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EXISTENCE OF GENERALIZED ALMOST PERIODIC AND ASYMPTOTIC ALMOST PERIODIC SOLUTIONS TO ABSTRACT VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The main aim of this paper is to study the asymptotic almost periodicity, Stepanov almost periodicity, and asymptotic Stepanov almost periodicity of various classes of regularized solution operator families appearing in the theory of abstract Volterra integro-differential equations. Subgenerators of these solution operator families, which can be degenerate or non-degenerate in time, are multivalued linear operators. We contemplate the results established by many other authors, providing also a great number of original contributions, illustrative examples and applications of our abstract results.

1. Introduction and preliminaries

The notion of an asymptotically almost periodic strongly continuous semigroup was introduced by Ruess and Summers [48] in 1986, while the notion of an (asymptotically) Stepanov almost periodic strongly continuous semigroup was introduced by Henríquez [28] in 1990. As mentioned in the abstract, the main aim of this paper is to continue our recent research studies [34]-[36] by enquiring into the most important asymptotically almost periodic properties, Stepanov almost periodic properties and asymptotically Stepanov almost periodic properties of abstract (degenerate) Volterra integro-differential equations in Banach spaces. Concerning almost periodic properties and asymptotically almost periodic properties of abstract Volterra integro-differential equations that are degenerate in time, the existing literature is very limited: as already mentioned in [34], we have been able to locate only two research papers in this direction, [59] by Vu and [37] by Lan. In both of these papers, the authors have analyzed the abstract degenerate differential equations of first order (for almost periodic properties and asymptotically almost periodic properties of various types of abstract non-degenerate Volterra integro-differential equations, one refers to [1, 2], [4]-[8], [13, 16, 19, 21, 25, 36, 39], [43, Section 11.4, Section 11.6]-[44], [48, 49, 51, 53, 60, 63, 66, 67], and especially, to the monographs [29] by Hino, Naito, Minh, Shin and [15] by Cheban). Concerning (asymptotically) Stepanov almost periodic properties of abstract Volterra integro-differential equations, our results seem to be completely new and not considered elsewhere in degenerate case

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(cf. [12, 28, 46, 47] for some results established so far in non-degenerate case). We pay special attention to the analysis of generalized almost periodicity and asymptotical almost periodicity of solutions to the abstract (degenerate) Cauchy problems of first and second order.

The organization and main ideas of this paper can be briefly described as follows. In Subsection 1.1 and Subsection 1.2, we recall the basic facts about multivalued linear operators in Banach spaces and various types of solution operator families subgenerated by them. We open Section 2 by examining various questions about generalized almost periodicity and asymptotical almost periodicity, proving especially that the asymptotical almost periodicity is preserved under the action of subordination principle discovered by Bazhlekova [10, Theorem 3.1] (see Theorem 2.6(iii)) and that for each number $\alpha \in (0,2) \setminus \{1\}$ the only Stepanov almost periodic non-degenerate (g_α, C)-resolvent family ($C \in L(X)$ injective, with dense range) is that one generated by the zero operator (see Theorem 2.10). In Example 2.4, we will see that the notion of asymptotical almost periodicity is much more appropriate for dealing with the abstract fractional differential equations than that of almost periodicity, proving also that the main results of investigation [28], established for semigroups and cosine operator functions, are no longer true for fractional resolvent families of Caputo order $\alpha \in (0,2) \setminus \{1\}$ (cf. Proposition 2.1, Proposition 2.3, Proposition 2.5 and Theorem 2.8 for some other results given in this part). After proving Theorem 2.10, we break down the material in three separate subsections. In Subsection 2.1, we investigate the Stepanov (asymptotically) almost periodic properties of convolution products appearing in the variation of parameters formulae; in this subsection, we present the most important examples and applications of our abstract theoretical results. Subsection 2.2 is written in a half-expository manner: its aim is to extend the results of Henríquez [28] and Rao [46] as well as some results of Casarino [13], Cioranescu, Ubilla [14] and Vesentini [58] to (degenerate) C-(semi)groups and C-cosine operator functions in Banach spaces. In Subsection 2.3, we reconsider some structural results from the research study [63] by Xie, Li, Huang and from our recent joint research study with Pilipović and Velinov [36]; in this subsection, we analyze the subspace asymptotical almost periodicity of non-degenerate C-distribution semigroups and C-distribution cosine functions ($C \in L(X)$ injective). Primarily from the time and space limitations, we have decided to examine the Stepanov (asymptotically) almost periodic properties of semilinear Cauchy problems and inclusions in a new separate paper. We propose several open problems to our researchers.

We use the standard notation throughout the paper. By X we denote a complex Banach space. If Y is also such a space, then by $L(X,Y)$ we denote the space of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. If A is a linear operator acting on X, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph; $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm (here we assume that A is closed). The injectiveness of operator $C \in L(X)$, if necessary, will be explicitly emphasized. The dual space of X is denoted by X^* and the adjoint operator of A by A^*.

Given $s \in \mathbb{R}$ in advance, set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. Define $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$ ($\alpha \in (0,\pi]$). The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ ($\zeta > 0$, $t > 0$), $0^0 := 1$, $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$ and $g_0(t) :=$ the Dirac δ-distribution. By $C_b([0,\infty) : X)$ we denote the space consisted of all bounded continuous functions from $[0,\infty)$ into X; the symbol $C_0([0,\infty) : X)$ denotes the closed subspace of $C_b([0,\infty) : X)$ consisting of functions vanishing at infinity. By $BUC([0,\infty) : X)$ we denote the space consisted of all bounded uniformly continuous functions from $[0,\infty)$ to X. This space becomes one of Banach's when equipped with the sup-norm.

We refer the reader to [36] for the notion and notation of vector-valued distribution spaces used henceforth (more details can be found in [54]-[55]). Let us only recall that, for every $t \in \mathbb{R}$, we define the Dirac distribution centered at point t, δ_t for short, by $\delta_t(\varphi) := \varphi(t)$, $\varphi \in \mathcal{D}$.

Fractional calculus and fractional differential equations are popular fields of research nowdays (cf. [10], [23], [30]-[33], [52] and references cited therein for further information in this direction). The Mittag-Leffler function $E_{\alpha,\beta}(z)$, defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C},$$

plays a crucial role in the analysis of fractional differential equations ($\alpha > 0$, $\beta \in \mathbb{R}$). Set $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$.

In the sequel, we assume that the function $k(t)$ is a scalar-valued continuous kernel on $[0,\infty)$. The following condition on function $k(t)$ will be used occasionally:

(A1) $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} k(t)\,dt := \int_0^\infty e^{-\lambda t} k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $abs(k) :=\inf\{\Re\lambda : \tilde{k}(\lambda)$ exists$\}$, $\tilde{\delta}(\lambda) := 1$ and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

Let $\text{abs}(k) = 0$. Following Batty, van Neerven and Räbiger [9], we say that a point $\lambda = i\nu \in i\mathbb{R}$ is a regular point for $k(t)$ if there is an open neighborhood U of λ in $\mathbb{C}$ and a holomorphic function $g : U \to X$ such that $g(z) = \tilde{k}(z)$ whenever $z \in U \cap \mathbb{C}_+$. The singular set E of $k(t)$ is the set consisting of all points of $i\mathbb{R}$ which are not regular points. For further information concerning the vector-valued Laplace transform, the reader may consult [4], [61, Chapter 1] and [32, Section 1.2].

Suppose that $\gamma \in (0,1)$ and $\omega \in \mathbb{R}$. Then the Wright function $\Phi_\gamma(\cdot)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geq 0$ (cf. [10] and references cited therein for more details on the subject).

The concept of almost periodicity was introduced by Bohr in 1925 and later generalized by many other authors (cf. [17, 22, 26, 27, 38, 64] for more details on the subject). Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, if and only if for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$. The space consisted of all almost periodic functions from the interval I into X will be denoted by $AP(I : X)$.

The notion of an asymptotically almost periodic function was introduced by Fréchet in 1941 (for more details concerning the vector-valued asymptotically almost periodic functions and asymptotically almost periodic differential equations, see e.g. [9, 15, 20, 22, 26, 27, 49, 50, 53, 63, 65]). A function $f \in C_b([0,\infty) : X)$ is said to be asymptotically almost periodic if for every $\epsilon > 0$ we can find numbers $l > 0$ and $M > 0$ such that every subinterval of $[0, \infty)$ of length l contains, at least, one number τ such that $\|f(t+\tau) - f(t)\| \leq \epsilon$ for all $t \geq M$. The space consisted of all asymptotically almost periodic functions from $[0,\infty)$ into X will be denoted by $AAP([0,\infty) : X)$. It is well known that for any function $f \in C([0,\infty) : X)$, the following statements are equivalent:

(i) $f \in AAP([0,\infty) : X)$.
(ii) There exist uniquely determined functions $g \in AP([0,\infty) : X)$ and $\phi \in C_0([0,\infty) : X)$ such that $f = g + \phi$.
(iii) The set $H(f) := \{f(\cdot + s) : s \geq 0\}$ is relatively compact in $C_b([0,\infty) : X)$.

