Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 238, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

WELL-POSEDNESS OF A POROUS MEDIUM FLOW WITH FRACTIONAL PRESSURE IN SOBOLEV SPACES

XUHUAN ZHOU, WEILIANG XIAO

Abstract. We prove the existence of a non-negative solution for a linear degenerate diffusion transport equation from which we derive the existence and uniqueness of the solution for the fractional porous medium equation in Sobolev spaces H^α with nonnegative initial data, $\alpha > \frac{d}{2} + 1$. We also correct a mistake in our previous paper [14].

1. Introduction

We consider the porous medium type equation

$$\partial_t u = \nabla \cdot (u \nabla p), \quad p = (-\Delta)^{-s} u, \quad 0 < s < 1, \quad u(x,0) \geq 0, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $n \geq 2$, and $t > 0$ and the fractional Laplacian $(-\triangle)^{s/2} := \Lambda^s$ is given by the psuedo differential operator with symbol $|\xi|^s$, that is:

$$(-\triangle)^{s/2} f = \Lambda^s f = \mathcal{F}^{-1} |\xi|^s \mathcal{F} f.$$

Using the Riesz potential, one can also define this operator as

$$(-\triangle)^{s/2} f(x) = \Lambda^s f(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|y|^{n+s}} \mathrm{d}y.$$

This model is based on Darcy's law with pressure, p, is given by an inverse fractional Laplacian operator. It was first introduced by Caffarelli and Vázquez [4], in which they proved the existence of a weak solution when u_0 is a bounded function with exponential decay at infinity. For $\alpha = \frac{n}{n+2-2s}$, Caffarelli, Soria and Vázquez [3] proved that the bounded nonnegative solutions are C^α continuous in a strip of space-time for $s \neq 1/2$. And same conclusion for the index $s = 1/2$ was proved by Caffarelli and Vázquez in [5]. [7, 6, 13] give a detailed description of the large-time asymptotic behaviour of the solutions of (1.1). [2, 12] consider degenerate cases and show the existence and properties of self-similar solutions. Allen, Caffarelli and Vasseur [1] study the equation with fractional time derivative, and proved the Hölder continuity for its weak solutions.

In this paper, we study the existence and uniqueness of solutions of (1.1) in Sobolev spaces. Unlike considering the existence of weak solution in L^∞ or constructing approximate solutions of linear transport systems, we solve equation (1.1) by constructing solutions to a linear degenerate diffusion transport systems. The

2010 *Mathematics Subject Classification.* 35K55, 35K65, 76S05.
Key words and phrases. Fractional porous medium equation; Sobolev space; degenerate diffusion transport equation.

Submitted December 22, 2016. Published October 3, 2017.

well-posedness and properties of the linear degenerate diffusion transport are interesting results by themselves and lead us to proving that for $s \in [\frac{1}{2}, 1)$, $\alpha > \frac{d}{2} + 1$, $u_0 \in H^\alpha(\mathbb{R}^n)$ nonnegative, and some $T_0 > 0$, the equation (1.1) in $\mathbb{R}^n \times [0, T_0]$ has a unique solutions. Besides, using the methods and results in this paper, we correct a mistake in our previous paper [14].

2. Preliminaries

Define $\rho \in C_c^\infty(\mathbb{R}^n)$ by

$$\rho(x) = \begin{cases} c_0 \exp(-\frac{1}{1-|x|^2}), |x| < 1, \\ 0, |x| \geq 1, \end{cases}$$

where c_0 is selected such that $\int \rho(x)dx = 1$. Let J_ϵ be defined by

$$J_\epsilon u = \rho_\epsilon * u = \epsilon^{-n} \rho(\frac{\cdot}{\epsilon}) * u.$$

This operator satisfies the following properties.

Proposition 2.1. (1) $\Lambda^s J_\epsilon u = J_\epsilon \Lambda^s u$, $s \in \mathbb{R}$.

(2) *For all* $u \in L^p(\mathbb{R}^n)$, $v \in H^\alpha(\mathbb{R}^n)$, *with* $\frac{1}{p} + \frac{1}{q} = 1$, $\int (J_\epsilon f) g = \int f(J_\epsilon g)$.

