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DETERMINATION OF OBSTACLES IN STOKES FLOW BY BOUNDARY MEASUREMENT

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Abstract. We study the determination of some obstacles in a Stokes flow domain with overdetermined boundary data. We use a method based on the topological sensitivity technique associated to the reciprocity gap function concept. We develop an asymptotic formula between the flow parameters and the boundary data. The obtained formula is interesting and serve as a useful tool to develop an accurate and robust numerical method in geometry inverse problems.

1. Introduction

Let Ω be a regular domain in $\mathbb{R}^3$ occupied by a homogeneous incompressible fluid flow. We assume that the fluid flow is in laminar regime in such way that the convection term can be neglected and the Navier-Stokes equations can be approximated by the Stokes system.

The velocity fluid w and the pressure q describing the fluid flow in Ω satisfy the following system

$$\begin{gathered} -\nu\Delta w + \nabla q = G \quad \text{in } \Omega \\ \nabla \cdot w = 0 \quad \text{in } \Omega \\ w = w_d \quad \text{on } \Gamma_d, \\ \sigma(w,q)n = g \quad \text{on } \Gamma_n, \end{gathered} \tag{1.1}$$

where G is a source term (gravitational force), ν is the fluid viscosity, w_d is a given boundary velocity and g is a given boundary force. Hence Γ_d and Γ_n are two parts of the boundary $\partial\Omega$ verifying $\overline{\partial\Omega} = \overline{\Gamma_d} \cup \overline{\Gamma_n}$ and $\Gamma_d \cap \Gamma_n = \emptyset$.

We suppose that the fluid flow domain Ω contains a finite number of unknowns obstacles $\mathcal{O}_i$, $i = 1, \dots, m$ that are well separated and not close to the boundary $\partial\Omega$. In this work we assume that each obstacle $\mathcal{O}_i$ is characterized by its center $\xi_i \in \Omega$, its size r_i and its shape S_i with $r_i > 0$ and $S_i \subset \mathbb{R}^3$ is a fixed bounded and smooth domain containing the origin. In other word, each obstacle $\mathcal{O}_i$ can be defined as $\mathcal{O}_i = \xi_i + r_i S_i$, $1 \le i \le m$.

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The problem that we consider can be formulated as follows:
• Given two boundaries data on the accessible part Γ_a of the boundary Γ_n a measured velocity w_m and an imposed force g.
• Find the unknown obstacle $\mathcal{O} = \cup_{i=1}^m \mathcal{O}_i$ such that the velocity field $w_{\mathcal{O}}$ and the pressure $q_{\mathcal{O}}$ in the perturbed domain $\Omega\backslash\overline{\mathcal{O}}$ satisfy the boundary value problem

$$\begin{gathered}
-\nu\Delta w_{\mathcal{O}} + \nabla q_{\mathcal{O}} = G \quad \text{in } \Omega\backslash\overline{\mathcal{O}} \\
\nabla\cdot w_{\mathcal{O}} = 0 \quad \text{in } \Omega\backslash\overline{\mathcal{O}} \\
w_{\mathcal{O}} = w_m \quad \text{on } \Gamma_a \quad \text{[accessible boundary]} \\
\sigma(w_{\mathcal{O}}, q_{\mathcal{O}})n = g \quad \text{on } \Gamma_a \quad \text{[accessible boundary]} \\
\sigma(w_{\mathcal{O}}, q_{\mathcal{O}})n = 0 \quad \text{on } \Gamma_i^1 \quad \text{(in and out) [inaccessible boundary]} \\
w_{\mathcal{O}} = 0 \quad \text{on } \Gamma_i^2 \quad \text{(the wall) [inaccessible boundary]} \\
w_{\mathcal{O}} = 0 \quad \text{on } \partial\mathcal{O}.
\end{gathered}$$

In this formulation, the fluid flow domain $\Omega\backslash\overline{\mathcal{O}}$ is unknown since the obstacle geometry is unknown. It is well known that this kind of problem is ill-posed in the sense of Hadamard. The majority of investigation focusing on this type of problems fall into the category of shape optimization and utilize the shape derivation technics.

In this work, we suggest a new formulation of the above inverse problem based on the reciprocity gap concept [1, 2, 3] and the topological sensitivity analysis method [4, 5, 6, 7, 8, 10, 11, 13]. More precisely, we will derive an asymptotic formula connecting the known boundary data and the unknown obstacle properties (its location ξ_i, its size r_i and its shape δ_i).

