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HEAT AND LAPLACE TYPE EQUATIONS WITH COMPLEX SPATIAL VARIABLES IN WEIGHTED BERGMAN SPACES

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Abstract. In a recent book, the authors of this paper have studied the classical heat and Laplace equations with real time variable and complex spatial variable by the semigroup theory methods, under the hypothesis that the boundary function belongs to the space of analytic functions in the open unit disk and continuous in the closed unit disk, endowed with the uniform norm. The purpose of the present note is to show that the semigroup theory methods works for these evolution equations of complex spatial variables, under the hypothesis that the boundary function belongs to the much larger weighted Bergman space $B^p_\alpha(D)$ with $1 \le p < +\infty$, endowed with a L^p-norm. Also, the case of several complex variables is considered. The proofs require some new changes appealing to Jensen's inequality, Fubini's theorem for integrals and the L^p-integral modulus of continuity. The results obtained can be considered as complex analogues of those for the classical heat and Laplace equations in $L^p(\mathbb{R})$ spaces.

1. Introduction

Extending the method of semigroups of operators in solving the evolution equations of real spatial variable, a way of "complexifying" the spatial variable in the classical evolution equations is to "complexify" their solution semigroups of operators, as it was summarized in the book [4]. In the cases of heat and Laplace equations and their higher order correspondents, the results obtained can be summarized as follows.

Let $D = \{z \in \mathbb{C}; |z| < 1\}$ be the open unit disk and $A(D) = \{f : \overline{D} \to \mathbb{C};$ f is analytic on D, continuous on $\overline{D}\}$, endowed with the uniform norm $\|f\| = \sup\{|f(z)| : z \in \overline{D}\}$. It is well-known that $(A(D), \|\cdot\|)$ is a Banach space. Let $f \in A(D)$ and consider the operator

$$W_t(f)(z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(ze^{-iu}) e^{-u^2/(2t)} du, \quad z \in \overline{D}. \tag{1.1}$$

In [3] (see also [4, Chapter 2], for more details) it was proved that $(W_t, t \ge 0)$ is a (C_0)-contraction semigroup of linear operators on $A(D)$ and that the unique

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solution $u(t,z)$ (that belongs to $A(D)$, for each fixed $t \geq 0$) of the Cauchy problem

$$\frac{\partial u}{\partial t}(t,z) = \frac{1}{2}\frac{\partial^2 u}{\partial \varphi^2}(t,z), \quad (t,z) \in (0,+\infty) \times D,\ z = re^{i\varphi},\ z \neq 0, \tag{1.2}$$

$$u(0,z) = f(z), \quad z \in \overline{D},\ f \in A(D), \tag{1.3}$$

is exactly

$$u(t,z) = W_t(f)(z). \tag{1.4}$$

In the same contribution [3], setting

$$Q_t(f)(z) := \frac{t}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{u^2+t^2}\,du, \quad z \in \overline{D}, \tag{1.5}$$

we proved that $(Q_t, t \geq 0)$ is a (C_0)-contraction semigroup of linear operators on $A(D)$. Consequently, the unique solution $u(t,z)$ (that belongs to $A(D)$, for each fixed t) of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2}(t,z) + \frac{\partial^2 u}{\partial \varphi^2}(t,z) = 0, \quad (t,z) \in D \times (0,+\infty),\ z = re^{i\varphi},\ z \neq 0, \tag{1.6}$$

$$u(0,z) = f(z), \quad z \in \overline{D}, \quad f \in A(D), \tag{1.7}$$

is exactly

$$u(t,z) = Q_t(f)(z). \tag{1.8}$$

The goal of the present note is to show that the well-posedness of the above problems in the space $A(D)$, can be replaced by well-posedness in some (larger) weighted Bergman spaces defined in what follows. For $0 < p < +\infty$, $1 < \alpha < +\infty$ and $\rho_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$, the weighted Bergman space $B^p_\alpha(D)$, is the space of all analytic functions in D, such that

