_{\sigma_q} < \overline{M}$. The same argument can prove that Λ_2^- is closed. The proof of Lemma 3.4 is complete. □

Lemma 3.5. *(i) If $\beta_1 < c_1$, then the minimization problem* (3.2) *achieves its infimum at a point which defines a sign-changing critical point of I.*

(ii) If $\beta_2 < c_1$, then the same conclusion follows for the minimization problem (3.2)*.*

Proof. We only prove (i). Part (ii) of the lemma can be proved by a similar argument. By Lemma 3.4, we can use Ekeland variational principle to construct a minimizing sequence $\{u_n\} \subset \Lambda_1^-$ with the following properties:

(1) $I(u_n) \to \beta_1$,
(2) $I(z) \ge I(u_n) - \frac{1}{n}\|u_n - z\|$ for all $z \in \Lambda_1^-$.

Firstly, we claim that $\|u_n^-\| \ge d > 0$. Indeed, if to the contrary, there is a subsequence (still denoted by $\{u_n^-\}$) such that $\|u_n^-\| \to 0$, then

$$\beta_1 + o(1) = I(u_n) = I(u_n^+) + I(-u_n^-) \ge c_1 + o(1),$$

which is a contradiction with assumption $\beta_1 < c_1$. Secondly, we claim $I'(u_n) \to 0$ in H^{-1}. Indeed, set $0 < \rho < \rho_n \equiv \rho_{u_n}$, $g_n^\pm \equiv g_{u_n}^\pm$, where ρ_{u_n} and $g_{u_n}^\pm$ are given by

Lemma 3.3 so that for $v_\rho = \rho v$ with $\|v\| = 1$, there holds

$$z_\rho = g_n^+(v_\rho)(u_n - v_\rho)^+ - g_n^-(v_\rho)(u_n - v_\rho)^- \in \Lambda_1^-.$$

Consequently,

$$\begin{aligned} \frac{1}{n}\|z_\rho - u_n\| &\ge \langle I'(u_n), u_n - z_\rho\rangle + o(1)\|z_\rho - u_n\| \\ &= \langle I'(u_n), u_n - v_\rho - z_\rho\rangle + \rho\langle I'(u_n), v\rangle + o(1)\|z_\rho - u_n\| \\ &= (1 - g_n^+(v_\rho))\langle I'(u_n), (u_n - v_\rho)^+\rangle + \rho\langle I'(u_n), v\rangle \\ &\quad - (1 - g_n^-(v_\rho))\langle I'(u_n), (u_n - v_\rho)^-\rangle + o(1)\|z_\rho - u_n\|. \end{aligned} \tag{3.1}$$

It is trivial to show $\{u_n^+\}$ is bounded, and so we may assume that $u_n^+ \rightharpoonup w_0^+$ in H for some $w_0^+ \in H$. Since $\{u_n\} \subset \Lambda_1^-$, one has

$$(2^* - 2)\|u_n^+\|^2 - (2^* - q)\int K(x)a(x)|u_n^+|^q > 0.$$

This together with $\lim_{n\to\infty} \int K(x)a(x)|u_n^+|^q = \int K(x)a(x)|w_0^+|^q$ (see [10]) imply

$$(2^* - 2)\liminf_{n\to\infty} \|u_n^+\|^2 - (2^* - q)\int K(x)a(x)|w_0^+|^q \ge 0.$$

At this point, we show that for $\|a\|_{\sigma_q} < \overline{M}$,

$$(2^* - 2)\liminf_{n\to\infty} \|u_n^+\|^2 - (2^* - q)\int K(x)a(x)|w_0^+|^q > 0. \tag{3.2}$$

To prove that, we employ the method used in [16] and suppose to the contrary that

$$(2^* - 2)\liminf_{n\to\infty} \|u_n^+\|^2 = (2^* - q)\int K(x)a(x)|w_0^+|^q.$$

In view of (A1) and the fact $\|u_n^+\| \ge d > 0$, we have $\int K(x)a(x)|w_0^+|^q > 0$ and so

$$\liminf_{n\to\infty}\Big[\frac{(2^* - 2)\|u_n^+\|^2}{(2^* - q)\int K(x)a(x)|u_n^+|^q}\Big] = \frac{\liminf_{n\to\infty}\big[(2^* - 2)\|u_n^+\|^2\big]}{(2^* - q)\int K(x)a(x)|w_0^+|^q} = 1 \tag{3.3}$$