Let $1 \leq p < \infty$. Then we say that a function $f \in L^p_{\rm loc}(I : X)$ is Stepanov p-bounded, S^p-bounded shortly, if

$$\|f\|_{S^p} := \sup_{t \in I} \Big(\int_t^{t+1} \|f(s)\|^p \, ds \Big)^{1/p} < \infty.$$

The space $L^p_S(I : X)$ consisted of all S^p-bounded functions becomes a Banach space when equipped with the above norm. A function $f \in L^p_S(I : X)$ is said to be Stepanov p-almost periodic, S^p-almost periodic shortly, if and only if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I, \ s \in [0,1]$$

is almost periodic (cf. [3] for more details). It is said that $f \in L^p_S([0,\infty) : X)$ is asymptotically Stepanov p-almost periodic, asymptotically S^p-almost periodic shortly, if and only if $\hat{f} : [0,\infty) \to L^p([0,1] : X)$ is asymptotically almost periodic.

It is a well-known fact that if $f(\cdot)$ is an almost periodic (respectively, a.a.p.) function then $f(\cdot)$ is also S^p-almost periodic (resp., asymptotically S^p-a.a.p.) for $1 \leq p < \infty$. The converse statement is false, however. The notion of a scalary S^p-almost periodic function, slightly different from the notion of usually considered weakly S^p-almost periodic function, is given as follows: A function $f \in L^p_S(I : X)$ is said to be scalarly Stepanov p-almost periodic if and only if for each $x^* \in X^*$ we have that the function $x^*(f) : [0,\infty) \to \mathbb{C}$ defined by $x^*(f)(t) := x^*(f(t))$, $t \geq 0$ is Stepanov p-almost periodic.

In the case that the value of p is irrelevant, then we simply say that the function under our consideration is (asymptotically, scalarly) Stepanov almost periodic.

We need the assertion of [28, Lemma 1]:

Lemma 1.1. *Suppose that $f : [0,\infty) \to X$ is an asymptotically S^p-almost periodic function. Then there are two locally p-integrable functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:*

(i) *g is S^p-almost periodic,*
(ii) *$\hat{q}$ belongs to the class $C_0([0,1] : L^p([0,1] : X))$,*
(iii) *$f(t) = g(t) + q(t)$ for all $t \geq 0$.*

Moreover, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim_{n\to\infty} t_n = \infty$ and $g(t) = lim_{n\to\infty} f(t+t_n)$ a.e. $t \geq 0$.

Denote by $\mathfrak{T}_p$ the class of all locally p-integrable functions $q : [0,\infty) \to X$ satisfying that $\hat{q}$ belongs to $C_0([0,1] : L^p([0,1] : X))$.

Let $(R(t))_{t\geq 0} \subseteq L(X)$ be a strongly continuous operator family, and let $\oplus$ denote any of (asymptotically) almost periodic properties mentioned above. Then we say that $(R(t))_{t\geq 0}$ is $\oplus$ (asymptotically) almost periodic if and only if the mapping $t \mapsto R(t)x$, $t \geq 0$ is $\oplus$ (asymptotically) almost periodic for all $x \in X$.

The reader may consult the monographs by Carroll, Showalter [11], Favini, Yagi [24], Melnikova, Filinkov [40], Sviridyuk, Fedorov [56] and Kostić [33] for the theory of abstract degenerate differential equations. In the following subsection, we will present a brief recollection of results and definitions from the theory of multivalued linear operators in Banach spaces.

1.1. **Multivalued linear operators in Banach spaces.** Suppose that X and Y are Banach spaces. Let us recall that a multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) if and only if the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;
(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$, $x, y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then we say that $\mathcal{A}$ is an MLO in X.

The fundamental equality used below says that, if $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, then $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. Assuming $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear submanifold of Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. Then the set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted by either $N(\mathcal{A})$ or $\mathrm{Kern}(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It can be simply checked that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$; $\mathcal{A}$ is said to be injective if and only if $\mathcal{A}^{-1}$ is single-valued.

For any mapping $\mathcal{A} : X \to P(Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}),\ y \in \mathcal{A}x\}$. Then $\mathcal{A}$ is an MLO if and only if $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., if and only if $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$.

Assume that $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs. Then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It is clear that $\mathcal{A} + \mathcal{B}$ is likewise an MLO.

Let $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is also a Banach space. The product of operators $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A} : X \to P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A} : X \to P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z,\ \omega \in \mathbb{C}$.

Assume that X' is a linear subspace of X, and $\mathcal{A} : X \to P(Y)$ is an MLO. The restriction of operator $\mathcal{A}$ to the subspace X', $\mathcal{A}_{|X'}$ for short, is defined by $D(\mathcal{A}_{|X'}) := D(\mathcal{A}) \cap X'$ and $\mathcal{A}_{|X'}x := \mathcal{A}x$, $x \in D(\mathcal{A}_{|X'})$. It is evident that $\mathcal{A}_{|X'} : X' \to P(Y)$ is an MLO.

If $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are two MLOs, then we write $\mathcal{A} \subseteq \mathcal{B}$ if and only if $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

Let $\mathcal{A}$ be an MLO in X. Then we say that a point $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ if and only if there exists a vector $x \in X \setminus \{0\}$ such that $\lambda x \in \mathcal{A}x$; we call x an eigenvector of operator $\mathcal{A}$ corresponding to the eigenvalue λ. Observe that, in purely multivalued case, a vector $x \in X \setminus \{0\}$ can be an eigenvector of operator $\mathcal{A}$

corresponding to different values of scalars λ. The point spectrum of $\mathcal{A}$, $\sigma_p(\mathcal{A})$ for short, is defined as the union of all eigenvalues of $\mathcal{A}$.

We say that an MLO operator $\mathcal{A} : X \to P(Y)$ is closed if and only if for any two sequences (x_n) in $D(\mathcal{A})$ and (y_n) in Y such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ we have that the preassumptions $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

We will use the following crucial lemma from [33].

Lemma 1.2. *Let Ω be a locally compact, separable metric space, and let μ be a locally finite Borel measure defined on Ω. Suppose that $\mathcal{A} : X \to P(Y)$ is a closed* MLO. *Let $f : \Omega \to X$ and $g : \Omega \to Y$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f\, d\mu \in D(\mathcal{A})$ and $\int_\Omega g\, d\mu \in \mathcal{A}\int_\Omega f\, d\mu$.*

Assume now that $\mathcal{A}$ is an MLO in X, as well as that $C \in L(X)$ and $C\mathcal{A} \subseteq \mathcal{A}C$ (observe that we allow C to be possibly non-injective). Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued bounded operator on X.

The spectrum $\sigma_C(\mathcal{A})$ of $\mathcal{A}$ is defined as the complement of $\rho_C(\mathcal{A})$ in $\mathbb{C}$; $\sigma(\mathcal{A}) := \sigma_I(\mathcal{A})$, where I denotes the identity operator on X. The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). The basic properties of C-resolvent sets of single-valued linear operators continue to hold in our framework ([31]-[33]).

Of concern is the following abstract degenerate Volterra inclusion:

$$\mathcal{B}u(t) \subseteq \mathcal{A}\int_0^t a(t-s)u(s)\, ds + \mathcal{F}(t),\ t \in [0,\tau),$$

where $a \in L^1_{\rm loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ and $\mathcal{B} : X \to P(X)$ are given multivalued linear operators, and $\mathcal{F} : X \to P(X)$ is a given mutivalued mapping, as well as the following fractional Sobolev inclusions:

$$\begin{gathered} \mathbf{D}_t^\alpha Bu(t) \in \mathcal{A}u(t) + \mathcal{F}(t), \quad t \geq 0, \\ (Bu)^{(j)}(0) = Bx_j, \quad 0 \leq j \leq \lceil \alpha \rceil - 1, \end{gathered} \tag{1.1}$$

where we assume that $B = \mathcal{B}$ is single-valued, and

$$\begin{gathered} \mathcal{B}\mathbf{D}_t^\alpha u(t) \subseteq \mathcal{A}u(t) + \mathcal{F}(t), \quad t \geq 0, \\ u^{(j)}(0) = x_j, \quad 0 \leq j \leq \lceil \alpha \rceil - 1, \end{gathered} \tag{1.2}$$

where $\alpha > 0$ and $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative ([10, 32]). For the notions of various types of solutions of the above Cauchy problems, we refer the reader to [33].

1.2. **Degenerate (a,k)-regularized C-resolvent family.** The following definitions have been recently introduced in [33]:

Definition 1.3. Assume that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\rm loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$.

(i) Then we say that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)} \subseteq L(Y,X) \times L(X)$ if and only if the mappings $t \mapsto R_1(t)y$, $t \geq 0$ and $t \mapsto R_2(t)x$, $t \in [0,\tau)$ are continuous for every fixed $x \in X$ and $y \in Y$, as well as the following conditions hold:
$$\Big(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1y\Big) \in \mathcal{A}, \tag{1.3}$$
for $t \in [0,\tau)$, $y \in Y$, and
$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2x, \tag{1.4}$$
whenever $t \in [0,\tau)$ and $(x,y) \in \mathcal{A}$.

(ii) Suppose that $(R_1(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is strongly continuous. Then we say that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ if and only if (1.3) holds.

(iii) Suppose that $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$ is strongly continuous; then we say that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ if and only if (1.4) holds.

Definition 1.4. Assume that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\rm loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C \in L(X)$ and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ if and only if $(R(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized C-uniqueness family having $\mathcal{A}$ as subgenerator, as well as $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in [0,\tau))$.

Any (a,k)-regularized C-resolvent family analyzed below will be likewise a mild (a,k)-regularized C-existence family and the condition $0 \in \operatorname{supp}(a)$ will be assumed.

It is said that an (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ is exponentially bounded (bounded) if and only if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\} \subseteq L(X)$ is bounded. If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geq 0$, then it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family. We pay special attention to the case $a(t) \equiv 1$, resp. $a(t) \equiv t$, when we say that $(R(t))_{t\geq 0}$ is an α-times integrated C-semigroup (C-(regularized) semigroup, if $\alpha = 0$), resp. an α-times integrated C-cosine function (C-(regularized) cosine function, if $\alpha = 0$).