$$\Big[\int_D |f(z)|^p dA_\alpha(z)\Big]^{1/p} < +\infty,$$

where $dA_\alpha(z) = \rho_\alpha(z)dA(z)$, with

$$dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta, \quad z = x+iy = re^{i\theta},$$

the normalized Lebesgue area measure on the unit disk of the complex plane. Setting

$$\|f\|_{p,\alpha} = (\int_D |f(z)|^p dA_\alpha(z))^{1/p},$$

for $1 \leq p < +\infty$, it is well-known that $(B^p_\alpha(D), \|\cdot\|_{p,\alpha})$ is a Banach space, while for $0 < p < 1$, $B^p_\alpha(D)$ is a complete metric space with metric $d(f,g) = \|f-g\|^p_{p,\alpha}$. For other details concerning Bergman spaces in the complex plane and their properties, we refer the reader to the books [6, Chapter 1, Section 1.1] and [2, pp. 30-32].

The results obtained can be considered as complex analogues of those for the classical heat and Laplace equations in $L^p(\mathbb{R})$ spaces (see, e.g., [5, p. 23]). In Section 2, we reconsider (1.1), (1.2), (1.3), (1.4) assuming that the boundary function $f \in B^p_\alpha(D)$ with $1 \leq p < +\infty$. Section 3 treats (1.5), (1.6), (1.7), (1.8) under the same hypothesis for the boundary function f. It is worth mentioning that since the uniform norm used in the case of the space $A(D)$ is now replaced with the L^p-type norm in the Bergman space $B^p_\alpha(D)$, the proofs of these results require now some changes based on new tools, like the Jensen's inequality, the Fubini's theorem for integrals and the L^p -integral modulus of continuity.

2. Heat-type equations with complex spatial variables

The first main result of this section is concerned with the heat equation of complex spatial variable.

Theorem 2.1. *Let $1 \le p < +\infty$ and consider $W_t(f)(z)$ given by (1.1), for $z \in D$. Then, $(W_t, t \ge 0)$ is a (C_0)-contraction semigroup of linear operators on $B^p_\alpha(D)$ and the unique solution $u(t,z)$ that belongs to $B^p_\alpha(D)$ for each fixed t, of the Cauchy problem (1.2) with the initial condition*

$$u(0,z) = f(z), \quad z \in D,\ f \in B^p_\alpha(D),$$

is given by $u(t,z) = W_t(f)(z)$.

Proof. By [3, Theorem 2.1] (see also [4, Theorem 2.2.1, p. 27]), $W_t(f)(z)$ is analytic in D and for all $z \in D$, $t, s \ge 0$, we have $W_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-k^2 t/2} z^k$ and $W_{t+s}(f)(z) = W_t[W_s(f)](z)$.

In what follows, we apply the following well-known Jensen type inequality for integrals: if $\int_{-\infty}^{+\infty} G(u)du = 1$, $G(u) \ge 0$ for all $u \in \mathbb{R}$ and $\varphi(t)$ is a convex function over the range of the measurable function of real variable F, then

$$\varphi\Big(\int_{-\infty}^{+\infty} F(u)G(u)du\Big) \le \int_{-\infty}^{+\infty} \varphi(F(u))G(u)du.$$

Now, since $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/(2t)}du = 1$, by the above Jensen's inequality with $\varphi(t) = t^p$, $F(u) = |f(ze^{-iu})|$ and $G(u) = \frac{1}{\sqrt{2\pi t}} e^{-u^2/(2t)}du$, we find

$$|W_t(f)(z)|^p \le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} |f(ze^{-iu})|^p e^{-u^2/(2t)}du.$$

Multiplying this inequality by $\rho_\alpha(z) = (\alpha+1)(1+|z|^2)^\alpha$, integrating on D with respect to $dA(z)$ (the normalized Lebesgue's area measure) and taking into account the Fubini's theorem, we obtain

$$\int_D |W_t(f)(z)|^p dA_\alpha(z) \le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \Big[\int_D |f(ze^{-iu})|^p dA_\alpha(z)\Big] e^{-u^2/(2t)}du.$$

