Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 232, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LOCAL WELL-POSEDNESS FOR AN ERICKSEN-LESLIE'S PARABOLIC-HYPERBOLIC COMPRESSIBLE NON-ISOTHERMAL MODEL FOR LIQUID CRYSTALS

JISHAN FAN, TOHRU OZAWA

Communicated by Mitsuharo Otani

Abstract. In this article we prove the local well-posedness for an Ericksen-Leslie's parabolic-hyperbolic compressible non-isothermal model for nematic liquid crystals with positive initial density.

1. Introduction

We consider the following Ericksen-Leslie system modeling the hydrodynamic flow of compressible nematic liquid crystals [1, 2, 3, 4, 5]:

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\begin{aligned}&\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho,\theta) - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u \\ &= -\nabla\cdot\Big(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\Big),\end{aligned} \tag{1.2}$$

$$\partial_t(\rho e) + \operatorname{div}(\rho u e) + p\operatorname{div} u - \Delta\theta = \frac{\mu}{2}|\nabla u + \nabla u^t|^2 + \lambda(\operatorname{div} u)^2 + |\dot d|^2, \tag{1.3}$$

$$\ddot d - \Delta d = d(|\nabla d|^2 - |\dot d|^2),\ |d| = 1, \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.4}$$

$$(\rho,u,\theta,d,d_t)(\cdot,0) = (\rho_0,u_0,\theta_0,d_0,d_1) \quad \text{in } \mathbb{R}^3,\ |d_0| = 1,\ d_0\cdot d_1 = 0. \tag{1.5}$$

Here ρ, u, θ is the density, velocity and temperature of the fluid, and d represents the macroscopic average of the nematic liquid crystals orientation field. $e := C_V\theta$ is the internal energy and $p := R\rho\theta$ is the pressure with positive constants C_V and R. The viscosity coefficients μ and λ of the fluid satisfy $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$. The symbol $\nabla d\odot\nabla d$ denotes a matrix whose (i,j)th entry is $\partial_i d\partial_j d$, $\mathbb{I}_3$ is the identity matrix of order 3, and it is easy to see that

$$\operatorname{div}\Big(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\Big) = -\sum_k \nabla d_k\Delta d_k, \dot d := d_t + u\cdot\nabla d.$$

u^t is the transpose of vector u and $\partial_t u \equiv u_t$.

System (1.1)-(1.3) is the well-known full compressible Navier-Stokes-Fourier system. When $u = 0$, (1.4) reduces to the wave maps system, which is one of the most beautiful and challenging nonlinear hyperbolic system. It has captured the

2010 *Mathematics Subject Classification.* 35Q30, 35Q35, 76N10.
Key words and phrases. Compressible; liquid crystals; local well-posedness.

Submitted February 23, 2017. Published September 25, 2017.

attention of mathematicians for more than thirty years now. Moreover, the wave maps system is nothing other than the Euler-Lagrange system for the nonlinear sigma model, which is one of the fundamental problems in classical field theory.

When θ is a positive constant and the equation (1.4) is replaced by a harmonic heat flow

$$\dot{d} - \Delta d = d|\nabla d|^2, \tag{1.6}$$

this problem has received many studies. Huang, Wang and Wen [6, 7] (see also [8, 9]) show the local well-posedness of strong solutions with vacuum and prove some regularity criteria. Ding, Huang, Wen and Zi [10] (also see [11, 12]) studied the low Mach number limit. Jiang, Jiang and Wang [13] (see also [14]) proved the global existence of weak solutions in $\mathbb{R}^2$.

When the fluid is incompressible, i.e., $\operatorname{div} u = 0$, the similar model has been studied in [15, 16].

The aim of this article is to prove a local-well posedness result when $\inf \rho_0 \geq 1/C$, we will prove the following result.

Theorem 1.1. *Let* $1/C \leq \rho_0 \leq C$, $0 \leq \theta_0$, $\nabla\rho_0 \in H^2$, $u_0, \theta_0, \dot{d}_0, \nabla d_0 \in H^3$, *with* $|d_0| = 1, d_0 \cdot d_1 = 0$. *Then problem* (1.1)-(1.5) *has a unique strong solution* (ρ, u, θ, d) *satisfying*

$$\begin{gathered}
\frac{1}{C} \leq \rho \leq C, \quad 0 \leq \theta, \quad |d| = 1, \\
\nabla\rho \in L^\infty(0,T;H^2), \quad u, \theta, \dot{d}, \nabla d \in L^\infty(0,T;H^3), \\
u, \theta \in L^2(0,T;H^4), \quad u_t, \theta_t \in L^2(0,T;H^2)
\end{gathered} \tag{1.7}$$

for some $T > 0$.