(3) *For all* $u \in H^\alpha(\mathbb{R}^n)$,

$$\lim_{\epsilon \to 0} \|J_\epsilon u - u\|_{H^\alpha} = 0, \quad \lim_{\epsilon \to 0} \|J_\epsilon u - u\|_{H^{\alpha-1}} \leq C\|u\|_{H^\alpha}.$$

(4) *For all* $u \in H^\alpha(\mathbb{R}^n)$, $s \in \mathbb{R}$, $k \in \mathbb{Z} \cup \{0\}$, *then*

$$\|J_\epsilon u\|_{H^{\alpha+k}} \leq \frac{C_{\alpha k}}{\epsilon^k} \|u\|_{H^\alpha}, \quad \|J_\epsilon D^k u\|_{L^\infty} \leq \frac{C_k}{\epsilon^{\frac{n}{2}+k}} \|u\|_{H^\alpha}$$

The following propositions can be found in [8, 9].

Proposition 2.2. *Suppose that* $s > 0$ *and* $1 < p < \infty$. *If* $f, g \in \mathcal{S}$, *the Schwartz class, then we have*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq c\|\nabla f\|_{L^{p_1}} \|g\|_{\dot{H}^{s-1,p_2}} + c\|g\|_{L^{p_4}} \|f\|_{\dot{H}^{s,p_3}},$$
$$\|\Lambda^s(fg)\|_{L^p} \leq c\|f\|_{L^{p_1}} \|g\|_{\dot{H}^{s,p_2}} + c\|g\|_{L^{p_4}} \|f\|_{\dot{H}^{s,p_3}}$$

with $p_2, p_3 \in (1, +\infty)$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Proposition 2.3. *If* $0 \leq s \leq 2$, $f \in \mathcal{S}(\mathbb{R}^n)$, *then*

$$2f(x)\Lambda^s f(x) \geq \Lambda^s f^2(x) \quad \textit{for all } x \in \mathbb{R}^n.$$

Proposition 2.4. *Let* α_1 *and* α_2 *be two real numbers such that* $\alpha_1 < \frac{n}{2}$, $\alpha_2 < \frac{n}{2}$ *and* $\alpha_1 + \alpha_2 > 0$. *Then there exists a constant* $C = C_{\alpha_1,\alpha_2} \geq 0$ *such that for all* $f \in \dot{H}^{\alpha_1}$ *and* $g \in \dot{H}^{\alpha_2}$,

$$\|fg\|_{\dot{H}^\alpha} \leq C\|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}},$$

where $\alpha = \alpha_1 + \alpha_2 - \frac{n}{2}$.

3. Main results

Theorem 3.1. *If $s \in [1/2,1]$, $T>0$, $\alpha > \frac{n}{2}+1$, $u_0 \in H^\alpha(\mathbb{R}^n)$, $v \geq 0$ and $v \in C([0,T];H^\alpha(\mathbb{R}^n))$, then the linear initial-value problem*

$$\begin{gathered} \partial_t u = \nabla u \cdot \nabla(-\Delta)^{-s} v - v(-\Delta)^{1-s} u, \\ u(x,0) = u_0. \end{gathered} \tag{3.1}$$

has a unique solution $u \in C^1([0,T];H^\alpha(\mathbb{R}^n))$. If the initial data $u_0 \geq 0$, then $u \geq 0, (x,t) \in \mathbb{R}^n \times [0,T]$.

Proof. For any $\epsilon > 0$, we consider the linear problem

$$\begin{gathered} \partial_t u^\epsilon = F_\epsilon(u^\epsilon) = J_\epsilon(\nabla J_\epsilon u^\epsilon \cdot \nabla(-\Delta)^{-s} v) - J_\epsilon(v(-\Delta)^{1-s} J_\epsilon u^\epsilon), \\ u^\epsilon(x,0) = u_0. \end{gathered} \tag{3.2}$$

By Propositions 2.1 and 2.2 and by $s \geq \frac12$ we can estimate

$$\begin{aligned} &\|F_\epsilon(u_1^\epsilon) - F_\epsilon(u_2^\epsilon)\|_{H^\alpha} \\ &= \|J_\epsilon(\nabla J_\epsilon(u_1^\epsilon - u_2^\epsilon)\cdot \nabla(-\Delta)^{-s} v) - J_\epsilon(v(-\Delta)^{1-s} J_\epsilon(u_1^\epsilon - u_2^\epsilon)\|_{H^\alpha} \\ &\leq C(\epsilon, \|v\|_{H^\alpha})\|u_1^\epsilon - u_2^\epsilon\|_{H^\alpha}. \end{aligned}$$