But writing $z = re^{i\theta}$ in polar coordinates and taking into account that $dA(z) = \frac{1}{\pi} r dr d\theta$, some simple calculations lead to the equality

$$\int_D |f(ze^{-iu})|^p dA_\alpha(z) = \int_D |f(z)|^p dA_\alpha(z), \text{ for all } u \in \mathbb{R}. \tag{2.1}$$

This immediately implies

$$\|W_t(f)\|_{p,\alpha} \le \|f\|_{p,\alpha}.$$

So $W_t(f) \in B^p_\alpha(D)$ and W_t is a contraction. Next, for $f \in B^p_\alpha(D)$ let us introduce the integral modulus of continuity (see, e.g., [1])

$$\omega_1(f;\delta)_{B^p_\alpha} = \sup_{0 \le |h| \le \delta} \Big(\int_D |f(ze^{ih}) - f(z)|^p dA_\alpha(z)\Big)^{1/p}.$$

Reasoning as above and taking into account that $\int_D 1 dA(z) = \frac{1}{\pi}\int_D dxdy = 1$, we obtain

$$\int_D |W_t(f)(z) - f(z)|^p dA_\alpha(z)$$

$$
\begin{aligned}
&\le \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\Big[\int_D |f(ze^{-iu})-f(z)|^p dA_\alpha(z)\Big]e^{-u^2/(2t)}du\\
&\le \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\omega_1(f;|u|)^p_{B^p_\alpha}e^{-u^2/(2t)}\,du\\
&\le \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\omega_1(f;\sqrt{t})^p_{B^p_\alpha}\Big(\frac{|u|}{\sqrt{t}}+1\Big)^p e^{-u^2/(2t)}\,du\\
&= 2\frac{\sqrt{t}}{\sqrt{2\pi t}}\cdot\int_0^{\infty}\omega_1(f;\sqrt{t})^p_{B^p_\alpha}(v+1)^p e^{-v^2/2}dv\\
&= C_p\omega_1(f;\sqrt{t})^p_{B^p_\alpha}.
\end{aligned}
$$

Consequently, we get

$$
\|W_t(f)-f\|_{B^p_\alpha}\le C'_p\omega_1(f;\sqrt{t})_{B^p_\alpha},
$$

and so it yields that $\lim_{t\searrow 0}\|W_t(f)-f\|_{B^p_\alpha}=0$. Since it is well-known that (see, e.g., [6, pg. 3, proof of Proposition 1.2]) the convergence in the Bergman space means uniform convergence in any compact subset of D, it follows that $\lim_{t\searrow 0}W_t(f)(z)=f(z)$, for all $z\in D$.

Now, let $s\in(0,+\infty)$, V_s be a small neighborhood of s, both fixed, and take an arbitrary $t\in V_s$, $t\neq s$. Applying the reasonings in the proof of [3, Theorem 2.1, (iii)] (see also [4, Theorem 2.2.1, (iii)]), we get

$$
\begin{aligned}
|W_t(f)(z)-W_s(f)(z)| &\le \int_{-\infty}^{+\infty}|f(ze^{-iu})|\Big|\frac{e^{-u^2/(2t)}}{\sqrt{2\pi t}}-\frac{e^{-u^2/(2s)}}{\sqrt{2\pi s}}\Big|du\\
&\le \frac{1}{\sqrt{2\pi}}|t-s|\int_{-\infty}^{+\infty}|f(ze^{-iu})|e^{-u^2/c^2}\Big|\frac{2u^2}{c^4}-\frac{1}{c^2}\Big|du,
\end{aligned}
$$

where c depends on s (and not on t).