This article is organized as follows. In section 2, we introduce the reciprocity gap functional. A preliminary estimate describing the variation of the reciprocity gap functional with respect to the presence of an obstacle $\mathcal{O} = \xi + rs$ inside the fluid flow domain Ω is presented in Proposition 1. To derive the expected formula, we start our analysis by studying the influence of the presence of the obstacle on the velocity field. We derive a high order asymptotic expansion of the perturbed velocity with respect to the obstacle size r in section 3. Finally, section 4 is devoted to the derivation of a high order topological sensitivity analysis for the reciprocity gap function.

2. Reciprocity gap functional and Stokeslet sub-space

The reciprocity gap function is a function defined on the boundary $\partial\Omega$. it describes the fluid response to an imposed force on the boundary. This function associated to the presence of an obstacle $\mathcal{O}_{\xi,r}$ in the flow domain Ω is defined by $\mathcal{F}_{\xi,r} : H^1(\Omega)\times L^2(\Omega)\to\mathbb{R}$:

$$\mathcal{F}_{\xi,r}(u,p) = \int_{\partial\Omega} \sigma(u,p)nw_r\,ds - \int_{\partial\Omega}\sigma(w_r,q_r)nu\,ds,$$

where w_r, q_r is the solution of the Stokes problem in the presence of an obstacle $\mathcal{O}_{\xi,r}$,

$$\begin{gathered}
-\nu\Delta w_r + \nabla q_r = G \quad \text{in } \Omega\setminus\overline{\mathcal{O}_{\xi,r}} \\
\nabla\cdot w_r = 0 \quad \text{in } \Omega\setminus\overline{\mathcal{O}_{\xi,r}} \\
w_r = w_d \quad \text{on } \Gamma_d,
\end{gathered}$$

$$\sigma(w_r, q_r)n = g \quad \text{on } \Gamma_n.$$

In the absence of the obstacle $\mathcal{O}_{\xi,r}$, the reciprocity gap function is denoted by $\mathcal{F}_0$ and is defined on $H^1(\Omega) \times L^2(\Omega)$ by

$$\mathcal{F}_0(u,p) = \int_{\partial\Omega} \sigma(u,p)nw_0\,ds - \int_{\partial\Omega} \sigma(w_0,q_0)nu\,ds,$$

where (w_0, q_0) is the solution of (1.1).

Our goal is to establish a relation between the boundary data and the obstacles $\mathcal{O}_{\xi,r}$ properties ξ, r and S. We begin this study by the following estimation.

2.1. **Preliminary estimations.** We consider the subspace

$$\mathcal{V} = \{(u,p) \in H^1(\Omega) \times L^2(\Omega); -\nu\Delta u + \nabla p = 0 \text{ in } \Omega \text{ and } \nabla \cdot u = 0 \ \text{ in}\, \Omega\}.$$

The restriction of the reciprocity gap function $\mathcal{F}_{\xi,r}$ to the subspace $\mathcal{V}$ gives the following estimation.

Proposition 2.1. *For all $(u,p) \in \mathcal{V}$, we have*

$$\mathcal{F}_{\xi,r}(u,p) - \mathcal{F}_0(u,p) = -\int_{\mathcal{O}_{\xi,r}} \nu\nabla u : \nabla w_0\,dx + \int_{\partial\mathcal{O}_{\xi,r}} \sigma(w_r - w_0, q_r - q_0)nu\,ds. \tag{2.1}$$

Proof. Using Green's formula and the fact that $w_r = 0$ on $\partial\mathcal{O}_{\xi,r}$, one can obtain

$$\int_{\partial\Omega} \sigma(u,p)nw_r\,ds = \int_{\Omega_{\xi,r}} \nabla u : \nabla w_r\,dx,$$
$$\int_{\partial\Omega} \sigma(w_r,q_r)nu\,ds = \int_{\Omega_{\xi,r}} \nabla u : \nabla w_r\,dx - \int_{\partial\mathcal{O}_{\xi,r}} \sigma(w_r,q_r)nu\,ds$$

which implies

$$\mathcal{F}_{\xi,r}(u,p) = -\int_{\partial\mathcal{O}_{\xi,r}} \sigma(w_r,q_r)nu\,ds \quad \forall (u,p) \in \mathcal{V}. \tag{2.2}$$