Using Picard iterations, for any $\alpha > \frac{n}{2}+1, \epsilon > 0$, there exists a $T_\epsilon = T_\epsilon(u_*) > 0$, problem (3.2) has a unique solution $u^\epsilon \in C^1([0,T_\epsilon);H^\alpha)$. By Propositions 2.1 and 2.3,

$$\begin{aligned} \frac12 \frac{d}{dt}\|u^\epsilon\|_{L^2}^2 &= \int \nabla J_\epsilon u^\epsilon \cdot \nabla(-\Delta)^{-s} v J_\epsilon u^\epsilon - \int v(-\Delta)^{1-s} J_\epsilon u^\epsilon J_\epsilon u^\epsilon \\ &\leq \frac12 \int \nabla |J_\epsilon u^\epsilon|^2 \cdot \nabla(-\Delta)^{-s} v - \frac12 \int v(-\Delta)^{1-s}|J_\epsilon u^\epsilon|^2 \\ &\leq \frac12 \int |J_\epsilon u^\epsilon|^2(-\Delta)^{1-s} v - \frac12 \int |J_\epsilon u^\epsilon|^2 (-\Delta)^{1-s} v = 0. \end{aligned}$$

Moreover, for any $\alpha > 0$,

$$\begin{aligned} &\frac12 \frac{d}{dt}\|\Lambda^\alpha u^\epsilon\|_{L^2}^2 \\ &= \int \Lambda^\alpha(\nabla J_\epsilon u^\epsilon \cdot \nabla(-\Delta)^{-s} v) J_\epsilon \Lambda^\alpha u^\epsilon - \int \Lambda^\alpha(v(-\Delta)^{1-s} J_\epsilon u^\epsilon)\Lambda^\alpha J_\epsilon u^\epsilon \\ &\leq C\|[\Lambda^\alpha, \nabla(-\Delta)^{-s} v]\nabla J_\epsilon u^\epsilon\|_{L^2}\|\Lambda^\alpha u^\epsilon\|_{L^2} + \int \nabla(-\Delta)^{-s} v \Lambda^\alpha \nabla J_\epsilon u^\epsilon \Lambda^\alpha J_\epsilon u^\epsilon \\ &\quad + C\|[\Lambda^\alpha, v](-\Delta)^{1-s} J_\epsilon u^\epsilon\|_{L^2}\|\Lambda^\alpha u^\epsilon\|_{L^2} - \int v \Lambda^\alpha(-\Delta)^{1-s} J_\epsilon u^\epsilon \Lambda^\alpha J_\epsilon u^\epsilon. \end{aligned}$$

By Proposition 2.2 and Sobolev embeddings,

$$\begin{aligned} &\|[\Lambda^\alpha, \nabla(-\Delta)^{-s} v]\nabla J_\epsilon u^\epsilon\|_{L^2} \\ &\leq C\|(-\Delta)^{1-s} v\|_{L^\infty}\|\Lambda^{\alpha-1}\nabla J_\epsilon u^\epsilon\|_{L^2} + \|\nabla(-\Delta)^{-s} v\|_{\dot H^\alpha}\|\nabla J_\epsilon u^\epsilon\|_{L^\infty} \\ &\leq C\|v\|_{H^\alpha}\|u^\epsilon\|_{H^\alpha}, \end{aligned}$$

$$\begin{aligned} &\|[\Lambda^\alpha, v](-\Delta)^{1-s} J_\epsilon u^\epsilon\|_{L^2} \\ &\leq C\|\nabla v\|_{L^\infty}\|(-\Delta)^{1-s} J_\epsilon u^\epsilon\|_{\dot H^{\alpha-1}} + \|v\|_{\dot H^\alpha}\|(-\Delta)^{1-s} J_\epsilon u^\epsilon\|_{L^\infty} \\ &\leq C\|v\|_{H^\alpha}\|u^\epsilon\|_{H^\alpha}. \end{aligned}$$