The symbol $\chi(R)$ stands for the set consisting of all subgenerators of $(R(t))_{t\in[0,\tau)}$. Clearly, for each subgenerator $\mathcal{A} \in \chi(R)$ we have $\overline{\mathcal{A}} \in \chi(R)$. The set $\chi(R)$ can have infinitely many elements; furthermore, if $\mathcal{A} \in \chi(R)$, then $\mathcal{A} \subseteq \mathcal{A}_{int}$, where the integral generator of $(R(t))_{t\in[0,\tau)}$ is defined by
$$\mathcal{A}_{int} := \Big\{(x,y) \in X \times X : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds \text{ for all } t \in [0,\tau)\Big\}.$$
The integral generator $\mathcal{A}_{int}$ of $(R(t))_{t\in[0,\tau)}$ is a closed subgenerator of $(R(t))_{t\in[0,\tau)}$, provided that $\tau = \infty$. If $\mathcal{A}$ and $\mathcal{B}$ are two subgenerators of $(R(t))_{t\in[0,\tau)}$ and α, $\beta \in \mathbb{C}$ with $\alpha + \beta = 1$, then $C(D(\mathcal{A})) \subseteq D(\mathcal{B})$, $\mathcal{A}_{int} \subseteq C^{-1}\mathcal{A}C$ and $\alpha\mathcal{A} + \beta\mathcal{B}$ is also a subgenerator of $(R(t))_{t\in[0,\tau)}$; furthermore, if C is injective, then $\mathcal{A}_{int} = C^{-1}\mathcal{A}C$.

We refer the reader to [33] for the notion of an (exponentially) bounded analytic (a,k)-regularized C-resolvent family subgenerated by a multivalued linear operator.

2. Generalized almost periodic properties and asymptotically almost periodic properties of abstract Volterra integro-differential equations

We start this section by stating the following simple consequence of [9, Theorem 4.1], which provides proper extensions of [63, Theorem 3.7, Theorem 3.9] (cf. also [6, Theorem 4.1]).

Theorem 2.1. *Suppose that* $\operatorname{abs}(k)=\operatorname{abs}(|a|)=0$, $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$ *for* $\Re\lambda>0$, $C_1\in L(Y,X)$, *and* $\mathcal{A}$ *is a closed subgenerator of a mild* (a,k)*-regularized* C_1*-existence family* $(R_1(t))_{t\geq 0}$. *Let* $y\in Y$ *be such that the mapping* $t\mapsto R_1(t)y$, $t\geq 0$ *is bounded and uniformly continuous. If the operator* $I-\tilde{a}(\lambda)\mathcal{A}$ *is injective for* $\Re\lambda>0$, *then*

$$\mathfrak{H}(\lambda):=\frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\Big(\frac{1}{\tilde{a}(\lambda)}-\mathcal{A}\Big)^{-1}C_1y=\int_0^{\infty}e^{-\lambda t}R_1(t)y\,dt,\quad \Re\lambda>0. \tag{2.1}$$

Suppose that the singular set iS *of mapping* $\lambda\mapsto\mathfrak{H}(\lambda)$, $\Re\lambda>0$, *where* $S\subseteq\mathbb{R}$, *is at most countable. If for every* $\mu\in S$, *we have that* $\lim_{\lambda\to 0+}\lambda\int_0^{\infty}e^{-(\lambda+i\mu)t}R_1(t+s)y\,dt$ *exists, uniformly in* $s\geq 0$, *then the mapping* $t\mapsto R_1(t)y$, $t\geq 0$ *is asymptotically almost periodic.*

Proof. Applying Lemma 1.2 and the Laplace transform, we get that

$$\tilde{k}(\lambda)C_1y\in\big(I-\tilde{a}(\lambda)\mathcal{A}\big)\int_0^{\infty}e^{-\lambda t}R_1(t)y\,dt,\quad \Re\lambda>0$$

i.e.,

$$\frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\Big(\frac{1}{\tilde{a}(\lambda)}-\mathcal{A}\Big)^{-1}C_1y=\tilde{k}(\lambda)\big(I-\tilde{a}(\lambda)\mathcal{A}\big)^{-1}C_1y\ni\int_0^{\infty}e^{-\lambda t}R_1(t)y\,dt,$$

for $\Re\lambda>0$. Since the operator $I-\tilde{a}(\lambda)\mathcal{A}$ is injective for $\Re\lambda>0$, we have that the set $((1/\tilde{a}(\lambda))-\mathcal{A})^{-1}C_1x$ is singleton, so that the last equality immediately implies (2.1). Now the proof follows from a simple application of [9, Theorem 4.1]. $\square$

Remark 2.2. Suppose $\alpha>0$, $\beta\geq 0$, $a(t)=g_{\alpha}(t)$, $k(t)=g_{\beta+1}(t)$, $X=Y$, $C_1=I$ and the set of all $\lambda\in i\mathbb{R}\setminus\{0\}$ such that $\lambda^{\alpha}\in\sigma(\mathcal{A})$ is at most countable. Then the singular set iS of mapping $\lambda\mapsto\mathfrak{H}(\lambda)$, $\Re\lambda>0$ is at most countable, as well.

Further information on connections between countable spectrum of operators and asymptotical almost periodicity can be obtained by consulting the monograph [4]. We continue by stating the following important proposition, which is very similar to [34, Proposition 4.1]:

Proposition 2.3. *Suppose that abs*$(|a|)<\infty$, $\operatorname{abs}(k)<\infty$ *and* $\mathcal{A}$ *is a subgenerator of a mild, strongly Laplace transformable,* (a,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\geq 0}$. *Denote by* D *the set consisting of all eigenvectors* x *of operator* $\mathcal{A}$ *which corresponds to eigenvalues* $\lambda\in\mathbb{C}$ *of operator* $\mathcal{A}$ *for which the mapping*

$$f_{\lambda,x}(t):=\mathcal{L}^{-1}\Big(\frac{\tilde{k}(z)}{1-\lambda\tilde{a}(z)}\Big)(t)C_2x,\quad t\geq 0$$

is asymptotically almost periodic. Then the mapping $t\mapsto R_2(t)x$, $t\geq 0$ *is asymptotically almost periodic for all* $x\in\operatorname{span}(D)$; *furthermore, the mapping* $t\mapsto R_2(t)x$,

$t \geq 0$ *is asymptotically almost periodic for all* $x \in \overline{\operatorname{span}(D)}$ *provided additionally that* $(R_2(t))_{t\geq 0}$ *is bounded.*

Proof. Suppose that $x \in D$ is an eigenvector of operator $\mathcal{A}$ which corresponds to an eigenvalue $\lambda \in \sigma_p(\mathcal{A})$. Using the identity

$$\lambda \int_0^t a(t-s)R_2(s)x\,ds = R_2(t)x - k(t)C_2x, \quad t \geq 0,$$

and performing the Laplace transform, we obtain that $R_2(t)x = f_{\lambda,x}(t)$, $t \geq 0$. This immediately implies the result since $AAP([0,\infty) : X)$ is a closed subspace of $BUC([0,\infty) : X)$. $\square$

As explained in [34, Remark 4.3], it is very complicated to apply [34, Proposition 4.1] in the case that $a(t) = g_\alpha(t)$, where $\alpha \in (0,\infty) \setminus \mathbb{N}$; cf. also Theorem 2.10 below. The situation is completely different if we consider the asymptotical almost periodicity, when Proposition 2.3 can be essentially applied:

Example 2.4. Suppose that $\alpha \in (0,2)$ and $\theta = \pi - \pi\alpha/2$. Let us consider the fractional Cauchy problem:

$$\mathbf{D}_t^\alpha u(t,x) = e^{i\theta}u_{xx}(t,x), \quad 0 < x < 1,\ t \geq 0,$$

proposed already by Bazhlekova in her doctoral dissertation [10, Example 2.20] and equipped with initial boundary conditions like for the general problem of form (1.2) with $B = I$.

Let $X := L^2[0,1]$ and $A := e^{i\theta}\Delta$, where Δ denotes the Dirichlet Laplacian. It is well known that A is the integral generator of a bounded (g_α, I)-resolvent family $(R(t))_{t\geq 0}$. Since A has eigenvalues $\lambda_n = e^{i\alpha\pi/2}n^2\pi^2$ and eigenfunctions $x_n = \sin n\pi x$, $n \in \mathbb{N}$, the Laplace transform identity [10, (1.26)] shows that

$$f_{\lambda_n,x_n}(t) := E_\alpha\big(e^{i\alpha\pi/2}n^2\pi^2t^\alpha\big)(t)x_n, \quad t \geq 0,\ n \in \mathbb{N}.$$

In the case that $\alpha = 1$, the above simply implies by [34, Theorem 3.1] that $(R(t))_{t\geq 0}$ is almost periodic. The situation is quite different in the case that $\alpha \in (0,2) \setminus \{1\}$: Then the asymptotic expansion formulae for the Mittag-Leffler functions [(1.27)-(1.28)] imply that the mapping $t \mapsto f_{\lambda_n,x_n}(t)$, $t \geq 0$ is asymptotically almost periodic for all $n \in \mathbb{N}$, because the mapping $t \mapsto \alpha^{-1}e^{itn^{2/\alpha}\pi^{2/\alpha}}$, $t \geq 0$ is almost periodic and the mapping $t \mapsto E_\alpha\big(e^{i\alpha\pi/2}n^2\pi^2t^\alpha\big)(t) - \alpha^{-1}e^{itn^{2/\alpha}\pi^{2/\alpha}}$, $t \geq 0$ is continuous, tending to zero as t goes to infinity. Hence, $(R(t))_{t\geq 0}$ is asymptotically almost periodic (since $AP([0,\infty) : X) \cap C_0([0,\infty) : X) = \{0\}$, the mapping $t \mapsto R(t)x$, $t \geq 0$ cannot be almost periodic for $x \in \operatorname{span}(\{x_n : n \in \mathbb{N}\})$; this is a very intriguing fact for fractional resolvent families of order α close to $2-$ because H. R. Henríquez has proved [28, Theorem 3] that a strongly continuous cosine operator function $(C(t))_{t\geq 0}$ is almost periodic if and only if $(C(t))_{t\geq 0}$ is asymptotically almost periodic if and only if $(C(t))_{t\geq 0}$ is Stepanov asymptotically almost periodic). Finally, we would like to observe that we can similarly examine asymptotically almost periodic solutions of certain classes of abstract muti-term fractional Cauchy problems with Caputo derivatives, see e.g. [32, Example 2.10.32].