In the same way we obtain

$$\int_{\partial\Omega} \sigma(u,p)nw_0\,ds = \int_{\Omega_{\xi,r}} \nabla u : \nabla w_0\,dx - \int_{\partial\mathcal{O}_{\xi,r}} \sigma(u,p)nw_0\,ds,$$
$$\int_{\partial\Omega} \sigma(w_0,q_0)nu\,ds = \int_{\Omega_{\xi,r}} \nabla w_0 : \nabla u\,dx - \int_{\partial\mathcal{O}_{\xi,r}} \sigma(w_0,q_0)nu\,ds.$$

It follows that

$$\mathcal{F}_0(u,p) = -\int_{\partial\mathcal{O}_{\xi,r}} \sigma(u,p)nw_0\,ds + \int_{\partial\mathcal{O}_{\xi,r}} \sigma(w_0,q_0)nu\,ds. \tag{2.3}$$

Using (2.2), (2.3) and the fact that $-\nu\Delta u + \nabla p = \ in\ \mathcal{O}_{\xi,r}$ and $\nabla . u = 0\ in\ \mathcal{O}_{\xi,r}$ we deduce

$$\mathcal{F}_{\xi,r}(u,p) - \mathcal{F}_0(u,p) = -\int_{\Omega_{\xi,r}} \nabla w_0 : \nabla u\,dx + \int_{\partial\mathcal{O}_{\xi,r}} \sigma(w_r - w_0, q_r - q_0)nu\,ds.$$

$\square$

2.2. **Stokeslet sub-space.** To make relation (2.1) more explicit we introduce the so called Stokeslet sub-space [9, 12].

Definition 2.2. We call Stokeslet of size $b \in \mathbb{R}^3$ and location $\eta \in \mathbb{R}^3$ the vectorial function $\mathcal{S}_{\eta,b}$ defined on $\mathbb{R}^3$ by

$$\mathcal{S}_{\eta,b}(x) = \Big(U(x-\eta)b, P(x-\eta).b\Big) \quad \forall x \in \mathbb{R}^3,$$

where $x \mapsto (U(x-\eta), P(x-\eta))$ is the fundamental solution of the Stokes operator with regards to a Dirac mass at point η.

Remark 2.3. For all $1 \le i \le 3$, the function $x \mapsto (U^i(x-\eta)b, P^i(x-\eta))$ is the solution of

$$\begin{gathered} -\nu\Delta_x U^i(x-\eta) + \nabla_x P^i(x-\eta) = \delta_\eta e_i \quad \text{in } \mathbb{R}^3 \\ \nabla_x . U^i(x-\eta) = 0 \quad \text{in } \mathbb{R}^3 \end{gathered}$$

with U^i is the i^{th} column of U and e_i is the i^{th} vector of canonical basis of $\mathbb{R}^3$.

We introduce know the so-called Stokeslet sub-space $\mathcal{V}_{\mathcal{S},\Omega}$. It is a sub-space of $\mathcal{V}$ defined by the functions of Stokeslet type localized outside Ω

$$\mathcal{V}_{\mathcal{S},\Omega} = \{x \mapsto \mathcal{S}_{\eta,b}|_{\Omega},\ \eta \in \mathbb{R}^3 \backslash \overline{\Omega}, b \in \mathbb{R}^3\}.$$

We remark that the sub-space $\mathcal{V}_{\mathcal{S},\Omega}$ is generated by the functions $(U^i(x-\eta)b, P^i(x-\eta));\ 1 \le i \le 3$.

For all $1 \le i \le 3$, we denote by $\mathcal{F}^i_{\xi,r}$ the reciprocity gap function associated to the Stokeslet $\mathcal{S}_{\eta,e_i}$ defined by

$$\mathcal{F}^i_{\xi,r} = \int_{\partial\Omega} \sigma(U^i(x,\eta), P^i(x,\eta)) n w_r \, ds - \int_{\partial\Omega} \sigma(w_r,p_r) n U^i(x,\eta)\, ds \quad \forall \eta \in \mathbb{R}^3 \backslash \overline{\Omega}.$$

From Proposition 2.1, we deduce the following result.