By Proposition 2.3,

$$\begin{aligned}
&\int \nabla(-\Delta)^{-s} v \Lambda^\alpha \nabla J_\epsilon u^\epsilon \Lambda^\alpha J_\epsilon u^\epsilon - \int v \Lambda^\alpha (-\Delta)^{1-s} J_\epsilon u^\epsilon \Lambda^\alpha J_\epsilon u^\epsilon \\
&\le \frac{1}{2} \int \nabla(-\Delta)^{-s} v \nabla (\Lambda^\alpha J_\epsilon u^\epsilon)^2 - \frac{1}{2} \int v (-\Delta)^{1-s} (\Lambda^\alpha J_\epsilon u^\epsilon)^2 \\
&\le C \|v\|_{H^\alpha} \|u^\epsilon\|_{H^\alpha}^2.
\end{aligned}$$

Combining the above estimates,

$$\frac{d}{dt} \|u^\epsilon(\cdot,t)\|_{H^\alpha} \le C \|v\|_{H^\alpha} \|u^\epsilon\|_{H^\alpha}.$$

By Gronwall's inequality,

$$\|u^\epsilon(\cdot,t)\|_{H^\alpha} \le \|u_0\|_{H^\alpha} \exp(C \sup_{0\le t\le T} \|v\|_{H^\alpha}).$$

Such the solution u^ϵ exists on $[0,T]$. Similarly,

$$\frac{d}{dt} \|u^\epsilon(\cdot,t)\|_{H^{\alpha-1}} \le C \|v\|_{H^\alpha} \|u^\epsilon\|_{H^\alpha} \le C(\|v\|_{H^\alpha}, \|u_0\|_{H^\alpha}, T).$$

By Aubin compactness theorem [11], there is a subsequence of $\{u^{\frac{1}{n}}\}_{n\ge1}$ that convergence strongly to u in $C([0,T];H^\alpha)$. If $\alpha > \frac{d}{2}+1$, $H^\alpha \hookrightarrow C^1$, so u is a solution of (3.3).

If u and $\tilde u$ are two solutions of problem (3.3), then $w = u - \tilde u$ satisfies

$$\begin{gathered}
\partial_t w = \nabla w \cdot \nabla(-\Delta)^{-s} v - v(-\Delta)^{1-s} w, \\
w(x,0) = 0.
\end{gathered}$$

Similarly, we get $\frac{d}{dt}\|w\|_{L^2} \le 0$ and $\frac{d}{dt}\|w\|_{\dot H^\alpha} \le \|v\|_{H^\alpha}\|w\|_{H^\alpha}$, i.e, $\frac{d}{dt}\|w\|_{H^\alpha} \le \|v\|_{H^\alpha}\|w\|_{H^\alpha}$. By Gronwall's inequality, $u(x,t)=0$, $(x,t)\in\mathbb{R}^n\times[0,T]$.

Since $u_0 \ge 0$ then if there exists a first time t_0 where for some point x_0 we have $u(x_0,t_0)=0$, then (x_0,t_0) will correspond to a minimum point and therefore $\nabla u(x_0,t_0)=0$, and

$$(-\Delta)^{1-s} u(x) = c \int \frac{u(x)-u(y)}{|y|^{n+2-2s}} \mathrm{d}y \le 0.$$

Hence $u_t|_{(x_0,t_0)} \ge 0$. So $u(x,t)\ge 0$ for all $(x,t)\in\mathbb{R}^n\times[0,T]$. □

Theorem 3.2. *Let $n \ge 2$, $s \in [\frac{1}{2},1)$, $\alpha > \frac{d}{2}+1$, $u_0 \in H^\alpha(\mathbb{R}^n)$, and $u_0 \ge 0$. Then there the linear initial value problem*

$$\begin{gathered}
\partial_t u = \nabla \cdot (u \nabla(-\Delta)^{-s} u), \\
u(x,0) = u_0.
\end{gathered}$$

has a unique solution $u \in C^1([0,T_0], H^\alpha(\mathbb{R}^n))$. If the initial data $u_0 \ge 0$, then $u \ge 0, (x,t) \in \mathbb{R}^n \times [0,T_0]$.