Denoting $K_s=\int_{-\infty}^{+\infty}e^{-u^2/c^2}|\frac{2u^2}{c^4}-\frac{1}{c^2}|du$, i.e., where $0<K_s<+\infty$, by the proof of [4, Theorem 2.2.1, (iii)], it follows that

$$
\begin{aligned}
&\int_D |W_t(f)(z)-W_s(f)(z)|^p dA_\alpha(z)\\
&\le \Big(\frac{1}{\sqrt{2\pi}}\Big)^p|t-s|^pK_s^p\int_D\Big(\int_{-\infty}^{+\infty}|f(ze^{-iu})|\frac{1}{K_s}e^{-u^2/c^2}\Big|\frac{2u^2}{c^4}-\frac{1}{c^2}\Big|du\Big)^p dA_\alpha(z)
\end{aligned}
$$

and reasoning exactly as at the beginning of the proof, we obtain

$$
\int_D |W_t(f)(z)-W_s(f)(z)|^p dA_\alpha(z)\le\Big(\frac{1}{\sqrt{2\pi}}\Big)^p|t-s|^pK_s^p\|f\|^p_{p,\alpha}.
$$

This implies

$$
\|W_t(f)-W_s(f)\|_{p,\alpha}\le\frac{1}{\sqrt{2\pi}}|t-s|K_s\|f\|_{p,\alpha}.
$$

Therefore, $(W_t,t\ge 0)$ is a (C_0)-contraction semigroup of linear operators on $B^p_\alpha(D)$. Also, since the series representation for $W_t(f)(z)$ is uniformly convergent in any compact disk included in D, it can be differentiated term by term with respect to t and φ. We then easily obtain that $W_t(f)(z)$ satisfies the Cauchy problem in the statement of the theorem. We also note that in equation we must take $z\neq 0$ simply because $z=0$ has no polar representation, namely $z=0$ cannot be represented as function of φ. This completes the proof. $\square$

The partial differential equation (1.2) can equivalently be expressed in terms of t and z as follows.

Corollary 2.2. *Let $1 \le p < +\infty$. For each $f \in B^p_\alpha(D)$, the initial value problem*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Big(z\frac{\partial u}{\partial z} + z^2\frac{\partial^2 u}{\partial z^2}\Big) = 0, \quad (t, z) \in \mathbb{R}_+ \times D\backslash\{0\},\ u(0, z) = f(z), z \in D,$$

is well-posed and its unique solution is $W_t(f) \in C^\infty(\mathbb{R}_+; B^p_\alpha(D))$.

Proof. By [3, Theorem 2.1, (i)], we can compute the generator associated with $W_t(f)$, to get

$$\begin{aligned}
\big(\frac{d}{dt}W_t(f)(z)\big)_{|t=0} &= -\sum_{k=0}^{\infty}\frac{k^2}{2}a_k z^k = -\sum_{k=0}^{\infty}\big(\frac{k(k-1)}{2} + \frac{k}{2}\big)a_k z^k \\
&= -\frac{z^2}{2}f''(z) - \frac{z}{2}f'(z).
\end{aligned}$$

Therefore, the statement is an immediate consequence of a classical result of Hill (see, e.g., [4, Theorem 1.2.1, pg. 8]). □

Remark 2.3. Theorem 2.1 can be easily extended to functions of several complex variables, as follows. For $0 < p < \infty$, let $f : D^n \to \mathbb{C}$, $f(z_1, z_2, \dots, z_n)$ be analytic with respect to each variable $z_i \in D$, $i = 1, \dots, n$, such that

$$\int_{D^n} |f(z_1, \dots, z_n)|^p dA_\alpha(z_1)\dots dA_\alpha(z_n) < \infty.$$