Remark 1.2. When $n = 2$ and taking $d := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$, System (1.1)-(1.4) reduces to

$$\begin{gathered}
\partial_t \rho + \operatorname{div}(\rho u) = 0, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho, \theta) - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u \\
= -\nabla \cdot \Big(\nabla\phi \otimes \nabla\phi - \frac{1}{2}|\nabla\phi|^2 \mathbb{I}_2\Big), \\
\partial_t(\rho e) + \operatorname{div}(\rho u e) + p \operatorname{div} u - \Delta\theta = \frac{\mu}{2}|\nabla u + \nabla u^t|^2 + \lambda(\operatorname{div} u)^2 + |\dot{\phi}|^2, \\
\ddot{\phi} - \Delta\phi = 0.
\end{gathered}$$

And hence the well-known wave map

$$d_{tt} - \Delta d = d(|\nabla d|^2 - |d_t|^2)$$

reduces to the wave equation $\phi_{tt} - \Delta\phi = 0$.

Remark 1.3. Let d be a smooth solution to the system (1.4) with the initial data $(d, d_t)(\cdot, 0) = (d_0, d_1)$, if the initial data (d_0, d_1) obeys the conditions

$$|d_0| = 1, \quad d_0 \cdot d_1 = 0,$$

then we have $|d| = 1$ and $d \cdot d_t = 0$ for all times t.

Proof of Remark 1.3. Denote $w := |d|^2 - 1$, multiplying (1.4) by d, we see that

$$\ddot{w} - \Delta w = 2w(|\nabla d|^2 - |\dot{d}|^2).$$

Testing the above equation by $\dot{w}$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\dot{w}^2+|\nabla w|^2)dx\\
&=2\int w\dot{w}(|\nabla d|^2-|\dot{d}|^2)\,dx+\int\Delta w(u\cdot\nabla w)\,dx-\int(u\cdot\nabla)\dot{w}\cdot\dot{w}\,dx\\
&=2\int w\dot{w}(|\nabla d|^2-|\dot{d}|^2)\,dx-\sum_i\int\partial_j u_i\partial_i w\partial_j w\,dx+\frac{1}{2}\int|\nabla w|^2\operatorname{div}u\,dx\\
&\quad+\frac{1}{2}\int\dot{w}^2\operatorname{div}u\,dx\\
&\le C\int(w^2+\dot{w}^2+|\nabla w|^2)\,dx.
\end{aligned}$$

On the other hand, we observe that

$$\frac{1}{2}\frac{d}{dt}\int w^2\,dx=\int w(\dot{w}-u\cdot\nabla w)\,dx\le C\int(w^2+\dot{w}^2+|\nabla w|^2)\,dx.$$

Combining the above two estimates and using the Gronwall inequality, we finish the proof. □

We denote

$$\begin{aligned}
M(t):=&1+\sup_{0\le s\le t}\Big\{\|\frac{1}{\rho}(\cdot,s)\|_{L^\infty}+\|\rho(\cdot,s)\|_{L^\infty}+\|\nabla\rho(\cdot,s)\|_{H^2}+\|u(\cdot,s)\|_{H^3}\\
&+\|\theta(\cdot,s)\|_{H^3}+\|\dot{d}(\cdot,s)\|_{H^3}+\|\nabla d(\cdot,s)\|_{H^3}\Big\}\\
&+\|u\|_{L^2(0,t;H^4)}+\|u_t\|_{L^2(0,t;H^2)}+\|\theta\|_{L^2(0,t;H^4)}+\|\theta_t\|_{L^2(0,t;H^2)}.
\end{aligned}\tag{1.8}$$

Theorem 1.4. *Let T^* be the maximal time of existence for problem* (1.1)-(1.5) *in the sense of Theorem 1.1. Then for any $t\in[0,T^*)$, we have that*

$$M(t)\le C_0M(0)\exp(\sqrt{t}C(M(t)))\tag{1.9}$$

for some given nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

It follows from (1.9) [17, 18, 19] that

$$\sup_{0\le t\le T}M(t)\le C\tag{1.10}$$

for some $T\in(0,T^*)$.