Proof. Set $u^1 = u_0$. Note that $\partial_t u = \nabla\cdot(u\nabla(-\Delta)^{-s}u) = \nabla u \cdot \nabla(-\Delta)^{-s} u - u(-\Delta)^{1-s}u$, and let $\{u^n\}$ be the sequence defined by

$$\begin{gathered}
\partial_t u^{n+1} = \nabla u^{n+1} \cdot \nabla(-\Delta)^{-s} u^n - u^n(-\Delta)^{1-s} u^{(n+1)}, \\
u^{n+1}(x,0) = u_0.
\end{gathered} \tag{3.3}$$

By Theorem 3.1, $u^2 \in C([0,T);H^\alpha)$, for all $T<\infty$, satisfies $u^2 \geq 0$ and

$$\sup_{0\leq t\leq T} \|u^2\|_{H^\alpha} \leq \|u_0\|_{H^\alpha} \exp(C\|u^1\|_{H^\alpha}T).$$

If $\exp(2C\|u_1\|_{H^\alpha}T_0) \leq 2$, for example $T_0 = \frac{\ln 2}{2C(1+\|u_0\|_{H^\alpha})}$, we have

$$\sup_{0\leq t\leq T_0} \|u^2\|_{H^\alpha} \leq 2\|u_0\|_{H^\alpha}.$$

By the standard induction argument, if $u^n \in C([0,T_0];H^\alpha)$, $u^n \geq 0$ is a solution of (3.3) with $\|u^n\|_{H^\alpha} \leq 2\|u_0\|_{H^\alpha}$. By Theorem 3.1 $u^{n+1} \in C([0,T_0];H^\alpha)$, $u^{n+1} \geq 0$ and

$$\begin{gathered}
\sup_{0\leq t\leq T_0} \|u^{n+1}\|_{H^\alpha} \leq \|u_0\|_{H^\alpha} \exp(C\|u^n\|_{H^\alpha}T_0) \leq 2\|u_0\|_{H^\alpha},\\
\frac{d}{dt}\|u^{n+1}\|_{H^{\alpha-1}} \leq C\|u^n\|_{H^\alpha}\|u^{n+1}\|_{H^\alpha} \leq C\|u_0\|^2_{H^\alpha}.
\end{gathered}$$

By Aubin compactness theorem [11], there is a subsequence of u^n that convergence strongly to u in $C([0,T];H^\alpha)$. If $u \geq 0, \tilde{u} \geq 0$ are two solutions of problem (3.3), then $w = u - \tilde{u}$ satisfies

$$\begin{gathered}
\partial_t w = \nabla\cdot(w\nabla(-\Delta)^{-s}u) + \nabla\cdot(\tilde{u}\nabla(-\Delta)^{-s}w),\\
w(x,0)=0.
\end{gathered}$$

By Proposition 2.2,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2} &= \int w\nabla\cdot(w\nabla(-\Delta)^{-s}u) + \int w\nabla\tilde{u}\cdot\nabla(-\Delta)^{-s}w - \int w\tilde{u}(-\Delta)^{1-s}w\\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Note that

$$\begin{gathered}
\begin{aligned}
I_1 &= \int w\nabla w\cdot\nabla(-\Delta)^{-s}u = \frac{1}{2}\int \nabla w^2\cdot\nabla(-\Delta)^{-s}u\\
&= \frac{1}{2}\int w^2(-\Delta)^{1-s}u \leq C\|u\|_{H^\alpha}\|w\|^2_{L^2},
\end{aligned}\\
I_3 \leq -\frac{1}{2}\int \tilde{u}(-\Delta)^{1-s}w^2 = \int -\frac{1}{2}(-\Delta)^{1-s}\tilde{u}\cdot w^2 \leq C\|u\|_{H^\alpha}\|w\|^2_{L^2}.
\end{gathered}$$

When $s>1/2$,

$$\begin{aligned}
I_2 &\leq C\|w\|_{L^2}\|\nabla u\cdot\nabla(-\Delta)^{-s}w\|_{L^2}\\
&\leq C\|w\|_{L^2}\|\nabla u\|_{\dot{H}^{\frac{n}{2}+1-2s}}\|\nabla(-\Delta)^{-s}w\|_{\dot{H}^{2s-1}} \leq C\|u\|_{H^\alpha}\|w\|^2_{L^2}.
\end{aligned}$$

When $s=1/2$, the above estimates are still valid. Combining the above estimates we have $\frac{d}{dt}\|w\|_{L^2} \leq C\|w\|_{L^2}\|u\|_{H^\alpha}$. By Gronwall's inequality we can deduce $w(x,t)=0$ on $[0,T_0]$. □

4. Correction

In [14], trying to establish the well-posedness of (1.1) in Besov spaces the authors incurred in a mistake in page 9 when estimating the term J_4' in equation [14, (4.5)]. To correct the mistake, we modify our proof the following way.