Consider again the situation of Proposition 2.3. If $x \in D$, $\lambda x \in \mathcal{A}x$ and $C_2x \neq 0$, then the function

$$\vartheta(t) := t \mapsto \mathcal{L}^{-1}\Big(\frac{\tilde{k}(z)}{1 - \lambda\tilde{a}(z)}\Big)(t), \quad t \geq 0$$

needs to be asymptotically almost periodic. By our examinations from [34, Remark 4.3], the most important case in which the above holds is that there exist integer $n \in \mathbb{N}$, real numbers $r_1(\lambda), \dots, r_n(\lambda)$, positive real number $\omega(\lambda)$, complex numbers $\alpha_1(\lambda), \dots, \alpha_n(\lambda)$, and a function $f \in C_0([0,\infty))$, such that

$$\frac{\tilde{k}(z)}{1-\lambda\tilde{a}(z)} = \frac{\alpha_1(\lambda)}{z - ir_1(\lambda)} + \dots + \frac{\alpha_n(\lambda)}{z - ir_n(\lambda)} + \tilde{f}(z), \quad \Re z > \omega(\lambda). \tag{2.2}$$

It is clear that (2.2) holds for substantially large classes of kernels $a(t)$ and regularizing functions $k(t)$.

Propositions 2.3 and 2.5 below, whose proof is omitted, can be also formulated for the class of exponentially bounded (a,k)-regularized C-resolvent families generated by a pair of closed linear operators, see [33] for the notion and [34, Proposition 4.14] for the corresponding result in the case of almost periodicity.

Proposition 2.5. *Suppose that* $\operatorname{abs}(|a|) < \infty$, $\operatorname{abs}(k) < \infty$ *and* $\mathcal{A}$ *is a subgenerator of a mild, strongly Laplace transformable,* (a,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\geq 0}$*. Denote by* D *the set consisting of all eigenvectors* x *of operator* $\mathcal{A}$ *which corresponds to eigenvalues* $\lambda \in \mathbb{C}$ *of operator* $\mathcal{A}$ *for which the mapping*

$$f_{\lambda,x}(t) := \mathcal{L}^{-1}\Big(\frac{\tilde{k}(z)}{1-\lambda\tilde{a}(z)}\Big)(t)C_2x, \quad t \geq 0$$

is (asymptotically) Stepanov almost periodic. Then the mapping $t \mapsto R_2(t)x$, $t \geq 0$ *is (asymptotically) Stepanov almost periodic for all* $x \in \operatorname{span}(D)$*; furthermore, the mapping* $t \mapsto R_2(t)x$, $t \geq 0$ *is (asymptotically) Stepanov almost periodic for all* $x \in \overline{\operatorname{span}(D)}$ *provided additionally that* $(R_2(t))_{t\geq 0}$ *is bounded.*

Now we would like to inscribe some basic facts about asymptotical almost periodicity of subordinated fractional solution operator families (see e.g. [10, Theorem 3.1] for a pioneering result in this direction). Assume that $0 < \alpha < \beta$, $\gamma = \alpha/\beta$ and $(S_\beta(t))_{t\geq 0} \subseteq L(X)$ is a strongly continuous operator family satisfying $\|S_\beta(t)\| \leq Me^{\omega t}$, $t \geq 0$ for some constants $M \geq 1$ and $\omega \in \mathbb{R}$. A great number of subordination principles appearing in the theory of abstract (degenerate) Volterra integro-differential equations is closely connected with the following formula:

$$S_\alpha(t)x := \int_0^\infty \Phi_\gamma(s) S_\beta\big(st^\gamma\big)x\, ds, \ x \in X, \ t > 0 \text{ and } S_\alpha(0) := S_\beta(0). \tag{2.3}$$

Concerning the inheritance of asymptotical almost periodicity under the action of this subordination principle, we have the following result.

Theorem 2.6. (i) *Suppose that* $\omega < 0$*. Then* $\|S_\alpha(t)\| = O(t^{-\gamma})$, $t \geq 1$.

(ii) *Suppose that the mapping* $t \mapsto S_\beta(t)x$, $t \geq 0$ *belongs to* $C_0([0,\infty) : X)$*. Then the mapping* $t \mapsto S_\alpha(t)x$, $t \geq 0$ *belongs to* $C_0([0,\infty) : X)$*, as well.*

(iii) *Suppose that the mapping* $t \mapsto S_\beta(t)x$, $t \geq 0$ *belongs to* $AAP([0,\infty) : X)$*. Then the mapping* $t \mapsto S_\alpha(t)x$, $t \geq 0$ *belongs to* $AAP([0,\infty) : X)$*, as well.*

Proof. By [10, (1.31)], we have that

$$\int_0^\infty e^{-zs}\Phi_\gamma(s)\, ds = E_\gamma(-z), \quad z \in \mathbb{C}. \tag{2.4}$$

Keeping in mind this identity, (2.3) and the asymptotic expansion formulae for Mittag-Leffler functions [10, (1.27)-(1.28)], it readily follows that the assumption

$\omega < 0$ yields that, for every $x \in X$,

$$\big\|S_\alpha(t)x\big\| \leq M\|x\|\big|E_\gamma\big(\omega t^\gamma\big)\big| = M\|x\|\Big[-\frac{t^{-\gamma}}{\omega\Gamma(\gamma-1)} + O\big(t^{-2\gamma}\big)\Big], \quad t \to +\infty.$$

This proves (i). To prove (ii), choose a number $\epsilon > 0$ arbitrarily. Then there exists $M > 0$ such that $\|g(v)\| < \epsilon$, $v \geq M$. Suppose that $\|g(v)\| < M'$, $v \geq 0$ for some finite constant $M' > 0$. Then

$$\begin{aligned} \|S_\alpha(t)x\| &\leq \int_0^{Mt^{-\gamma}} \Phi_\gamma(s)\|g\big(st^\gamma\big)\|\,ds + \int_{Mt^{-\gamma}}^\infty \Phi_\gamma(s)\|g\big(st^\gamma\big)\|\,ds \\ &\leq \int_0^{Mt^{-\gamma}} \Phi_\gamma(s)M'\,ds + \int_{Mt^{-\gamma}}^\infty \Phi_\gamma(s)\epsilon\,ds \\ &\leq \int_0^{Mt^{-\gamma}} \Phi_\gamma(s)M'\,ds + \int_0^\infty \Phi_\gamma(s)\epsilon\,ds \\ &= \int_0^{Mt^{-\gamma}} \Phi_\gamma(s)M'\,ds + \epsilon < 2\epsilon, \quad t \to +\infty. \end{aligned}$$

It remains to be proved (iii). By (ii), it suffices to show that, for every function $f \in AP([0,\infty) : X)$, the function

$$F(t) := \int_0^\infty \Phi_\gamma(s)f\big(st^\gamma\big)\,ds, \quad t > 0;\ F(0) := f(0)$$

belongs to the space $AAP([0,\infty) : X)$. But, for every $n \in \mathbb{N}$, we can find a trigonometric polynomial $f_n(\cdot)$ such that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in $BUC([0,\infty) : X)$, as $n \to \infty$. Define

$$F_n(t) := \int_0^\infty \Phi_\gamma(s)f_n\big(st^\gamma\big)\,ds, \quad t > 0,\ F_n(0) := f_n(0).$$

By the proof of [10, Theorem 3.1], the function $F(\cdot)$ and functions $F_n(\cdot)$ are continuous ($n \in \mathbb{N}$). Furthermore, $\|F_n - F\|_\infty \leq \sup_{t\geq 0}\int_0^\infty \Phi_\gamma(s)\|f_n(st^\gamma) - f(st^\gamma)\|\,ds \leq \int_0^\infty \Phi_\gamma(s)\|f_n - f\|_\infty\,ds = \|f_n - f\|_\infty \to 0$, $n \to \infty$. Keeping in mind that $AAP([0,\infty) : X)$ is closed in the space $BUC([0,\infty) : X)$, it remains to be proved that $F_n \in AAP([0,\infty) : X)$ for all $n \in \mathbb{N}$ ($n \in \mathbb{N}$). This follows from an application of (2.4), showing that $F_n(\cdot)$ is a linear combination of functions like $E_\gamma(i\theta\cdot^\gamma) \otimes x$ ($\theta \in \mathbb{R}$, $x \in X$). The final conclusion is a consequence of the fact that $E_\gamma(i\theta\cdot^\gamma) = 1$ for $\theta = 0$ and $E_\gamma(i\theta\cdot^\gamma) \in C_0([0,\infty) : X)$ for $\theta \neq 0$, which follows from the asymptotic expansion formulae for the Mittag-Leffler functions [10, (1.27)-(1.28)]. $\square$

Remark 2.7. It is very non-trivial and difficult to say anything relevant about the invariance of asymptotical Stepanov almost periodicity under the action of this subordination principle.