We write this by $f \in B^p_\alpha(D^n)$ and

$$\|f\|_{B^p_\alpha(D^n)} = \Big(\int_{D^n} |f(z_1, \dots, z_n)|^p dA_\alpha(z_1)\dots dA_\alpha(z_n)\Big)^{1/p}$$

becomes a norm on $B^p_\alpha(D^n)$. We call the later the weighted Bergman space in several complex variables. Following now the model for the semigroup attached to the real multivariate heat equation (see, e.g., [5, pg. 69]), for $z_1, z_2, \dots, z_n \in D$, we may define the integral of Gauss-Weierstrass type by

$$H_t(f)(z_1, \dots, z_n) = (2\pi t)^{-n/2}\int_{D^n} f(z_1 e^{-iu_1}, \dots, z_n e^{-iu_n})e^{-|u|^2/(2t)}du_1\dots du_n,$$

where $u = (u_1, \dots, u_n)$ and $|u| = \sqrt{u_1^2 + \dots + u_n^2}$. One can reason exactly as in the proof of Theorem 2.1 to deduce that $(H_t,\, t \ge 0)$ is (C_0)-contraction semigroup of linear operators on $B^p_\alpha(D^n)$ and that $u(t, z_1, \dots, z_n) = H_t(f)(z_1, \dots, z_n)$ is the unique solution of the Cauchy problem

$$\begin{gathered}
\frac{\partial u}{\partial t}(t, z_1, \dots, z_n) = \frac{1}{2}\big[\frac{\partial^2 u}{\partial {\varphi_1}^2}(t, z_1, \dots, z_n) + \dots + \frac{\partial^2 u}{\partial {\varphi_n}^2}(t, z_1, \dots, z_n)\big], \quad t > 0, \\
z_1 = r_1 e^{i\varphi_1}, \dots, z_n = r_n e^{i\varphi_n} \in D, \quad z_1, \dots, z_n \ne 0, \\
u(0, z_1, \dots, z_n) = f(z_1, \dots, z_n).
\end{gathered}$$

Remark 2.4. Reasoning as in the proof of Corollary 2.2 and calculating

$$\Big(\frac{d}{dt}H_t(f)(z_1, \dots, z_n)\Big)\Big|_{t=0},$$

we easily find that the initial value problem

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sum_{k=1}^{n}\Big(z_k\frac{\partial u}{\partial z_k}+z_k^2\frac{\partial^2 u}{\partial z_k^2}\Big)=0,\quad (t,z_1,\dots,z_n)\in\mathbb{R}_+\times(D\backslash\{0\})^n,$$
$$u(0,z_1,\dots,z_n)=f(z_1,\dots,z_n),\quad z_1,\dots,z_n\in D,$$

is well-posed and its unique solution is $H_t(f)\in C^\infty(\mathbb{R}_+;B^p_\alpha(D^n))$.

3. Laplace-type equations with complex spatial variables

The first main result of this section is concerned with the Laplace equation of a complex spatial variable.

Theorem 3.1. *Let $1\le p<+\infty$ and consider $Q_t(f)(z)$ given by* (1.5)*, for $z\in D$. Then $(Q_t,t\ge 0)$ is a (C_0)-contraction semigroup of linear operators on $B^p_\alpha(D)$ and the unique solution $u(t,z)$ that belongs to $B^p_\alpha(D)$ for each fixed t, of the Cauchy problem* (1.6) *with the initial condition*

$$u(0,z)=f(z),\quad z\in D,\ f\in B^p_\alpha(D),$$

is given by $u(t,z)=Q_t(f)(z)$.

Proof. By [3, Theorem 3.1] (see also [4, Theorem 2.3.1, p. 27]), $Q_t(f)(z)$ is analytic in D and for all $z\in D$, $t,s\ge 0$ we have $Q_t(f)(z)=\sum_{k=0}^{\infty}a_ke^{-kt}z^k$ and $Q_{t+s}(f)(z)=Q_t[Q_s(f)](z)$. Now, since $\frac{t}{\pi}\int_{-\infty}^{+\infty}\frac{1}{u^2+t^2}du=1$, by Jensen's inequality, we get

$$|Q_t(f)(z)|^p\le\frac{t}{\pi}\int_{-\infty}^{+\infty}|f(ze^{-iu})|^p\frac{1}{u^2+t^2}du,$$

which multiplied on both sides by $\rho_\alpha(z)$, then integrated on D with respect to the Lebesgue's area measure $dA(z)$ and applying the Fubini's theorem, gives

$$\int_D|Q_t(f)(z)|^pdA_\alpha(z)\le\frac{t}{\pi}\int_{-\infty}^{+\infty}\Big[\int_D|f(ze^{-iu})|^pdA_\alpha(z)\Big]\frac{1}{u^2+t^2}du.$$