[14, Theorem 1.1] Let $n \geq 2, s \in [\frac{1}{2},1], \alpha > n+1$. If the initial data $u_0 \in B^\alpha_{1,\infty}$, then there exists $T = T(\|u_0\|_{B^\alpha_{1,\infty}})$ such that (1.1) has a unique solution in

$[0,T]\times\mathbb{R}^n$. Such a solution belongs to $C^1([0,T];B^{\alpha+2s-2}_{1,\infty})\cap L^\infty([0,T];B^{\beta}_{1,\infty})$, with $\beta\in[\alpha+2s-2,\alpha]$.

Proof. First we construct the approximate equation

$$\begin{gathered} u_t^{(n+1)}=\nabla u^{(n+1)}\cdot\nabla(-\triangle)^{-s}u_\epsilon^{(n)}-u_\epsilon^{(n)}(-\triangle)^{1-s}u^{(n+1)};\\ u^{(n+1)}(0)=\sigma_\epsilon*u_0,\quad u^{(1)}=\sigma_\epsilon*u_0. \end{gathered}\tag{4.1}$$

By the argument in section 2, there exists a sequence $u^{(n)}$ that solves the linear systems (4.1). Assuming that $u_0\ge 0$, we prove that $u^{(n+1)}\ge 0$. Inspired by [4], we assume that x_0 is a point of minimum of $u^{(n+1)}$ at time $t=t_0$. This indicates that $\nabla u^{(n+1)}(x_0)=0$, and

$$(-\triangle)^{1-s}u^{(n+1)}(x_0)=c\int\frac{u(x_0)-u(y)}{|y|^{n+2(1-s)}}dy\le 0.$$

Thus we deduce that $\frac{\partial}{\partial t}u^{(n+1)}\big|_{t=t_0}\ge 0$, and by induction we have $u^{(n+1)}\ge 0$. Arguing as in [14], taking $\triangle_j$ on (4.1), we obtain

$$\begin{aligned} \partial_t\triangle_j u^{(n+1)}&=\sum[\triangle_j,\partial_i(-\triangle)^{-s}u_\epsilon^{(n)}]\partial_i u^{n+1}+\sum\partial_i(-\triangle)^{-s}u_\epsilon^{(n)}\triangle_j(\partial_i u^{(n+1)})\\ &\quad-[\triangle_j,u_\epsilon^{(n)}](-\triangle)^{1-s}u^{(n+1)}-u_\epsilon^{(n)}\triangle_j(-\triangle)^{1-s}u^{(n+1)}. \end{aligned}$$

Multiplying both sides by $\frac{\triangle_j u^{(n+1)}}{|\triangle_j u^{(n+1)}|}$, and integrating over $\mathbb{R}^d$, then denote the corresponding part in the right side by J_1',J_2',J_3',J_4', respectively. We obtain the estimates,

$$\begin{aligned} J_1'&\le C2^{-j\alpha}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}\|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}}\\ J_2'&\le C2^{-j\alpha}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}\|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}}\\ J_3'&\le C2^{-j\alpha}\|u^{(n)}\|_{B^\alpha_{1,\infty}}\|u^{(n+1)}\|_{B^{\alpha+1-2s}_{1,\infty}}. \end{aligned}$$

The estimate for the term J_4' is now replaced by

$$\begin{aligned} J_4'&=-\int u^{(n)}\triangle_j(-\triangle)^{1-s}u^{(n+1)}\frac{\triangle_j u^{(n+1)}}{|\triangle_j u^{(n+1)}|}\\ &\le-\int u^{(n)}(-\triangle)^{1-s}|\triangle_j u^{(n+1)}|\\ &\le-\int(-\triangle)^{1-s}u^{(n)}|\triangle u^{(n+1)}|\\ &\le 2^{-j\alpha}\|u^n\|_{B^{r+2-2s}_{1,\infty}}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}. \end{aligned}$$

Here $r>d$ is any real number. The first inequality uses the following pointwise estimate.