Let $f : [0,\infty) \to X$ be Stepanov almost periodic. Then the Bohr-Fourier coefficients $P_r(f) := \lim_{t\to\infty}\frac{1}{t}\int_\alpha^{t+\alpha} e^{-irs}f(s)\,ds$ exists for all $r \in \mathbb{R}$, independently of $\alpha \in \mathbb{R}$, and the assumption $P_r(f) = 0$ for all $r \in \mathbb{R}$ implies that $f(t) = 0$ for a.e. $t \in \mathbb{R}$. Especially, with $r = \alpha = 0$, we obtain that the first antiderivative of $f(\cdot)$ is exponentially bounded so that the function $f(\cdot)$ satisfies (A1). Keeping these facts in mind, we can repeat literally the proof of [34, Theorem 4.5] in order to see that the following result holds true (cf. also [34, Remark 4.6]):

Theorem 2.8. *Let $\mathcal{A}$ be the integral generator of a Stepanov almost periodic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$, let $\overline{R(C)} = \overline{D(\mathcal{A})} = X$, and let $k(0) \neq 0$. Denote*

$$\mathcal{R} := \big\{ r \in \mathbb{R} : \tilde{a}(ir) \text{ exists} \big\}.$$

Suppose that $k(t)$ and $|a|(t)$ satisfy (A1)*, $\lim_{\Re z \to \infty} \tilde{a}(z) = 0$ as well as that*

$$P_r^k = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} k(s)\, ds = 0, \quad r \in \mathcal{R}.$$

Then we have

- *$P_r^R x \in \mathcal{A}[\tilde{a}(ir) P_r^R x]$, $r \in \mathcal{R}$, $x \in X$ and the mapping*

$$R(t)P_r^R x = \mathcal{L}^{-1}\Big(\frac{\tilde{k}(z)\tilde{a}(ir)}{\tilde{a}(ir) - \tilde{a}(z)} \Big)(t) C P_r^R x, \quad t \geq 0, \ x \in X,$$

  is Stepanov almost periodic for all $r \in \mathcal{R}$ and $x \in X$.

Suppose, in addition, that

$$R(t)P_r^R x = k(t) C P_r^R x, \quad t \geq 0, \ r \in \mathbb{R} \setminus \mathcal{R}, \ x \in X.$$

Then the set D consisting of all eigenvectors of operator $\mathcal{A}$ which corresponds to eigenvalues $\lambda \in \{0\} \cup \{\tilde{a}(ir)^{-1} : r \in \mathcal{R}, \ \tilde{a}(ir) \neq 0\}$ of operator $\mathcal{A}$ is total in X.

For the proof of Theorem 2.10 stated below, we need the following auxiliary lemma.

Lemma 2.9. *Suppose that $\alpha \in (0,2) \setminus \{1\}$ and $r \in \mathbb{R} \setminus \{0\}$. Then the function $t \mapsto E_\alpha((ir)^\alpha t^\alpha)$, $t \geq 0$ is not Stepanov almost periodic.*

Proof. Suppose to the contrary that the function $t \mapsto E_\alpha((ir)^\alpha t^\alpha)$, $t \geq 0$ is Stepanov almost periodic. By the asymptotic expansion formula for the Mittag-Leffler functions, we have that $E_\alpha((ir)^\alpha t^\alpha) = \alpha^{-1}(ir)^{1-\beta} e^{irt} + \epsilon_\alpha((ir)^\alpha t^\alpha)$, $t \geq 1$, where

$$\big|\epsilon_\alpha\big((ir)^\alpha t^\alpha\big)\big| = O\big(t^{-\alpha}\big), \quad t \geq 1. \tag{2.5}$$

Furthermore, the function $t \mapsto \epsilon_\alpha((ir)^\alpha t^\alpha)$, $t \geq 0$ needs to be Stepanov almost periodic since the function $t \mapsto \alpha^{-1}(ir)^{1-\beta} e^{irt}$, $t \geq 0$ is almost periodic. By (2.5), we get that there exist two finite constants $c_1, c_2 > 0$ such that

$$c_1 t^{-\alpha} \leq \big|\epsilon_\alpha\big((ir)^\alpha t^\alpha\big)\big| \leq c_2 t^{-\alpha}, \quad t \geq 1.$$

Using these estimates, we obtain

$$c_1 \int_t^{t+1} s^{-\alpha p}\, ds \leq \int_t^{t+1} \big|\epsilon_\alpha\big((ir)^\alpha s^\alpha\big)\big|^p\, ds \leq c_2 \int_t^{t+1} s^{-\alpha p}\, ds, \quad t \geq 1.$$

This simply implies $\epsilon_\alpha(\widehat{(ir)^\alpha \cdot^\alpha}) \in C_0([0,\infty) : L^p([0,1] : X))$, which is a contradiction since $\epsilon_\alpha(\widehat{(ir)^\alpha \cdot^\alpha}) \in AP([0,\infty) : L^p([0,1] : X))$. $\square$

Theorem 2.10. *Let $C \in L(X)$ be injective, let A be a closed single-valued linear operator, and let $\overline{R(C)} = X$. Suppose that $\alpha \in (0,2) \setminus \{1\}$ and A generates a Stepanov almost periodic (g_α, C)-resolvent family $(R(t))_{t\geq 0}$. Then $A = 0 \in L(X)$ and $R(t) = C$, $t \geq 0$.*

Proof. Suppose that $r \in \mathbb{R} \setminus \{0\}$ and $x \in X$ satisfies that $P_r^R x \neq 0$. By Theorem 2.8 and injectiveness of C, we get that the function

$$E_\alpha\big((ir)^\alpha t^\alpha\big) = \mathcal{L}^{-1}\Big(\frac{z^{\alpha-1}}{z^\alpha - (ir)^\alpha}\Big)(t)CP_r^R x, \quad t \geq 0$$

is Stepanov almost periodic. This is false due to Lemma 2.9, and therefore, $P_r^R x = 0$, $r \in \mathbb{R}\setminus\{0\}$, $x \in X$. Using dominated convergence and simple argumentation, this implies $\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-irs}R(s+\cdot)x\,ds = 0$ (in $L^p([0,1]:X)$), $r \in \mathbb{R}\setminus\{0\}$, $x \in X$. By spectral synthesis [4, Proposition 4.5.8], we get that $R(t+\cdot)x = Const.$ in $L^p([0,1]:X)$, $t \geq 0$ which simply implies by the continuity of mapping $t \mapsto R(t)x$, $t \geq 0$ and $R(0) = C$ that $R(t) = C$, $t \geq 0$. Therefore, the integral generator A of $(R(t))_{t\geq 0}$ is the zero operator. □

In [35], we have recently analyzed the Weyl-almost periodicity and asymptotical Weyl-almost periodicity of abstract Volterra integro-differential equations. It is worth noting that Theorem 2.10 holds true even if we replace the Stepanov almost periodicity in its formulation with Weyl-almost periodicity, as well as that several results from Subsection 2.2 below can be formulated for the Weyl class.

2.1. **Stepanov (asymptotically) almost periodic properties of convolution products.** In this subsection, we will investigate the Stepanov (asymptotically) almost periodic properties of various types of convolution products (for almost periodicity and asymptotical almost periodicity, see [1, Lemmas 2.12, 2.13] and [21, Lemma 4.1]). Our first result reads as follows.

Proposition 2.11. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying $M := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. If $g : \mathbb{R} \to X$ is S^p-almost periodic, then the function $G(\cdot)$, given by*

$$G(t) := \int_{-\infty}^{t} R(t-s)g(s)\,ds, \quad t \geq 0, \tag{2.6}$$

is well-defined and almost periodic.

Proof. It can be easily seen that, for every $t \geq 0$, we have that $G(t) = \int_0^\infty R(s)g(t-s)\,ds$ and that the last integral is absolutely convergent by the Hölder inequality and S^p-boundedness of function $g(\cdot)$:

$$\begin{aligned}\int_0^\infty \|R(s)\|\|g(t-s)\|\,ds &= \sum_{k=0}^{\infty}\int_k^{k+1} \|R(s)\|\|g(t-s)\|\,ds \\ &\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]}\|g\|_{S^p} = M\|g\|_{S^p}, \quad t \geq 0.\end{aligned}$$

Let a number $\epsilon > 0$ be given in advance. Then we can find a finite number $l > 0$ such that any subinterval I of $\mathbb{R}$ of length l contains a number $\tau \in I$ such that $\int_t^{t+1} \|g(s+\tau) - g(s)\|^p\,ds \leq \epsilon^p$, $t \in \mathbb{R}$. Applying Hölder inequality and this estimate, we get that

$$\begin{aligned}\|G(t+\tau) - G(t)\| &\leq \int_0^\infty \|R(r)\| \cdot \|g(t+\tau-r) - g(t-r)\|\,dr \\ &= \sum_{k=0}^{\infty}\int_k^{k+1} \|R(r)\| \cdot \|g(t+\tau-r) - g(t-r)\|\,dr\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \Big(\int_k^{k+1} \|g(t+\tau-r) - g(t-r)\|^p \, dr \Big)^{1/p} \\
&= \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \Big(\int_{t-k-1}^{t-k} \|g(s+\tau) - g(s)\|^p \, ds \Big)^{1/p} \\
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \epsilon = M\epsilon, \quad t \geq 0,
\end{aligned}
$$

which clearly implies that the set of all ϵ-periods of $G(\cdot)$ is relatively dense in $\mathbb{R}$. It remains to be proved the continuity of $G(\cdot)$. Since $\hat{g}(\cdot)$ is uniformly continuous, we have the existence of a number $\delta \in (0,1)$ such that

$$
\int_0^1 \|g(t+s) - g(t'+s)\|^p \, ds \leq \epsilon^p, \quad \text{provided } t,\ t' \in \mathbb{R} \text{ and } |t-t'| < \delta. \tag{2.7}
$$