Notice that

$$\frac{(2^* - 2)\|u_n^+\|^2}{(2^* - q)\int K(x)a(x)|u_n^+|^q} > 1, \tag{3.4}$$

for $n = 1, 2, \dots$. Combining with (3.3) and (3.4), we obtain that there exists a subsequence $\{u_{n_k}^+\}$ of $\{u_n^+\}$ such that

$$\frac{(2^* - 2)\|u_{n_k}^+\|^2}{(2^* - q)\int K(x)a(x)|u_{n_k}^+|^q} \to 1$$

as $k \to \infty$. Hence,

$$\|u_{n_k}^+\|^2 \to \frac{2^* - q}{2^* - 2}\int K(x)a(x)|w_0^+|^q,$$

$$\int K(x)b(x)|u_{n_k}^+|^{2^*} \to \frac{2 - q}{2^* - 2}\int K(x)a(x)|w_0^+|^q$$

and so we have that for $\|a\|_{\sigma_q} < \overline{M}$,

$$0 < \Big[\Big(\frac{2 - q}{2^* - q}\Big)\Big(\frac{2^* - 2}{2^* - q}\Big)^{\frac{2^*-2}{2-q}} \|a\|_{\sigma_q}^{-\frac{2^*-2}{2-q}} S_{q\sigma_q'}^{\frac{q}{2}\frac{2^*-2}{2-q}} - |b|_\infty S^{-\frac{2^*}{2}}\Big]\int K(x)|u_{n_k}^+|^{2^*}$$

$$
\begin{aligned}
&\le \frac{2-q}{2^*-q}\Big(\frac{2^*-2}{2^*-q}\Big)^{\frac{2^*-2}{2-q}} \frac{\|u_{n_k}^+\|^{\frac{2(2^*-q)}{2-q}}}{\big(\int K(x)a(x)|u_{n_k}^+|^q\big)^{\frac{2^*-2}{2-q}}} - \int K(x)b(x)|u_{n_k}^+|^{2^*} \\
&\to \frac{2-q}{2^*-q}\Big(\frac{2^*-2}{2^*-q}\Big)^{\frac{2^*-2}{2-q}} \frac{\big[\frac{2^*-q}{2^*-2}\int K(x)a(x)|w_0^+|^q\big]^{\frac{2^*-q}{2-q}}}{\big(\int K(x)a(x)|w_0^+|^q\big)^{\frac{2^*-2}{2-q}}} \\
&\quad - \frac{2-q}{2^*-2}\int K(x)a(x)|w_0^+|^q = 0,
\end{aligned}
$$

namely, $u_{n_k}^+ \to 0$ in $L_K^{2^*}(\mathbb{R}^N)$, and consequently $w_0^+ \equiv 0$, which leads to a contradiction. Thus, (3.2) follows. From (3.2) we can further obtain that there is a suitable positive constant d for n large enough

$$
(2^*-2)\|u_n^+\|^2 - (2^*-q)\int K(x)a(x)|u_n^+|^q \ge d > 0.
$$

Therefore, by Lemma 3.3 and the boundness of $\{u_n^+\}$, we conclude that $\|(g_n^+)'(0)\| \le d_1$. Since $0 < d_2 \le \|u_n^-\| \le d_3$, a similar argument can show $\|(g_n^-)'(0)\| \le d_4$. For fixed n, since

$$
\begin{gathered}
(1-g_n^+(v_\rho)) = \rho\langle (g_n^+)'(0), v\rangle, \\
(1-g_n^-(v_\rho)) = \rho\langle (g_n^-)'(0), v\rangle, \\
\|z_\rho - u_n\| \le \rho + d(|1-g_n^+(v_\rho)| + |1-g_n^-(v_\rho)|),
\end{gathered}
$$

$\langle I'(u_n), u_n^\pm\rangle = 0$ and $(u_n - v_\rho)^\pm \to u_n^\pm$ as $\rho \to 0$, letting $\rho \to 0$ in (3.1) we obtain

$$
\langle I'(u_n), v\rangle \le \frac{d}{n}.
$$

From the above discussion, we can conclude that $I'(u_n) \to 0$ in H^{-1} as $n \to \infty$. By applying [8, Proposition 3.2], we obtain that the sequence $\{u_n\}$ indeed satisfies the following

(i) $I(u_n) \to \beta_1 < c_1 < c_0 + \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2}$,
(ii) $I'(u_n) \to 0$ in H^{-1}.