Corollary 2.4. *For all $1 \le i \le 3$, the function $\mathcal{F}^i_{\xi,r}$ satisfies: for all $\eta \in \mathbb{R}^3 \backslash \overline{\Omega}$,*

$$\begin{aligned} &\mathcal{F}^i_{\xi,r}(\eta) - \mathcal{F}^i_0(\eta) \\ &= -\int_{\mathcal{O}_{\xi,r}} \nu \nabla w_0 : \nabla u^i(x,\eta)\, dx + \int_{\partial\mathcal{O}_{\xi,r}} \sigma(w_r - w_0, q_r - q_0) n U^i(x-\eta)\, ds. \end{aligned} \tag{2.4}$$

3. Main results

We derive an asymptotic formula linking the unknown properties of the obstacle (its position z, size r and form $\mathcal{O}$) and the boundary data. We begin by studying the influence of the obstacle on the flow state.

3.1. **Estimate of the perturbed fluid flow.** We give an estimate of the solution (w_r, q_r) describing the flow in presence of the obstacle $\mathcal{O}_{\xi,r}$.

Proposition 3.1. *In presence of the obstacle $\mathcal{O}_{\xi,r}$ inside the fluid flow domain Ω, the Stokes solution (w_r, q_r) admits the following estimate: $\forall x \in \Omega \backslash \overline{\mathcal{O}_{\xi,r}}$*

$$w_r(x) = \sum_{k=0}^{N} r^k \big[W_k(x) + Z_k\big(\frac{x-z}{r}\big)\big] + O(r^{N+1}),$$

$$q_r(x) = \sum_{k=0}^{N} r^k \big[Q_k(x) + S_k\big(\frac{x-z}{r}\big)\big] + O(r^{N+1}),$$

where $(W_k, Q_k)_{0\le k\le N}$ are regular functions defined in Ω and solutions of a sequence of Stokes problems; $(Z_k, S_k)_{0\le k\le N}$ are regular functions, solutions of a sequence of Stokes problems in the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$.

3.2. **Preliminary calculus.** We give an estimate of each term in variation (2.4).

Lemma 3.2. *The first integral term in (2.4), admits the estimate*

$$\int_{\mathcal{O}} \nu\nabla w_o(x) : U^i(x-\eta)\,dx = \sum_{j=0}^{N-3} r^{j+3}\mathcal{I}^{i,j}_{\eta,\mathcal{O}}(z) + o(r^N), \quad \eta\in\mathbb{R}^3\backslash\overline{\Omega},$$

where the functions $z\mapsto \mathcal{I}^{i,j}_{\eta,\mathcal{O}}(z)$, $0\le j\le N-3$ are defined by

$$\mathcal{I}^{i,j}_{\eta,\mathcal{O}}(z) = -\sum_{q=0}^{j}\frac{1}{j!(j-q)!}\int_{\mathcal{O}}\nu(\nabla^{(q+1)}w_0(z)y^q)\cdot(\nabla^{(j-q+1)}U^i(z-\eta)y^{(j-q)})\,dy$$

with $y^q=(y,\dots,y)\in(\mathbb{R}^3)^q$ and $\nabla^{(p)}\varphi(z)$ is the p^{th} derivative of φ at the point z.

Lemma 3.3. *The second integral term in (2.4) satisfies the estimate*

$$\sum_{j=1}^{N} r^j\int_{\partial\mathcal{O}_{\xi,r}}\sigma(W_j,Q_j)nU^i(x-\eta)\,ds = \sum_{j=0}^{N-3} r^{j+3}\mathcal{K}^{i,j}_{\eta,\mathcal{O}}(z) + o(r^N) \quad \forall\eta\in\mathbb{R}^3\backslash\overline{\Omega},$$

where the functions $z\mapsto\mathcal{K}^{i,j}_{\eta,\mathcal{O}}(z)$ are defined by

$$\mathcal{K}^{i,j}_{\eta,\mathcal{O}}(z) = \sum_{k=0}^{j}\sum_{l=0}^{k}\frac{1}{l!(k-l)!}\int_{\partial\mathcal{O}}[\mathcal{A}^{(l)}_{j-k+1}(z)(y)n(y)][\nabla^{(k-l)}U^i(z-\eta)(y)^{(k-l)}]\,ds(y)$$

with $\mathcal{A}^{(l)}_{j-k+1}(z)(y)$ is the matrix defined by

$$\Big(\mathcal{A}^{(l)}_{j-k+1}(z)(y)\Big)_{p,q} = \nabla^{(l)}(\sigma(W_j,Q_j))_{p,q}(z)(y^l) \quad \forall 1\le p,q\le 3.$$