Proposition 4.1 ([10]). *If $0\le\alpha\le 2$, $p\ge 1$, then*

$$p|f(x)|^{p-2}f(x)\Lambda^\alpha f(x)\ge\Lambda^\alpha|f(x)|^p.$$

for any $f\in\mathcal{S}(\mathbb{R}^d)$

Taking r such that $r+2-2s<\alpha$, e.g. set $r=\alpha-1$, we conclude

$$\frac{d}{dt}\|u^{(n+1)}\|_{B^\alpha_{1,,\infty}}\le\|u^{(n)}\|_{B^\alpha_{1,,\infty}}\|u^{(n+1)}\|_{B^\alpha_{1,,\infty}}.$$

The other parts of the proof need no modification. □

Acknowledgement. We are very grateful to Ph. D. Mitia Duerinckx for pointing out our mistake and share some good views on this problem. This paper is supported by the NNSF of China under grants No. 11601223 and No.11626213.

References

[1] M. Allen, L. Caffarelli, A. Vasseur; *Porous medium flow with both a fractional potential pressure and fractional time derivative.* arXiv preprint arXiv:1509.06325, 2015.

[2] P. Biler, C. Imbert, G. Karch; *The nonlocal porous medium equation: Barenblatt profiles and other weak solutions.* Archive for Rational Mechanics and Analysis, 2015, 215(2): 497-529.

[3] L. Caffarelli, F. Soria, J. L. Vázquez; *Regularity of solutions of the fractional porous medium flow.* arXiv preprint arXiv:1201.6048, 2012.

[4] L. Caffarelli, J. L. Vázquez; *Nonlinear porous medium flow with fractional potential pressure.* Archive for rational mechanics and analysis, 2011, 202(2): 537-565.

[5] L. Caffarelli, J. L. Vázquez; *Regularity of solutions of the fractional porous medium flow with exponent* 1/2. arXiv preprint arXiv:1409.8190, 2014.

[6] L. Caffarelli, J. L. Vázquez; *Asymptotic behaviour of a porous medium equation with fractional diffusion.* arXiv preprint arXiv:1004.1096, 2010.

[7] J. A. Carrillo, Y. Huang, M. C. Santos, J. L. Vázquez; *Exponential convergence towards stationary states for the 1D porous medium equation with fractional pressure.* Journal of Differential Equations, 2015, 258(3): 736-763.

[8] A. Córdoba, D. Córdoba; *A maximum principle applied to quasi-geostrophic equations.* Communications in mathematical physics, 2004, 249(3): 511-528.

[9] N. Ju; *Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space.* Communications in Mathematical Physics, 2004, 251(2): 365-376.

[10] C. Miao, J. Wu, Z. Zhang; *Littlewood-Paley Theory and Applications to Fluid Dynamics Equations.* Monographson Modern Pure Mathematics, No. 142 (Science Press, Beijing, 2012).

[11] J. Simon. *Compact sets in the space* $L^p(0,T;B)$ Ann. Mat. Pura. Appl., (4) 146, 1987, 65-96.

[12] D. Stan, F. del Teso, J. L. Vázquez; *Transformations of self-similar solutions for porous medium equations of fractional type.* Nonlinear Analysis: Theory, Methods Applications, 2015, 119: 62-73.

[13] J. L. Vázquez; *Nonlinear diffusion with fractional Laplacian operators.* Nonlinear Partial Differential Equations. Springer Berlin Heidelberg, 2012: 271-298.

[14] X. Zhou, W. Xiao, J. Chen; *Fractional porous medium and mean field equations in Besov spaces*, Electron. J. Differential Equations, 2014 (2014), no. 199, 1-14.

Xuhuan Zhou
Department of Information Technology, Nanjing Forest Police College, 210023 Nanjing, China
E-mail address: zhouxuhuan@163.com

Weiliang Xiao
School of Applied Mathematics, Nanjing University of Finance and Economics, 210023 Nanjing, China
E-mail address: xwltc123@163.com