As in the proof of Theorem 3.1, writing $z=re^{i\theta}$ (in polar coordinates) and taking into account that $dA(z)=\frac{1}{\pi}rdrd\theta$, some simple calculations lead to the same equality (2.1). Hence, we get $\|Q_t(f)\|_{p,\alpha}\le\|f\|_{p,\alpha}$. This implies that $Q_t(f)\in B^p_\alpha(D)$ and that Q_t is a contraction.

To prove that $\lim_{t\searrow 0}Q_t(f)(z)=f(z)$, for any $f\in B^p_\alpha(D)$ and $z\in D$, let $f=U+iV$, $z=re^{ix}$ be fixed with $0<r\le\rho<1$ and denote

$$F(v)=U[r\cos(v),r\sin(v)],G(v)=V[r\cos(v),r\sin(v)].$$

We can write

$$Q_t(f)(z)=\frac{t}{\pi}\int_{-\infty}^{+\infty}F(x-u)\frac{1}{t^2+u^2}du+i\frac{t}{\pi}\int_{-\infty}^{+\infty}G(x-u)\frac{1}{t^2+u^2}du.$$

From the maximum modulus principle, when estimating the quantity $|Q_t(f)(z)-f(z)|$, for $|z|\le\rho<1$, we may take $r=\rho$. Now, passing to limit as $t\to 0^+$ and taking into account the property in the real case (see, e.g., [5, p. 23, Exercise 2.18.8]), we find

$$\lim_{t\searrow 0}|Q_t(f)(z)-f(z)|\le\lim_{t\searrow 0}\Big|\frac{t}{\pi}\int_{-\infty}^{+\infty}F(x-u)\frac{1}{t^2+u^2}du-F(x)\Big|$$

$$+\lim_{t\searrow 0}\Big|\frac{t}{\pi}\int_{-\infty}^{+\infty}G(x-u)\frac{1}{t^2+u^2}du-G(x)\Big|=0,$$

which holds uniformly with respect to $|z|\le\rho$. Consequently, it follows that $\lim_{t\searrow 0}Q_t(f)(z)=f(z)$, uniformly in any compact subset of D.

Now, let $s\in(0,+\infty)$, V_s be a small neighborhood of s, both fixed, and take an arbitrary $t\in V_s$, $t\neq s$. Applying the same reasoning as in the proof of [3, Theorem 3.1, (ii)] (see also [4, Theorem 2.3.1, (ii)]), we get

$$\begin{aligned}|Q_t(f)(z)-Q_s(f)(z)|&\le\frac{1}{\pi}\int_{-\infty}^{+\infty}|f(ze^{-iu})|\big|\frac{t}{t^2+u^2}-\frac{s}{s^2+u^2}\big|du\\&=\frac{1}{\pi}|t-s|\int_{-\infty}^{+\infty}|f(ze^{-iu})|\big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\big|du.\end{aligned}$$

Setting

$$K_s=\int_{-\infty}^{+\infty}\Big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\Big|du,$$

i.e.,

$$1=\frac{1}{K_s}\int_{-\infty}^{+\infty}\big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\big|du$$

(where $0<K_s<+\infty$), by the proof of [4, Theorem 2.3.1, (ii)], it follows that

$$\begin{aligned}&\int_D|Q_t(f)(z)-Q_s(f)(z)|^p dA_\alpha(z)\\&\le\big(\frac{1}{\pi}\big)^p|t-s|^pK_s^p\int_D\Big(\int_{-\infty}^{+\infty}|f(ze^{-iu})|\frac{1}{K_s}\big|\frac{u^2-ts}{(t^2+u^2)(s^2+u^2)}\big|du\Big)^p dA_\alpha(z).\end{aligned}$$