In the proofs below, we will use the following bilinear commutator and product estimates due to Kato-Ponce [20]:

$$\|D^s(fg)-fD^sg\|_{L^p}\le C(\|\nabla f\|_{L^{p_1}}\|D^{s-1}g\|_{L^{q_1}}+\|D^sf\|_{L^{p_2}}\|g\|_{L^{q_2}}),\tag{1.11}$$

$$\|D^s(fg)\|_{L^p}\le C(\|f\|_{L^{p_1}}\|D^sg\|_{L^{q_1}}+\|D^sf\|_{L^{p_2}}\|g\|_{L^{q_2}})\tag{1.12}$$

with $s>0$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}$ and $1<p<\infty$.

The proof of the uniqueness part is standard, we omit it here.

It is easy to prove Theorem 1.1 by the Galerkin method if we have (1.9) [6], thus we only need to show a priori estimates (1.9).

2. Proof of Theorem 1.4

Since the physical constants C_V and R do not bring any essential difficulties in our arguments, we shall take $C_V = R = 1$. First, it follows from (1.1) that

$$\rho(x,t) = \rho_0(y(0;x,t)) \exp\Big\{ -\int_0^t \operatorname{div} u(y(s;x,t),s)ds \Big\}, \tag{2.1}$$

where $y(s;x,t)$ is the characteristic curve defined by

$$\frac{dy}{ds} = u(y,s), \quad y(t;x,t) = x.$$

Then (2.1) gives

$$\rho, \frac{1}{\rho} \le C_0 \exp(tC(M)). \tag{2.2}$$

Applying ∇ to (1.1), testing by $\nabla\rho$, we see that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla\rho|^2\,dx = -\int \nabla \operatorname{div}(\rho u)\nabla\rho\,dx \le C(M),$$

which yields

$$\|\nabla\rho(\cdot,t)\|_{L^2} \le C_0 + tC(M). \tag{2.3}$$

Applying D^3 to (1.1), testing by $D^3\rho$, using (1.11) and (1.12), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (D^3\rho)^2\,dx \\
&= -\int (D^3(u\nabla\rho) - u\cdot\nabla D^3\rho)D^3\rho\,dx - \int u\cdot\nabla D^3\rho\cdot D^3\rho\,dx \\
&\quad - \int D^3(\rho \operatorname{div} u)D^3\rho\,dx \\
&\le C(\|\nabla u\|_{L^\infty}\|D^3\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|D^3u\|_{L^2})\|D^3\rho\|_{L^2} \\
&\quad + C(\|\rho\|_{L^\infty}\|D^3 \operatorname{div} u\|_{L^2} + \|\operatorname{div} u\|_{L^\infty}\|D^3\rho\|_{L^2})\|D^3\rho\|_{L^2} \\
&\le C(M) + C(M)\|D^3 \operatorname{div} u\|_{L^2},
\end{aligned}$$

which leads to

$$\|D^3\rho(\cdot,t)\|_{L^2} \le C + \sqrt{t}C(M). \tag{2.4}$$

It is easy to show that

$$\|u(\cdot,t)\|_{H^2} = \|u_0 + \int_0^t u_t ds\|_{H^2} \le C_0 + \sqrt{t}C(M), \tag{2.5}$$

$$\|\theta(\cdot,t)\|_{H^2} \le C_0 + \sqrt{t}C(M). \tag{2.6}$$

Testing (1.4) by $\dot d$ and using $d\cdot\dot d = 0$, we infer that

$$\frac{1}{2}\frac{d}{dt}\int (|\dot d|^2 + |\nabla d|^2)dx = \int u\cdot\nabla d\cdot\Delta d\,dx - \int (u\cdot\nabla)\dot d\cdot\dot d\,dx \le C(M),$$

which implies

$$\|\dot d(\cdot,t)\|_{L^2} + \|\nabla d(\cdot,t)\|_{L^2} \le C_0 + tC(M). \tag{2.7}$$

Taking D^3 to (1.2), testing by D^3u and using (1.1), we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|D^3u|^2\,dx + \mu\int |\nabla D^3u|^2\,dx + (\lambda+\mu)\int (\operatorname{div} D^3u)^2\,dx\\
&= \int D^3p\cdot \operatorname{div} D^3u\,dx - \int (D^3(\rho u\cdot\nabla u) - \rho u\cdot\nabla D^3u)D^3u\,dx\\
&\quad - \int (D^3(\rho u_t) - \rho D^3u_t)D^3u\,dx - \int (D^3(\nabla d\cdot\Delta d) - \nabla d\cdot\Delta D^3 d)D^3u\,dx\\
&\quad - \int (D^3u\cdot\nabla)d\cdot\Delta D^3d\,dx\\
&=: I_1 + I_2 + I_3 + I_4 - I_5.
\end{aligned}
\tag{2.8}
$$