Lemma 3.4. *The third integral term in (2.4) admits the estimate*

$$\sum_{j=0}^{N} r^j\int_{\partial\mathcal{O}_{\xi,r}}\sigma(Z_j,S_j)\big(\frac{x-z}{r}\big)n(x)U^i(x-\eta)\,ds(x) = \sum_{j=0}^{N-1} r^{j+1}\mathcal{L}^{i,j}_{\eta,\mathcal{O}}(z) + o(r^N)$$

for all $\eta\in\mathbb{R}^3\backslash\overline{\Omega}$, where the functions $z\mapsto\mathcal{L}^{i,j}_{\eta,\mathcal{O}}(z)$ are defined by

$$\mathcal{L}^{i,j}_{\eta,\mathcal{O}}(z) = \sum_{q=0}^{j}\frac{1}{q!}\int_{\partial\mathcal{O}}[\sigma(Z_{j-q},S_{j-q})(y)n(y)]\cdot[\nabla^{(q)}U^i(z-\eta)(y^{(q)})]\,ds(y) + o(r^N)$$

for all $\eta\in\mathbb{R}^3\backslash\overline{\Omega}$.

3.3. **Asymptotic formula for the reciprocity gap function.** In this section, we derive an asymptotic formula describing the variation of the reciprocity gap function with respect to the presence of the obstacle $\mathcal{O}_{\xi,r}$ in the flow domain Ω.

From corollary 2.4 and proposition 3.1, we have

$$\begin{aligned}
&\mathcal{F}^i_{\xi,r}(\eta)-\mathcal{F}^i_0(\eta)\\
&= -\int_{\mathcal{O}_{\xi,r}}\nu\nabla w_0 : \nabla u^i(x-\eta)\,dx + \sum_{j=1}^{N} r^j\int_{\partial\mathcal{O}_{\xi,r}}\sigma(W_j,Q_j)nU^i(x-\eta)\,ds
\end{aligned}$$

$$+\sum_{j=0}^{N} r^j \int_{\partial\mathcal{O}_{\xi,r}} \sigma(Z_j,S_j)(\frac{x-z}{r})n(x)U^i(x-z)\,ds + o(r^N).$$

Using Lemmas 3.2, 3.3 and 3.4, we obtain the following theorem.

Theorem 3.5. *Let $\mathcal{O}_{\xi,r} = z + r\mathcal{O}$ an obstacle immersed in the fluid flow domain Ω. For each $1 \le i \le 3$ the reciprocity gap function $\mathcal{F}^i_{\eta,\mathcal{O}}$ satisfies the following asymptotic formula*

$$\mathcal{F}^i_{\eta,\mathcal{O}}(\eta) - \mathcal{F}^i_0(\eta) = \sum_{j=1}^{N} r^j \Psi^{i,j}_{\eta,\mathcal{O}}(z) + o(r^N) \quad \forall \eta \in \mathbb{R}^3\backslash\overline{\Omega}. \tag{3.1}$$

where $\Psi^{i,j}_{\eta,\mathcal{O}}(z)$, $1 \le i \le 3$, $1 \le j \le N$ are defined by

$$\Psi^{i,j}_{\eta,\mathcal{O}}(z) = \begin{cases} \mathcal{L}^{i,j-1}_{\eta,\mathcal{O}}(z) & \text{if } 1 \le j \le 2,\\ \mathcal{L}^{i,j-1}_{\eta,\mathcal{O}}(z) + \mathcal{K}^{i,j-3}_{\eta,\mathcal{O}}(z) + \mathcal{I}^{i,j-3}_{\eta,\mathcal{O}}(z) & \text{if } 3 \le j \le N. \end{cases}$$

4. Conclusion

The asymptotic formula derived in Theorem 3.5 can be used as the basis of a numerical algorithm serving to reconstruct an unknown obstacle $\mathcal{O}_{\xi,r}$ from boundary measured data. In fact

- The force $\sigma(w_r, q_r)n$ is imposed on Γ_n and measured on Γ_d.
- The velocity field w_r is imposed on Γ_d and measured on Γ_n.