Then, we may use (i), (ii) and [8, Lemma 3.1] to guarantee a convergent subsequence for $\{u_n\}$ whose strong limit will give the desired minimizer. $\square$

Clearly, Lemma 3.5 would give the conclusion for Theorem 1.1 only if the given relations $\beta_1 < c_1$ or $\beta_2 < c_1$ could be established. While it is not sure whether or not such inequalities should hold, we shall use these values to compare with another minimization problem. Namely set

$$
\Lambda_*^- = \Lambda_1^- \cap \Lambda_2^- \subset \Lambda^-
$$

and then define

$$
c_2 = \inf_{u \in \Lambda_*^-} I(u). \tag{3.5}
$$

It is easy to see that $c_2 \ge c_1$. Since I satisfies (PS) condition only locally, we need the following upper bound for c_2.

Lemma 3.6. *(i) For any fixed $\varepsilon > 0$, then there are $s > 0$ and $t \in \mathbb{R}$ such that $su_1 - tU_\varepsilon \in \Lambda_*^-$.*

(ii) For $\varepsilon > 0$ sufficiently small , if $N \geq 7$, $(3N-2)/(2N-4) < q < 2$ and $\alpha > (N-2)/2$, then we have

$$c_2 \leq \sup_{s\geq 0, t\in\mathbb{R}} I(su_1 - tU_\varepsilon) < c_1 + \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2}.$$

Proof. (i) It suffices to show that there are $s > 0$ and $t \in \mathbb{R}$ so that

$$s(u_1 - tU_\varepsilon)^+ \in \Lambda^- \quad \text{and} \quad -s(u_1 - tU_\varepsilon)^- \in \Lambda^-.$$

To prove that, we set

$$t_2 = \max_{\mathbb{R}^N} \frac{u_1}{U_\varepsilon} \quad \text{and} \quad t_1 = \min_{\mathbb{R}^N} \frac{u_1}{U_\varepsilon}.$$

For each $t \in (t_1, t_2)$, we denote by $s^+(t)$ and $s^-(t)$ the positive values given by Lemma 3.1. Then one has

$$s^+(t)(u_1 - tU_\varepsilon)^+ \in \Lambda^- \quad \text{and} \quad -s^-(t)(u_1 - tU_\varepsilon)^- \in \Lambda^-.$$

Notice that $s^+(t)$ is continuous with respect to t satisfying

$$\lim_{t\to t_1^+} s^+(t) = t^+(u_1 - t_1U_\varepsilon) < +\infty \quad \text{and} \quad \lim_{t\to t_2^-} s^+(t) = +\infty.$$

Moreover, $s^-(t)$ is also continuous with respect to t and

$$\lim_{t\to t_1^+} s^-(t) = +\infty \quad \text{and} \quad \lim_{t\to t_2^-} s^-(t) = t^+(t_2U_\varepsilon - u_1) < +\infty.$$

By the continuity of $s^\pm(t)$, we conclude that there exists $t_0 \in (t_1, t_2)$ such that $s^+(t_0) = s^-(t_0) = s_0 > 0$. This proves (i) with $t = t_0$ and $s = s_0$.

(ii). Obviously, it suffices to estimate $I(su_1 - tU_\varepsilon)$ for $s \geq 0$ and $t \in \mathbb{R}$. Since ε can be now sufficiently small, we let $U_\varepsilon = v_\varepsilon$. From the structure of I, we can take $R_1 > 0$ possible large such that $I(su_1 - tv_\varepsilon) \leq c_1$ for all $s^2 + t^2 \geq R_1^2$. Hence, we only need to estimate $I(su_1 - tv_\varepsilon)$ for all $s^2 + t^2 \leq R_1^2$. It follows from Lemma 2.3 and the elementary inequality

$$|s+t|^m \geq |s|^m + |t|^m - d(|s|^{m-1}|t| + |s||t|^{m-1}), \quad \text{for any } s,\, t \in \mathbb{R},\, m > 1$$

that

$$\begin{aligned}
&I(su_1 - tv_\varepsilon)\\
&\leq I(su_1) + I(tv_\varepsilon) - st\int K(x)a(x)|u_1|^{q-2}u_1v_\varepsilon - st\int K(x)b(x)|u_1|^{2^*-2}u_1v_\varepsilon\\
&\quad + d\Big(\int K(x)b(x)|su_1|^{2^*-1}|tv_\varepsilon| + \int K(x)b(x)|su_1||tv_\varepsilon|^{2^*-1}\Big)\\
&\quad + d\Big(\int K(x)a(x)|su_1|^{q-1}|tv_\varepsilon| + \int K(x)a(x)|su_1||tv_\varepsilon|^{q-1}\Big)\\
&\leq I(su_1) + I(tv_\varepsilon) + O\big(\varepsilon^{\frac{(N-2)(q-1)}{4}}\big) + O\big(\varepsilon^{\frac{N-2}{4}}\big).
\end{aligned}$$