Applying D^3 to (1.4) and testing by $D^3\dot d$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (|D^3\dot d|^2 + |\nabla D^3 d|^2)\,dx\\
&= -\int (D^3(u\cdot\nabla\dot d) - u\cdot\nabla D^3\dot d)D^3\dot d\,dx - \int (u\cdot\nabla)D^3\dot d\cdot D^3\dot d\,dx\\
&\quad + \int D^3(d(|\nabla d|^2 - |\dot d|^2))D^3\dot d\,dx + \int \Delta D^3 d\cdot(u\cdot\nabla D^3 d)dx\\
&\quad + \int \Delta D^3 d\cdot(D^3(u\cdot\nabla d) - u\cdot\nabla D^3 d - (D^3u\cdot\nabla)d)\,dx + I_5\\
&=: \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + I_5.
\end{aligned}
\tag{2.9}
$$

Summing (2.8) and (2.9), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (\rho|D^3u|^2 + |D^3\dot d|^2 + |\nabla D^3 d|^2)\,dx\\
&+ \mu\int |\nabla D^3u|^2\,dx + (\lambda+\mu)\int (\operatorname{div} D^3u)^2\,dx\\
&= \sum_{i=1}^4 (I_i + \ell_i) + \ell_5.
\end{aligned}
\tag{2.10}
$$

Using (1.11) and (1.12), we bound I_i $(i = 1,\cdots,4)$ and ℓ_i $(i = 1,\cdots,5)$ as follows.

$$
\begin{gathered}
I_1 \le C(\|\rho\|_{L^\infty}\|D^3\theta\|_{L^2} + \|\theta\|_{L^\infty}\|D^3\rho\|_{L^2})\|\operatorname{div} D^3u\|_{L^2} \le C(M)\|\operatorname{div} D^3u\|_{L^2};\\
I_2 \le C(\|\nabla(\rho u)\|_{L^\infty}\|D^3u\|_{L^2} + \|\nabla u\|_{L^\infty}\|D^3(\rho u)\|_{L^2})\|D^3u\|_{L^2} \le C(M);\\
I_3 \le C(\|\nabla\rho\|_{L^\infty}\|D^2u_t\|_{L^2} + \|u_t\|_{L^\infty}\|D^3\rho\|_{L^2})\|D^3u\|_{L^2} \le C(M)\|u_t\|_{H^2};\\
I_4 \le C\|\nabla^2 d\|_{L^\infty}\|D^4 d\|_{L^2}\|D^3u\|_{L^2} \le C(M);\\
\ell_1 \le C\|\nabla u\|_{L^\infty}\|D^3\dot d\|_{L^2}^2 + C\|\nabla\dot d\|_{L^\infty}\|D^3u\|_{L^2}\|D^3\dot d\|_{L^2} \le C(M);\\
\ell_2 = \frac{1}{2}\int |D^3\dot d|^2\operatorname{div} u\,dx \le C(M);
\end{gathered}
$$

$$
\begin{aligned}
\ell_3 &\le C[\|d\|_{L^\infty}\|D^3(|\nabla d|^2 - |\dot d|^2)\|_{L^2} + (\|\nabla d\|_{L^\infty}^2 + \|\dot d\|_{L^\infty}^2)\|D^3 d\|_{L^2}]\|D^3\dot d\|_{L^2}\\
&\le C(M);
\end{aligned}
$$

$$\begin{aligned}
\ell_4 &= \sum_{i,j}\int u_i\partial_i D^3 d\partial_j^2 D^3 d\,dx\\
&= -\sum_{i,j}\int \partial_j u_i\partial_i D^3 d\partial_j D^3 d\,dx + \sum_{i,j}\frac{1}{2}\int \partial_i u_i(\partial_j D^3 d)^2\,dx\\
&\le C\|\nabla u\|_{L^\infty}\|D^4 d\|_{L^2}^2 \le C(M);
\end{aligned}$$

$$\begin{aligned}
\ell_5 &= \int \Delta D^3 d(C_1 D^2 u\cdot\nabla Dd + C_2 Du\cdot\nabla D^2 d)dx\\
&= -\sum_i\int \partial_i D^3 d\partial_i(C_1 D^2 u\nabla Dd + C_2 Du\cdot\nabla D^2 d)dx\\
&\le C\|D^4 d\|_{L^2}(\|D^3 u\|_{L^2}\|\nabla^2 d\|_{L^\infty} + \|D^2 u\|_{L^3}\|D^3 d|_{L^6} + \|\nabla u\|_{L^\infty}\|D^4 d\|_{L^2})\\
&\le C(M).
\end{aligned}$$