For any $\delta' \in (0,\delta)$, we have by the foregoing arguments that

$$
\begin{aligned}
&\|G(t+\delta') - G(t)\| \\
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \Big(\int_k^{k+1} \|g(t+\delta'-s) - g(t-s)\|^p \, ds \Big)^{1/p} \\
&\quad + \int_0^{\delta'} \|R(s)\| \|g(t-s)\| \, ds \\
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \Big(\int_0^1 \|g(t+\delta'+s-k-1) - g(t+s-k-1)\|^p \, ds \Big)^{1/p} \\
&\quad + \|R(\cdot)\|_{L^q[0,1]} \Big(\int_t^{t+\delta'} \|g(s)\|^p \, ds \Big)^{1/p}, \quad t \in \mathbb{R},
\end{aligned}
$$

so that conclusion follows from (2.7) and the well-known fact that

$$
\lim_{\delta' \to 0} \int_t^{t+\delta'} \|g(s)\|^p \, ds = 0, \quad t \in \mathbb{R}.
$$

□

Remark 2.12. Let $t \mapsto \|R(t)\|$, $t \in (0,1]$ be an element of the space $L^q[0,1]$. Then the condition $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$ holds provided that $(R(t))_{t>0}$ is exponentially decaying at infinity or that there exists a finite number $\zeta < 0$ such that $\|R(t)\| = O(t^{\zeta})$, $t \to +\infty$ and

(i) $p = 1$ and $\zeta < -1$, or
(ii) $p > 1$ and $\zeta < (1/p) - 1$.

In this way, we have extended the assertion of [1, Lemma 2.12], where the authors have considered the case in which $\zeta < -1$ and $g : \mathbb{R} \to X$ is almost periodic.

Concerning asymptotical Stepanov almost periodicity, we can deduce the following proposition.

Proposition 2.13. *Suppose that* $1 \leq p < \infty$, $1/p + 1/q = 1$ *and* $(R(t))_{t>0} \subseteq L(X)$ *is a strongly continuous operator family satisfying that, for every* $s \geq 0$, *we have that* $m_s := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[s+k,s+k+1]} < \infty$. *Suppose, further, that* $f : [0,\infty) \to X$ *is asymptotically* S^p*-almost periodic as well as that the locally* p*-integrable functions*

$g : \mathbb{R} \to X$, $q : [0,\infty) \to X$ satisfy the conditions from Lemma 1.1 (here and hereafter, the use of symbol q will be clear from the context). Let there exist a finite number $M > 0$ such that the following holds:

(i) $\lim_{t\to+\infty} \int_t^{t+1} \big[\int_M^s \|R(r)\|\|q(s-r)\|\,dr\big]^p\,ds = 0$.

(ii) $\lim_{t\to+\infty} \int_t^{t+1} m_s^p\,ds = 0$.

Then the function $H(\cdot)$, given by

$$H(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geq 0,$$

is well-defined, bounded and asymptotically S^p-almost periodic.

Proof. It is obvious that the function $H(\cdot)$ is well-defined and bounded because $f(\cdot)$ is S^p-bounded and $m_0 < \infty$; cf. the proof of Proposition 2.11. Let the function $G(\cdot)$ be given by (2.6). Define

$$F(t) := \int_0^t R(t-s)q(s)\,ds - \int_t^\infty R(s)g(t-s)\,ds, \quad t \geq 0.$$

Then it is clear that $H(t) = G(t) + F(t)$ for all $t \geq 0$ and by Proposition 2.11 it suffices to show that the mapping $\hat{F} : [0,\infty) \to L^p([0,1] : X)$ belongs to the class $C_0([0,1] : L^p([0,1] : X))$. This mapping is clearly continuous and we need to prove that

$$\lim_{t\to+\infty} \int_t^{t+1} \|F(s)\|^p\,ds = 0. \tag{2.8}$$

Let $M > 0$ be such that (i) holds. Then it is clear that there exist finite constants $c_p > 0$ and $c_p' > 0$ such that, by Hölder inequality,

$$\begin{aligned}
\|F(s)\|^p &\leq c_p\Big[\Big(\int_0^{s-M} \|R(s-r)\|\|q(r)\|\,dr\Big)^p + \Big(\int_{s-M}^s \|R(s-r)\|\|q(r)\|\,dr\Big)^p \\
&\quad + \Big(\int_s^\infty \|R(r)\|\|g(s-r)\|\,dr\Big)^p\Big] \\
&\leq c_p'\Big[\Big(\int_0^{s-M} \|R(s-r)\|\|q(r)\|\,dr\Big)^p + \|R(\cdot)\|^p_{L^q[0,M]}\|q\|^p_{L^p[s-M,s]} \\
&\quad + \Big(\sum_{k=0}^\infty \|R(\cdot)\|_{L^q[s+k,s+k+1]}\Big)^p\|g\|^p_{S^p}\Big]
\end{aligned} \tag{2.9}$$

Let $\epsilon > 0$ be given. Then there exists $t_0(\epsilon) \geq 1$ such that $\int_t^{t+1} \|q(s)\|^p\,ds \leq \epsilon$, $t \geq t_0(\epsilon)$. This implies that for each $t \geq t_0(\epsilon) + M$ we have $\|q\|^p_{L^p[s-M,s]} \leq \lceil M\rceil\epsilon$. Using this estimate and integrating (2.9) along the interval $[t, t+1]$, we obtain with the help of (i)-(ii) that (2.8) holds, as claimed. □

Remark 2.14. The proof of Proposition 2.13 is similar to those of [1, Lemma 2.13] and [21, Lemma 4.1]. Below are listed some special situations in which the asymptotical S^p-almost periodicity of function $H(\cdot)$ is proved by applying directly Proposition 2.13 or by combining Proposition 2.11 and the proofs of the above-mentioned lemmae:

(i) Suppose that $(R(t))_{t\geq 0}$ is strongly continuous, exponentially decaying, $g : \mathbb{R} \to X$ is S^p-almost periodic and $q \in C_0([0,\infty) : X)$; then we can use Proposition

2.11, the proof of [21, Lemma 4.1], decomposition

$$\int_0^t R(t-s)q(s)\,ds = \int_0^{t/2} R(t-s)q(s)\,ds + \int_{t/2}^t R(t-s)q(s)\,ds, \quad t \geq 0$$

and the estimates for the term $\int_t^\infty R(s)g(t-s)\,ds$ given in the proof of Proposition 2.13, in order to see that the function $H(\cdot)$ is asymptotically almost periodic. The case in which the function $H(\cdot)$ is asymptotically S^p-almost periodic but not asymptotically almost periodic can also occur: if we accept all above assumptions with the exception of $q \in C_0([0,\infty) : X)$, and suppose in place of this condition that $\lim_{t\to+\infty} \int_t^{t+1} (\int_{s/2}^s \|q(r)\|\,dr)^p\,ds = 0$, then the same argumentation as above show that the function $H(\cdot)$ is only asymptotically S^p-almost periodic. In such a way, we have proved a proper extension of [21, Lemma 4.1], which can be further applied for stating a proper extension of [21, Theorem 4.2] and new results about inhomogeneous abstract Cauchy problems of third order,

$$\alpha u'''(t) + u''(t) - \beta Au(t) - \gamma Au'(t) = f(t), \quad \alpha, \beta, \gamma > 0,\ t \geq 0,$$

appearing in the theory of dynamics of elastic vibrations of flexible structures [21].

(ii) We can prove a proper extension of [1, Lemma 2.13] as explained below. Suppose that $(R(t))_{t\geq 0}$ is strongly continuous, $\zeta < -1$, $\|R(t)\| = O(1+t^\zeta)$, $t \geq 0$, $g : \mathbb{R} \to X$ is S^p-almost periodic and $q \in C_0([0,\infty) : X)$; then we can use the same arguments as above, with appealing to [1, Lemma 2.13] in place of [21, Lemma 4.1], to show that the function $H(\cdot)$ is asymptotically almost periodic. The only thing worth noting here is that $m_t \to 0$ as $t \to \infty$; for this, observe that there exists a finite number $M' \geq 1$ such that ($\alpha = -\zeta$):

$$\begin{aligned}
&\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[t+k,t+k+1]} \\
&\leq M' \sum_{k=0}^{\infty} \Big(\int_{t+k}^{t+k+1} \frac{dr}{r^{\alpha q}} \Big)^{1/q} \\
&= M'(\alpha\zeta - 1)^{-1} \sum_{k=0}^{\infty} \Big| \big(t+k+1\big)^{1-\alpha q} - \big(t+k\big)^{1-\alpha q} \Big|^{1/q} \\
&\leq M'(\alpha\zeta - 1)^{1/q-1} \sum_{k=0}^{\infty} \frac{1}{(t+k)^{\alpha}} \\
&\leq M'(\alpha\zeta - 1)^{1/q-1} \sum_{k=0}^{\infty} \frac{1}{t^{\nu\alpha} k^{(1-\nu)\alpha}} \\
&\leq \text{Const.} t^{-\nu\alpha}, \quad t > 0,
\end{aligned}$$

provided that $(1-\nu)\alpha > 1$. If we assume, as in the first part of this remark, that $\lim_{t\to+\infty} \int_t^{t+1} (\int_{s/2}^s \|q(r)\|\,dr)^p\,ds = 0$, then the function $H(\cdot)$ will be only asymptotically S^p-almost periodic.