Applying Jensen's inequality, we obtain

$$\int_D|Q_t(f)(z)-Q_s(f)(z)|^p dA_\alpha(z)\le\big(\frac{1}{\pi}\big)^p|t-s|^pK_s^p\|f\|_{p,\alpha}^p$$

and so

$$\|Q_t(f)-Q_s(f)\|_{p,\alpha}\le\frac{1}{\pi}|t-s|K_s\|f\|_{p,\alpha}.$$

Then $(Q_t,t\ge 0)$ is a (C_0)-contraction semigroup of linear operators on $B_\alpha^p(D)$. Also, since the series representation for $Q_t(f)(z)$ is uniformly convergent in any compact disk included in D, it can be differentiated term by term, with respect to t and φ. We then easily obtain that $Q_t(f)(z)$ satisfies the Cauchy problem in the statement of the theorem. We also note that in equation we must take $z\neq 0$ simply because $z=0$ has no polar representation. This completes the proof. □

Reasoning exactly as in the proof of [4, Theorem 2.3.1, (v), pp. 53-54], we immediately get the following result.

Corollary 3.2. *Let $1\le p<+\infty$. For each $f\in B_\alpha^p(D)$, the initial value problem*

$$\frac{\partial u}{\partial t}+z\frac{\partial u}{\partial z}=0,\quad (t,z)\in\mathbb{R}_+\times D\backslash\{0\},\quad u(0,z)=f(z),\ z\in D,$$

is well-posed and its unique solution is $Q_t(f)\in C^\infty(\mathbb{R}_+;B_\alpha^p(D))$.

Remark 3.3. The above results can be easily extended to several complex variables. We may define the complex Poisson-Cauchy integral by

$$P_t(f)(z_1,\dots,z_n)=\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}t\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}f(z_1e^{-iu_1},\dots,z_ne^{-iu_n})\\ \times\frac{1}{(t^2+u_1^2+\dots+u_n^2)^{(n+1)/2}}du_1\dots du_n.$$

Using similar arguments, as in the univariate complex case, we can prove that the unique solution of the Cauchy problem

$$\begin{gathered}\frac{\partial^2u}{\partial t^2}(t,z_1,\dots,z_n)+\frac{\partial^2u}{\partial{\varphi_1}^2}(t,z_1,\dots,z_n)+\dots+\frac{\partial^2u}{\partial{\varphi_n}^2}(t,z_1,\dots,z_n)=0,\quad t>0,\\ z_1=r_1e^{i\varphi_1},\dots,\ z_n=r_ne^{i\varphi_n}\in D,\quad z_1,\dots,z_n\neq 0,\\ u(0,z_1,\dots,z_n)=f(z_1,\dots,z_n),\quad\text{for } z_1,\dots,z_n\in D,\ f\in B_\alpha^p(D^n)\end{gathered}$$

is given by $u(t,z_1,\dots,z_n)=P_t(f)(z_1,\dots,z_n)$.

Remark 3.4. Reasoning as in the proof of Corollary 3.2 and calculating

$$\Big(\frac{d}{dt}P_t(f)(z_1,\dots,z_n)\Big)\Big|_{t=0},$$

we easily obtain the following result: For each $f\in B_\alpha^p(D^n)$, the initial value problem

$$\begin{gathered}\frac{\partial u}{\partial t}+\sum_{k=1}^n\big(z_k\frac{\partial u}{\partial z_k}\big)=0,\quad (t,z_1,\dots,z_n)\in\mathbb{R}_+\times(D\setminus\{0\})^n,\\ u(0,z_1,\dots,z_n)=f(z_1,\dots,z_n),\quad z_1,\dots,z_n\in D,\end{gathered}$$

is well-posed and its unique solution is $Q_t(f)\in C^\infty(\mathbb{R}_+;B_\alpha^p(D))$.

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