Inserting the above estimates into (2.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\rho|D^3 u|^2 + |D^3\dot d|^2 + |\nabla D^3 d|^2)dx\\
&+\mu\int|\nabla D^3 u|^2\,dx + (\lambda+\mu)\int(\operatorname{div} D^3 u)^2\,dx\\
&\le C(M)\|\operatorname{div} D^3 u\|_{L^2} + C(M) + C(M)\|u_t\|_{H^2}.
\end{aligned}\tag{2.11}$$

Integrating the above estimates in $[0,t]$, we arrive at

$$\begin{aligned}
&\|D^3 u(\cdot,t)\|_{L^2}^2 + \|D^3\dot d(\cdot,t)\|_{L^2}^2 + \|\nabla D^3 d(\cdot,t)\|_{L^2}^2 + \int_0^t\int|D^4 u|^2\,dxds\\
&\le C_0\exp(\sqrt{t}C(M)).
\end{aligned}\tag{2.12}$$

On the other hand, it follows from (1.2) that

$$u_t = -u\cdot\nabla u + \frac{1}{\rho}\Big[\mu\Delta u + (\lambda+\mu)\nabla\operatorname{div} u - \nabla p - \nabla\cdot(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3)\Big]$$

which easily implies

$$\|u_t\|_{L^2(0,t;H^2)} \le C_0\exp(\sqrt{t}C(M)). \tag{2.13}$$

Applying D^3 to (1.3), testing by $D^3\theta$ and using (1.1), (1.11), and (1.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho(D^3\theta)^2 dx + \int|\nabla D^3\theta|^2\,dx\\
&= -\int(D^3(\rho u\cdot\nabla\theta) - \rho u\cdot\nabla D^3\theta)D^3\theta\,dx - \int D^3(p\operatorname{div} u)\cdot D^3\theta\,dx\\
&\quad - \int(D^3(\rho\theta_t) - \rho D^3\theta_t)D^3\theta\,dx\\
&\quad + \int D^3\big[\frac{\mu}{2}|\nabla u + \nabla u^t|^2 + \lambda(\operatorname{div} u)^2 + |\dot d|^2\big]D^3\theta\,dx\\
&\le C(\|\nabla(\rho u)\|_{L^\infty}\|D^3\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|D^3(\rho u)\|_{L^2})\|D^3\theta\|_{L^2}\\
&\quad + C(\|p\|_{L^\infty}\|D^3\operatorname{div} u\|_{L^2} + \|\operatorname{div} u\|_{L^\infty}\|D^3 p\|_{L^2})\|D^3\theta\|_{L^2}\\
&\quad + C(\|\nabla\rho\|_{L^\infty}\|D^2\theta_t\|_{L^2} + \|\theta_t\|_{L^\infty}\|D^3\rho\|_{L^2})\|D^3\theta\|_{L^2}\\
&\quad + C(\|\nabla u\|_{L^\infty}\|D^4 u\|_{L^2} + \|\dot d\|_{L^\infty}\|D^3\dot d\|_{L^2})\|D^3\theta\|_{L^2}
\end{aligned}$$

$$\leq C(M) + C(M)\|D^3 \operatorname{div} u\|_{L^2} + C(M)\|\theta_t\|_{H^2} + C(M)\|D^4 u\|_{L^2},$$

which gives

$$\|D^3\theta(\cdot,t)\|_{L^2}^2 + \int_0^t \int |D^4\theta|^2 \,dx\,ds \leq C_0 \exp(\sqrt{t}C(M)). \tag{2.14}$$

On the other hand, from (1.3) if follows that

$$\theta_t = -u\cdot\nabla\theta - \frac{p}{\rho}\operatorname{div} u + \frac{1}{\rho}\Big[\frac{\mu}{2}|\nabla u + \nabla u^t|^2 + \lambda(\operatorname{div} u)^2 + |\dot{d}|^2\Big], \tag{2.15}$$

which easily leads to

$$\|\theta_t\|_{L^2(0,t;H^2)} \leq C_0 \exp(\sqrt{t}C(M)). \tag{2.16}$$

This completes the proof.