(iii) Proposition 2.13 can be applied provided that $(R(t))_{t>0}$ is exponentially decaying at infinity, $(R(t))_{t>0}$ has a certain growth order at zero, and $q : [0,\infty) \to X$ has a certain growth order. For the sake of illustration, we will examine only the case in which the multivalued linear operator $\mathcal{A}$ satisfies the condition [24, (P), p. 47]:

- There exist finite constants $c, M > 0$ and $\beta \in (0,1]$ such that
$$\Psi := \Psi_c := \{\lambda \in \mathbb{C} : \Re\lambda \geq -c(|\Im\lambda| + 1)\} \subseteq \rho(\mathcal{A})$$
and
$$\|R(\lambda : \mathcal{A})\| \leq M(1+|\lambda|)^{-\beta}, \quad \lambda \in \Psi.$$

Then it is well known that there exists a degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ such that $\|T(t)\| = O(t^{\beta-1})$, $t > 0$. Furthermore, the proof of [24, Theorem 3.1] combined with the integral computation given in the proof of [4, Theorem 2.6.1] shows that $\|T(t)\| = O(e^{-ct}t^{\beta-1})$, $t > 0$. This estimate enables one to see that the condition (ii) from the formulation of Proposition 2.13 holds. Therefore, if $\lim_{t\to+\infty}\int_t^{t+1}\big[\int_M^s e^{-cr}r^{\beta-1}\|q(s-r)\|\,dr\big]^p\,ds = 0$, then we can apply Proposition 2.13 to conclude that the function $H(\cdot)$ is asymptotically S^p-almost periodic (cf. [24, Theorem 3.7] and [24, Example 3.3, Example 3.6] for some applications in the study of inhomogeneous Poisson heat equation in the spaces $H^{-1}(\Omega)$ and $L^r(\Omega)$, where Ω is a bounded domain with smooth boundary and $1 < r < \infty$). It is clear that Proposition 2.11 can be also applied here, which can be simply incorporated in the study of existence and uniqueness of almost periodic solutions of the following differential inclusion of first order
$$u'(t) \in \mathcal{A}u(t) + g(t), \quad t \in \mathbb{R},$$
where $g : \mathbb{R} \to X$ is S^p-almost periodic. Details can be left to the interested reader.

Now we will continue the analysis of Ponce and Warma [42] concerning diffusion Volterra integro-differential equations with memory, proving the existence of some specific classes of exponentially decaying (a,k)-regularized resolvent families (possibilities for work exist even in the case that $C \neq I$ and X is not a Banach space). In the final part of example, we will consider asymptotically almost periodic solutions and Stepanov asymptotically almost periodic solutions.

Example 2.15. Suppose that $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$, $0 < \zeta \leq 1$ and $\omega \in \mathbb{R}$. Let any of the following two conditions hold:

(i) $\alpha > 0$, $\mathcal{A}$ is an MLO satisfying
$$\omega + \Sigma_{\zeta\pi/2} \subseteq \rho(\mathcal{A}) \text{ and } \|R(\lambda : \mathcal{A})\| = O(|\lambda-\omega|^{-1}), \quad \lambda \in \omega + \Sigma_{\zeta\pi/2}. \tag{2.10}$$

(ii) $\alpha < 0$, $\alpha + \beta^{\zeta} \geq |\alpha|$ and $\mathcal{A}$ is an MLO satisfying (2.10).

Then it is well-known that the operator $\mathfrak{A} \equiv \mathcal{A}_{|\overline{D(\mathcal{A})}}$ is single-valued, linear and densely defined in the Banach space $\overline{D(\mathcal{A})}$, as well as that (2.10) holds with the operator $\mathcal{A}$ replaced with the operator $\mathfrak{A}$; see e.g. [41, Lemma 4.1]. Set $a(t) := 1 + \int_0^t k(s)\,ds$, where $k(t) := \alpha e^{-\beta t}g_{\zeta}(t)$. Owing to the proof of [42, Theorem 2.1], we get that $\mathfrak{A}$ generates an exponentially bounded (a,I)-resolvent family $(S_{\omega}(t))_{t\geq 0}$ in $\overline{D(\mathcal{A})}$, provided that $\omega = 0$. In the general case $\omega \neq 0$, the perturbation result [33, Theorem 3.7.4] and decomposition $\mathfrak{A} = (\mathfrak{A} - \omega I_{|\overline{D(\mathcal{A})}}) + \omega I_{|\overline{D(\mathcal{A})}}$ show that $\mathfrak{A}$ generates an exponentially bounded (a,I)-resolvent family $(S(t))_{t\geq 0}$ in $\overline{D(\mathcal{A})}$, as well. This extends the assertion of [42, Corollary 2.2], and can be applied in the analysis of Poisson heat equation with memory, in the space $H^{-1}(\Omega)$; see e.g. [24, Example 3.3] and the analysis below. The proof of [42, Theorem 2.3] works in degenerate case, as well, and we may conclude the following: Let $\alpha \neq 0$, $\beta \geq 0$, $0 < \zeta < \tilde{\zeta} \leq 1$, $\omega < 0$ and $\beta + \omega \leq 0$. If (i) holds with the number ζ replaced with the number $\tilde{\zeta}$ therein, then $\|S(t)\| = O(e^{-\beta t})$, $t \geq 0$; if (ii) holds with the number ζ replaced with the number $\tilde{\zeta}$ therein, then $\|S(t)\| = O((1+\alpha\omega t^{\zeta+1})e^{-(\beta-(\alpha\omega)^{1/(\zeta+1)})t})$, $t \geq 0$.

Consider now the case in which $X := H^{-1}(\Omega)$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $m(x) \geq 0$ a.e. be a given function in $L^\infty(\Omega)$, and let Δ denotes the Dirichet Laplacian defined as usually. Let $\theta \in (-\pi, \pi)$, $0 < \epsilon < \pi - |\theta|$, and let $\mathcal{A} := e^{i\theta}\Delta m(x)^{-1}$. Then the analysis contained in [24, Example 3.3] shows that (2.10) holds with some number $\omega = -c < 0$ and the number ζ replaced by the number $\tilde{\zeta} := 2(\pi - \epsilon - |\theta|)/\pi$ therein. Let $\alpha \neq 0$, $0 < \beta \leq c$, $0 < \zeta < \tilde{\zeta} \leq 1$, and let $\beta - (-\alpha c)^{1/(\zeta+1)} > 0$ in the case of consideration (ii). Hence, $\mathfrak{A}$ generates an exponentially decaying (a, I)-resolvent family $(S(t))_{t\geq 0}$ in $\overline{D(\mathcal{A})}$. Suppose that $f \in C^1([0,\infty) : \overline{D(\mathcal{A})})$ and $f' \in AAP([0,\infty) : \overline{D(\mathcal{A})})$. Then the variation of parameters formula [33, Theorem 3.2.9(i)] (cf. also [43, Proposition 1.2(ii)]) and the proof of [21, Lemma 4.1] show that the mapping $t \mapsto S(t)f(0) + \int_0^t S(t-s)f'(s)\,ds$, $t \geq 0$ is an asymptotically almost periodic solution of the abstract Volterra inclusion $u(t) \in \mathcal{A}\int_0^t a(t-s)u(s)\,ds + f(t)$, $t \geq 0$. The corresponding analysis in the space $X = L^2(\Omega)$ falls out from the scope of this paper (it is clear that [21, Lemma 4.1] can be applied in the analysis of existence and uniqueness of asymptotically almost periodic solutions of a substantially large class of inhomogeneous abstract Cauchy problems whose solution operator families are exponentially decaying; for degenerate case, see also [62, Theorems 2.2, 2.4], [33, Theorem 2.2.20] with $\alpha = 1$, and [24, Theorems 3.7, 3.8]).

Suppose, finally, that $f \in C^1([0,\infty) : \overline{D(\mathcal{A})})$ and $f' : [0,\infty) \to \overline{D(\mathcal{A})}$ is asymptotically Stepanov almost periodic. Then the mapping $t \mapsto S(t)f(0) + \int_0^t S(t-s)f'(s)\,ds$, $t \geq 0$ is an asymptotically Stepanov almost periodic solution of the abstract Volterra inclusion $u(t) \in \mathcal{A}\int_0^t a(t-s)u(s)\,ds + f(t)$, $t \geq 0$, provided that the function $f'(\cdot)$ can be written as $f'(\cdot) = g(\cdot) + q(\cdot)$ (cf. Lemma 1.1 with the function $f(\cdot)$ replaced therein with the function $f'(\cdot)$), and $q(\cdot)$ satisfies the condition (i) of Proposition 2.13.

In [34], we have investigated the almost periodic entire solutions to the abstract Barenblatt-Zheltov-Kochina equation and the abstract linearized Boussinesq-Love equation (see [33, Example 2.3.48, Example 2.3.49] for more details). In the following illustrative example, we shall continue the analysis contained in [34, Example 4.16] by enquiring into the asymptotically almost periodic solutions of the abstract Boussinesq-Love equation.