Then the variation

$$\begin{aligned} L^i(\eta) &= \mathcal{F}^i_{\eta,\mathcal{O}}(\eta) - \mathcal{F}^i_0(\eta)\\ &= \int_{\partial\Omega} \sigma(U^i,P^i)n(w_r - w_s)\,ds - \int_{\partial\Omega} \sigma(w_r - w_0, q_r - q_0)nU^i(x-\eta)\,ds \end{aligned}$$

can be used as a measured datum on $\partial\Omega$ for all $\eta \in \mathbb{R}^3\backslash\overline{\Omega}$.

By neglecting the terms $o(r^N)$, Theorem 3.5 gives us a non linear system verified by the unknown parameters: the location z, the size r and the form $\mathcal{O}$:

$$\sum_{j=1}^{N} r^j \Psi^{i,j}_{\eta,\mathcal{O}}(z) = L^i(\eta) \quad \forall 1 \le i \le 3,\ \forall \eta \in \mathbb{R}^3\backslash\overline{\Omega}.$$

This system is difficult to solve but firstly, we can establish a numerical algorithm to identify the location z and the size r, then we can use this system to have an approximation of the form $\mathcal{O}$. This numerical work will be subject of a forthcoming paper.

5. Proofs of main results

Proof of Lemma 3.2. By the change of variable $x = z + ry$,

$$\int_{\mathcal{O}_{\xi,r}} \nu\nabla w_0(x) : \nabla U^i(x-\eta)\,dx = r^3 \int_{\mathcal{O}} \nu\nabla_x w_0(z+ry).\nabla_x U^i(z-\eta+ry)\,dy.$$

The functions w_0 and U^i are sufficiently regular in $\mathcal{O}_{\xi,r}$. Using the Taylor-Young formula, we obtain

$$\nabla w_0(z+ry) = \nabla w_0(z) + \sum_{j=1}^{N-1} \frac{r^j}{j!}\nabla^{(j+1)} w_0(z)(y^j) + O(r^N)$$

$$\nabla U^i(z-\eta+ry) = \nabla U^i(z-\eta) + \sum_{j=1}^{N-1} \frac{r^j}{j!} \nabla^{(j+1)} U^i(z-\eta)(y^j) + O(r^N).$$

Using the Cauchy formula for the product of two polynomials, we deduce

$$\begin{aligned}
&\int_{\mathcal{O}_{\xi,r}} \nu \nabla w_0(x) : \nabla U^i(x-\eta)\,dx \\
&= r^3 \int_{\mathcal{O}} \Big[\nu \sum_{j=0}^{N-1} \frac{r^j}{j!} \nabla^{(j+1)} w_0(z)(y^j)\Big]\Big[\sum_{j=0}^{N-1} \frac{r^j}{j!} \nabla^{(j+1)} U^i(z)(y^j)\Big] + O(r^{N+1}) \\
&= \sum_{j=0}^{N-3} r^{j+3} \Big[\sum_{q=0}^{j} \frac{1}{q!(j-q)!} \int_{\mathcal{O}} \nu \Big(\nabla^{(q+1)} w_0(z)(y^{(q)})\Big) \\
&\quad \cdot \Big(\nabla^{(j-q+1)} U^i(z-\eta)(y^{(j-q)})\Big)\,dy\Big] + O(r^{N+1}).
\end{aligned}$$

□

Proof of Lemma 3.3. We have

$$\begin{aligned}
&\int_{\partial\mathcal{O}_{\xi,r}} \sigma_x(W_j,Q_j) n U^i(x-\eta)\,ds(x) \\
&= r^2 \int_{\partial\mathcal{O}} \sigma_x(W_j,Q_j)(z+ry) n(z+ry).U^i(z-\eta+ry)\,ds(y)
\end{aligned}$$

Since the solution (W_j,Q_j) is regular, for all $1 \le p,q \le 3$ the function

$$y \mapsto [\sigma_x(W_j,Q_j)]_{p,q}(z+ry) = \frac{1}{2}\Big(\frac{\partial (W_j)_p}{\partial x_q} + \frac{\partial (W_j)_q}{\partial x_p}\Big) + Q_j \delta_{p,q}$$

is regular in the neighborhood of z, and

$$[\sigma(W_j,Q_j)]_{p,q}(z+ry) = \sum_{k=0}^{N} \frac{r^k}{k!} \nabla^{(k)} [\sigma(W_j,Q_j)]_{p,q}(z)(y^k) + O(r^N).$$