Since $\int K(x)v_\varepsilon^{2^*} = 1$, we have

$$\begin{aligned}
I(tv_\varepsilon) &= \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\int K(x)b(x)v_\varepsilon^{2^*} - \frac{t^q}{q}\int K(x)a(x)v_\varepsilon^q\\
&= \Big(\frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}|b|_\infty\Big) + \frac{t^{2^*}}{2^*}\int K(x)(|b|_\infty - b(x))v_\varepsilon^{2^*}
\end{aligned}$$

$$- \frac{t^q}{q} \int K(x)a(x)v_\varepsilon^q.$$

For any $\varepsilon > 0$, it is easy to verify that the function $t \to I(tv_\varepsilon)$ attains its maximum at a point $t_\varepsilon > 0$. Moreover, applying the arguments similar to that of [8, Proposition 3.2] and [7, Lemma 4.1], we can conclude that there are two positive constants d_1 and d_2 such that $0 < d_1 \le t_\varepsilon \le d_2$, independent of ε.

Let $h(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}|b|_\infty$. Clearly, $h(t)$ achieves its maximum at the point $t_* = (\|v_\varepsilon\|^2/|b|_\infty)^{(N-2)/4}$. In conclusion, we can deduce from $\int K(x)(|b|_\infty - b(x))v_\varepsilon^{2^*} = O(\varepsilon^{N/2})$ (see [10]), $\|v_\varepsilon\|^N \le S^{N/2} + O(\varepsilon^{\alpha/2}) + O(\varepsilon^{(N-2)/2})$ (see [9, 10]) and (2.8) that

$$\begin{aligned}
\max_{t>0} I(tv_\varepsilon) &\le h(t_\varepsilon) + \frac{(t_\varepsilon)^{2^*}}{2^*}\int K(x)(|b|_\infty - b(x))v_\varepsilon^{2^*} - \frac{(t_\varepsilon)^q}{q}\int K(x)a(x)v_\varepsilon^q \\
&\le h(t_*) + \frac{d_2^{2^*}}{2^*}\int K(x)(|b|_\infty - b(x))v_\varepsilon^{2^*} - \frac{d_1^q}{q}\int K(x)a(x)v_\varepsilon^q \\
&\le \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}}S^{N/2} + O(\varepsilon^{\alpha/2}) + O(\varepsilon^{(N-2)/2}) + O(\varepsilon^{N/2}) \\
&\quad - d\varepsilon^{\frac{N}{2}-\frac{N-2}{4}q} + O\big(\varepsilon^{\frac{(N-2)q}{4}}\big).
\end{aligned}$$

Furthermore, we can obtain that for $\varepsilon > 0$ small enough,

$$\begin{aligned}
&\max_{s>0,\, t\in\mathbb{R}} I(su_1 - tU_\varepsilon) \\
&\le \max_{s>0} I(su_1) + \max_{t\in\mathbb{R}} I(tv_\varepsilon) + O\big(\varepsilon^{\frac{(N-2)(q-1)}{4}}\big) + O\big(\varepsilon^{\frac{N-2}{4}}\big) \\
&\le c_1 + \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}}S^{N/2} + O(\varepsilon^{\alpha/2}) + O\big(\varepsilon^{(N-2)/2}\big) + O(\varepsilon^{N/2}) \\
&\quad - d\varepsilon^{\frac{N}{2}-\frac{N-2}{4}q} + O\big(\varepsilon^{\frac{(N-2)q}{4}}\big) + O\big(\varepsilon^{\frac{(N-2)(q-1)}{4}}\big) + O\big(\varepsilon^{\frac{N-2}{4}}\big) \\
&< c_1 + \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}}S^{N/2},
\end{aligned}$$

if $N \ge 7$, $\frac{3N-2}{2N-4} < q < 2$ and $\alpha > (N-2)/2$. This completes the proof. ☐

Lemma 3.7. *Assume $\beta_1 \ge c_1$ and $\beta_2 \ge c_1$. The minimization problem*

$$c_2 = \inf_{u\in\Lambda_*^-} I(u) \tag{3.6}$$

achieves its infimum at $u_2 \in \Lambda_*^-$ which defines a sign-changing critical point for I, provided $\|a\|_{\sigma_q} < M_2$ with some $M_2 > 0$.

Proof. Set $M_2 = \min\{M_1, \overline{M}\}$. As in the proof of Lemma 3.5, we can construct a minimizing sequence $\{u_n\} \subset \Lambda_*^-$ for (3.6) such that $I(u_n) \to c_2$ and $I'(u_n) \to 0$. Noting that $\{u_n\} \subset \Lambda_*^-$, we have

$$0 < d_1 \le \|u_n^\pm\| \le d_2 \tag{3.7}$$

for some positive constants d_1 and d_2. Thus, we may assume that $u_n^\pm \rightharpoonup u_2^\pm$ in H.