Acknowledgments. J. Fan is supported by the NSFC (Grant No. 11171154).

References

[1] J. L. Ericksen; *Conservation laws for liquid crystals*, Trans. Soc. Rheo., 5 (1961). 23-34.
[2] J. L. Ericksen; *Continuum thoery of nematic liquid crystals*, Res Mech., 21 (1987). 381-392.
[3] J. L. Ericksen; *Liquid crystals with variable degree of orientation*, Arch. Rational Mech. Anal., 113 (2) (1990), 97-120.
[4] F. Leslie; *Theory of Flow Phenomenum in Liquid Crystals*, Springer, New York, NY, USA, 1979.
[5] N. Jiang, Y. Luo; *On well-posedness of Ericksen-Leslie's parabolic-hyperbolic liquid crystal model*, arXiv: 1612.04616.
[6] T. Huang, C. Y. Wang, H. Y. Wen; *Strong solutions of the compressible nematic liquid crystal flow*, J. Differ. Equ. 252 (2012), 2222-2265.
[7] T. Huang, C. Y. Wang, H. Y. Wen; *Blow up criterion for compressible nematic liquid crystal flows in dimension three*, Arch. Ration. Mech. Anal., 204 (2012), 285-311.
[8] X. Li, B. Guo; *Well-posedness for the three-dimensional compressible liquid crystal flows*, Discrete and Continuous Dynamical Systems-Series S, 9 (6) (2016), 1913-1937.
[9] Y. Chu, X. Liu, X. Liu; *Strong solutions to the compressible liquid crystal system*, Pacific J. of Math., 257 (1) (2012), 37-52.
[10] S. J. Ding, J. R. Huang, H. Y. Wen, R. Z. Zi; *Incompressible limit of the compressible nematic liquid crystal flow*, J. Funct. Anal., 264 (2013), 1711-1756.
[11] G. Qi, J. Xu; *The low Mach number limit for the compressible flow of liquid crystals*, Appl. Math. Comput., 297 (2017), 39-49.
[12] X. Yang; *Uniform well-posedness and low Mach number limit to the compressible nematic liquid flows in a bounded domain*, Nonlinear Analysis: Theory, Methods & Applications, 120 (2015), 118-126.
[13] F. Jiang, S. Jiang, D. H. Wang; *On multi-dimensional compressible flow of nematic liquid crystals with large initial energy in a bounded domain*, J. Funct. Anal., 265 (12) (2013), 3369-3397.
[14] F. Lin, C. Wang; *Recent developments of analysis for hydrodynamic flow of nematic liquid crystals*, Phil. Trans. R. Soc. A Math. Phys. Eng. Sci., 372 (2029) (2014).
[15] E. Feireisl, M. Fremond, E. Rocca, G. Schimperna; *A new approach to non-isothermal models for nematic liquid crystals*, Arch. Ration. Mech. Anal., 205 (2) (2012), 651-672.
[16] W. Gu, J. Fan, Y. Zhou; *Regularity criteria for some simplified non-isothermal models for nematic liquid crystals*, Computers and Mathematics with Applications, 72 (2016), 2839-2853.
[17] T. Alazard; *Low Mach number limit of the full Navier-Stokes equations*, Arch. Ration. Mech. Anal., 180 (2006), 1-73.
[18] C. Dou, S. Jiang, Y. Ou; *Low Mach number limit of full Navier-Stokes equations in a 3D bounded domain*, J. Diff. Eqs., 258 (2015), 379-398.
[19] G. Metivier, S. Schochet; *The incompressible limit of the non-isentropic Euler equations*, Arch. Ration. Mech. Anal., 158 (2001), 61-90.

[20] T. Kato, G. Ponce; *Commutator estimates and the Euler and Navier-Stokes equations*. Commun. Pure Appl. Math., 41 (1988), 891-907.

Jishan Fan
Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China
E-mail address: fanjishan@njfu.edu.cn

Tohru Ozawa (corresponding author)
Department of Applied Physics, Waseda University, Tokyo, 169-8555, Japan
E-mail address: txozawa@waseda.jp