In the same way,

$$U^i(z-\eta+ry) = \sum_{k=0}^{N} \frac{r^k}{k!} \nabla^{(k)} U^i(z-\eta)(y^k) + O(r^N).$$

We deduce

$$\begin{aligned}
&\int_{\partial\mathcal{O}_{\xi,r}} \sigma(W_j,Q_j) n U^i(x-\eta)\,ds(x) \\
&= \sum_{k=0}^{N-2} r^{k+2} \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \int_{\partial\mathcal{O}} \mathcal{A}_j^{(l)}(z)(y) n(y) \cdot \nabla^{(k-l)} U^i(z-\eta) y^{(k-l)}\,ds(y),
\end{aligned}$$

where $\mathcal{A}_j^{(l)}(z)(y)$ is the matrix $[\mathcal{A}_j^{(l)}(z)(y)]_{p,q} = \nabla^{(l)}[\sigma(W_j,Q_j)]_{p,q}(z)(y^l)$ for $1 \le p,q \le 3$. Then we obtain

$$\begin{aligned}
&\sum_{j=1}^{N} r^j \int_{\partial\mathcal{O}_{\xi,r}} \sigma(W_j,Q_j)(x) n(x).U^i(x-\eta)\,ds(x) \\
&= \sum_{j=1}^{N} r^j \sum_{k=0}^{N-2} r^{k+2} \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \int_{\partial\mathcal{O}} \mathcal{A}_j^{(l)}(z)(y) n(y) \cdot \nabla^{(k-l)} U^i(z-\eta)(y^{k-l})\,ds(y)
\end{aligned}$$

$$
\begin{aligned}
&+ O(r^{N+1}) \\
&= \sum_{j=3}^{N} r^j \sum_{k=0}^{j-3} \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \int_{\partial \mathcal{O}} \mathcal{A}_{j-k+2}^{(l)}(z)(y)n(y) \cdot \nabla^{(k-l)} U^i(z-\eta)(y^{k-l})\, ds(y) \\
&+ O(r^{N+1}).
\end{aligned}
$$

□

Proof of Lemma 3.4. We have

$$
\int_{\partial \mathcal{O}_{\xi,r}} \sigma_x(Z_k, S_k)(\frac{x-z}{r}) n \cdot U^i(x-\eta)\, ds(x) = r \int_{\partial \mathcal{O}} \sigma_y(Z_k, S_k)(y) n U^i(z-\eta+ry)\, ds(y).
$$

As $\eta \in \mathbb{R}^3 \backslash \overline{\Omega}$, the function $x \mapsto U^i(x-\eta)$ is C^∞ in the neighborhood of z. We can derive the expansion

$$
U^i(x-\eta) = U^i(z-\eta) + \sum_{j=1}^{N-1} \frac{r^j}{j!} \nabla^{(j)} U^i(z-\eta)(y^j) + O(r^N).
$$

Then we deduce

$$
\begin{aligned}
&\int_{\partial \mathcal{O}_{\xi,r}} \sigma_x(Z_k, S_k)(\frac{x-z}{r}) n . U^i(x-\eta)\, ds(x) \\
&= \sum_{j=0}^{N-1} \frac{r^{j+1}}{j!} \int_{\partial \mathcal{O}} [\sigma_y(Z_k, S_k) n(y)].[\nabla^{(j)} U^i(z-\eta)(y^j)] + O(r^N).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{j=0}^{N} r^j \int_{\partial \mathcal{O}_{\xi,r}} \sigma(Z_j, S_j)(\frac{x-z}{r}) n . U^i(x-z)\, ds(x) \\
&= \sum_{j=1}^{N} r^j \sum_{q=0}^{j-1} \frac{1}{q!} \int_{\partial \mathcal{O}} [\sigma(Z_{j-q-1}, S_{j-q+1})(y) n(y)] \\
&\quad \cdot [\nabla^{(q)} U^i(z-\eta)(y^{(q)})]\, ds(y) + O(r^N).
\end{aligned}
$$

□

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