Claim. $u_2^\pm \ne 0$. Suppose to the contrary, we assume first that $u_2^+ = 0$, then we infer from $u_n^+ \in \Lambda^- \subset \Lambda$ and $\lim_{n\to\infty}\int K(x)a(x)|u_n^+|^q = \int K(x)a(x)|w_0^+|^q$ that

$$\|u_n\|^2 - \int K(x)b(x)|u_n^+|^{2^*} = o(1).$$

Combining this with (2.1) and (3.7), we can obtain that for n large enough

$$\int K(x)b(x)|u_n^+|^{2^*} \geq \frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2} + o(1),$$

and so

$$I(u_n^+) = \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2^*}\int K(x)b(x)|u_n^+|^{2^*} + o(1) \geq \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2} + o(1). \quad (3.8)$$

On the other hand, from the upper bound of c_2 and $I(-u_n^-) \geq c_1$, we have

$$I(u_n^+) \leq c_2 - c_1 + o(1) < \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2},$$

which is a contradiction to (3.8). Hence, $u_2^+ \neq 0$. Similarly, we can prove that $u_2^- \neq 0$.

Let $u_2 = u_2^+ - u_2^-$. Obviously, u_2 is sign-changing and $u_n \rightharpoonup u_2$ in H. Since for any $\phi \in H$ there holds $\langle I'(u_2), \phi\rangle = 0$, u_2 is a weak solution of (1.1). Now, to complete the proof of Theorem 1.1, we only need to show that $u_n \to u_2$ in H. Define $u_n^+ = u_2^+ + v_n^+$ and $u_n^- = u_2^- + v_n^-$, then we have $v_n^\pm \rightharpoonup 0$ in H. Combining this with $u_n^\pm \in \Lambda$ and $\langle I'(u_2^+), u_2^+\rangle = \langle I'(u_2^-), u_2^-\rangle = 0$, we can use the Brezis-Lieb Lemma [19] to obtain

$$\|v_n^\pm\|^2 - \int K(x)b(x)|v_n^\pm|^{2^*} = o(1). \quad (3.9)$$

Because the fact $c_1 < c_0 + \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2}$, it follows from Lemma 3.6 that

$$\begin{aligned}
\lim_{n\to\infty}(I(v_n^+) + I(-v_n^-)) &= \lim_{n\to\infty} I(u_n) - I(u_2) \leq c_2 - c_0 \\
&< \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2} + c_1 - c_0 \\
&< \frac{2}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2}.
\end{aligned}$$

Therefore, we must have

$$\lim_{n\to\infty} \min\{I(v_n^+),\, I(-v_n^-)\} < \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2}.$$

This and (3.9) imply

$$\|v_n^+\| \to 0 \quad \text{or} \quad \|v_n^-\| \to 0,$$

that is, $u_2 = u_2^+ - u_2^- \in \Lambda_1^-$ or $u_2 = u_2^+ - u_2^- \in \Lambda_2^-$. Thus, under the assumption $\beta_1 \geq c_1$ and $\beta_2 \geq c_1$, we get $I(u_2) \geq c_1$. Hence, if writing $u_n = u_2 + w_n$, we have $w_n \rightharpoonup 0$ in H. According to Brezis-Lieb Lemma, one has

$$I(u_n) = I(u_2 + w_n) = I(u_2) + o(1) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2^*}\int K(x)b(x)|w_n|^{2^*}. \quad (3.10)$$

Since u_2 is a weak solution of (1.1), it follows from $u_n \in \Lambda$ that

$$\|w_n\|^2 - \int K(x)b(x)|w_n|^{2^*} = o(1). \quad (3.11)$$

Now assume

$$\|w_n\|^2 \to l \geq 0, \quad \int K(x)b(x)|w_n|^{2^*} \to l \geq 0.$$

If $l \neq 0$, then (2.1) and (3.11) yield that $l \geq \frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2}$. Using (3.10), $I(u_2) \geq c_1$ and Lemma 3.6, we obtain that

$$c_1 + o(1) + \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2} \leq I(u_n) = c_2 + o(1) < c_1 + \frac{1}{N}\frac{1}{|b|_\infty^{(N-2)/2}} S^{N/2}$$

which is a contradiction. Therefore, $l = 0$, that is, $u_n \to u_2$ in H which defines a sign-changing solution of (1.1). □

The proof of Theorem 1.1 follows from Lemmas 3.5 and 3.7 and the symmetry of the functional I.

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