Example 2.16. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and let $X := L^2(\Omega)$. Denote by $\{\lambda_k\}$ $[= \sigma(\Delta)]$ the eigenvalues of the Dirichlet Laplacian Δ (recall that $0 < -\lambda_1 \leq -\lambda_2 \cdots \leq -\lambda_k \leq \cdots \to +\infty$ as $k \to \infty$; see [33] for further information), numbered in nonascending order with regard to multiplicities. By $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal [in the sense of $L^2(\Omega)$] eigenfunctions.

Of importance is the following Cauchy-Dirichlet problem for Boussinesq-Love equation:

$$
\begin{aligned}
&(\lambda - \Delta)u_{tt}(t,x) - \alpha(\Delta - \lambda')u_t(t,x) \\
&= \beta(\Delta - \lambda'')u(t,x) + f(t,x), \quad t \in \mathbb{R},\ x \in \Omega,
\end{aligned}
\tag{2.11}
$$

$$\begin{gathered} u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \quad (t,x)\in \mathbb{R}\times\Omega; \\ u(t,x)=0, \quad (t,x)\in\mathbb{R}\times\partial\Omega, \end{gathered} \tag{2.12}$$

where $\lambda,\lambda',\lambda''\in\mathbb{R}$, $\alpha,\beta\in\mathbb{R}$ and $\alpha,\beta\neq 0$.

Suppose that $\lambda\in\rho(\Delta)$. By [57, Theorem 5.1(i)], for every $u_0,\ u_1\in H^2(\Omega)\cap H_0^1(\Omega)$ and $f\in C^\infty(\mathbb{R}:X)$, there exists a unique solution $u(\cdot)$ of problem [(2.11)-(2.12)] and $u(\cdot)$ has the form

$$\begin{aligned} u(t) &= \sum_{k=1}^{\infty}\Big[\frac{\mu_k^1}{\mu_k^1-\mu_k^2}e^{\mu_k^1 t}-\frac{\mu_k^2}{\mu_k^1-\mu_k^2}e^{\mu_k^2 t}\Big]\langle\phi_k,u_0\rangle\phi_k \\ &\quad+\sum_{k=1}^{\infty}\frac{e^{\mu_k^1 t}-e^{\mu_k^2 t}}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)}\langle\phi_k,u_1\rangle\phi_k \\ &\quad+\sum_{k=1}^{\infty}\int_0^t\frac{e^{\mu_k^1(t-s)}-e^{\mu_k^2(t-s)}}{(\lambda-\lambda_k)(\mu_k^1-\mu_k^2)}\langle\phi_k,f(s)\rangle\phi_k\,ds, \quad t\in\mathbb{R}, \end{aligned}$$

where

$$\mu_k^{1,2}:=\frac{-\alpha(\lambda'-\lambda_k)\pm\sqrt{\alpha^2(\lambda'-\lambda_k)^2-4\beta(\lambda-\lambda_k)(\lambda''-\lambda_k)}}{2(\lambda-\lambda_k)}, \quad k\in\mathbb{N}.$$

Suppose that $\Re\big[\mu_k^{1,2}\big]\leq 0$, $k\in\mathbb{N}$. This condition clearly implies that the function $t\mapsto u(t)$, $t\in\mathbb{R}$ is an asymptotically almost periodic solution of the homogeneous counterpart of problem (2.11)-(2.12) for all $u_0,\ u_1\in\operatorname{span}(\{\phi_k:k\in\mathbb{N}\})$, which is dense in X (the case that $\lambda\in\sigma(\Delta)$ can be also analyzed, but then the set of all initial values $u_0,\ u_1$ for which there exist such a solution cannot be dense in X). Concerning the inhomogeneous term in the representation of $u(\cdot)$, with $u_0=u_1=0$, the most simplest case when it will be asymptotically almost periodic for $t\geq 0$ is that there exists a finite subset $L\subseteq\mathbb{N}$ such that $R(f)\subseteq\operatorname{span}(\{\phi_k:k\in L\})$, $\Re\big[\mu_k^{1,2}\big]<0$, $k\in L$ and the mappings $t\mapsto\langle\phi_k,f(t)\rangle$, $t\geq 0$ are asymptotically almost periodic for all $k\in L$ (by Proposition 2.13, the asymptotical S^p-almost periodicity of this term can also occur provided certain growth order of these mappings).

2.2. **Generalized (asymptotically) almost periodic properties of degenerate C-semigroups and degenerate C-cosine functions.** Concerning Stepanov almost periodicity of degenerate C-semigroups, we will first state the following simple result (as announced earlier, the operator C is allowed to be possibly non-injective):

Proposition 2.17. *Suppose $(S(t))_{t\geq 0}$ is a bounded C-regularized semigroup with the integral generator $\mathcal{A}$. If $x\in X$ satisfies that the mapping $t\mapsto S(t)x$, $t\geq 0$ is Stepanov almost periodic, then the mapping $t\mapsto S(t)Cx$, $t\geq 0$ is almost periodic.*

Proof. Let us recall that $(S(t))_{t\geq 0}\subseteq L(X)$ is a strongly continuous operator family commuting with C, and that $S(t)S(s)=S(t+s)C$, $t,\ s\in\mathbb{R}$. Since the mapping $t\mapsto S(t)x$, $t\geq 0$ is Stepanov almost periodic and $(S(t))_{t\geq 0}$ is uniformly bounded, we have that the mapping $t\mapsto S(t)Cx$, $t\geq 0$ is Stepanov almost periodic and bounded, so that it remains to be proved that the mapping $t\mapsto S(t)Cx$, $t\geq 0$ is uniformly continuous (any Stepanov almost periodic function $f\in BUC([0,\infty):X)$ has to be almost periodic). But, this follows from the uniform boundedness of $(S(t))_{t\geq 0}$ and the estimate $\|S(t)Cx-S(s)Cx\|\leq\|S(s)\|\cdot\|S(t-s)x-Cx\|$, $t,\ s\geq 0$, $t\geq s$. $\square$

If $R(C)$ is dense in X and the mapping $t \mapsto S(t)x$, $t \geq 0$ is Stepanov almost periodic for all $x \in X$, then Proposition 2.17 and the fact that $AP([0,\infty) : X)$ is a closed subspace of $BUC([0,\infty) : X)$ together imply that the mapping $t \mapsto S(t)x$, $t \geq 0$ is almost periodic for all $x \in X$, as long as $(S(t))_{t\geq 0}$ is bounded. But, the strong Stepanov almost periodicity of mapping $t \mapsto S(t)$, $t \geq 0$ does not imply a priori the boundedness of $(S(t))_{t\geq 0}$; in the present situation, the best we can do concerning this question is to prove the following slight extension of well-known Henríquez's result [28, Theorem 1]:

Theorem 2.18. *Suppose that $1 \leq p < \infty$ and $(S(t))_{t\geq 0}$ is a C-regularized semigroup with the integral generator $\mathcal{A}$. Then the following holds:*

(i) *Let $x \in X$ satisfy that the mapping $t \mapsto S(t)x$, $t \geq 0$ is S^p-bounded. Then the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is bounded.*

Suppose that the mapping $t \mapsto S(t)x$, $t \geq 0$ is S^p-bounded for all $x \in X$. Then we have the following:

(ii) *The mapping $t \mapsto S(t)C^2x$, $t \geq 0$ is bounded and uniformly continuous for all $x \in X$, and there exists a finite constant $M \geq 0$ such that $\|S(t)C\| \leq M$, $t \geq 0$. Therefore, if $x \in X$ satisfies that the mapping $t \mapsto S(t)C^2x$, $t \geq 0$ is Stepanov almost periodic, then it is almost periodic.*

(iii) *If $R(C)$ is dense in X, then the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is bounded and uniformly continuous for all $x \in X$. Therefore, if $x \in X$ satisfies that the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is Stepanov almost periodic, then it is almost periodic.*

Proof. Assume that (i) does not hold. Then there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[1,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \|S(t_n)Cx\| = \infty$. Let $N := \sup_{s\in[0,1]} \|S(s)\| < \infty$. Then $S(t_n)Cx = S(s)S(t_n - s)x$, $0 \leq s \leq 1$, $n \in \mathbb{N}$ and therefore $\|S(t_n - s)x\| \geq \|S(t_n)Cx\|/N$, $n \in \mathbb{N}$. Integrating this estimate over the interval $[0,1]$, we obtain

$$\int_{t_n-1}^{t_n} \|S(s)x\|^p\, ds \geq \frac{\|S(t_n)Cx\|^p}{N^p}, \quad n \in \mathbb{N},$$

contradicting the S^p-boundedness of mapping $t \mapsto S(t)x$, $t \geq 0$. This completes the proof of (i). For the remnant of proof, let us assume that the mapping $t \mapsto S(t)x$, $t \geq 0$ is S^p-bounded for all $x \in X$. By the uniform boundedness principle and (i), we have that there exists a finite constant $M \geq 0$ such that $||S(t)C\| \leq M$, $t \geq 0$. Now the uniform continuity of mapping $t \mapsto S(t)C^2x$, $t \geq 0$ for any $x \in X$ follows from the estimate $\|S(t)C^2x - S(s)C^2x\| \leq \|S(s)C\| \cdot \|S(t-s)x - Cx\|$, $t,\ s \geq 0$, $t \geq s$, which simply completes the proof of (ii). If $R(C)$ is dense in X, then for each $x \in X$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} Cx_n = x$. Hence, for every number $\epsilon > 0$ given in advance, we can find an integer $n_0 \in \mathbb{N}$ and a positive real number $\delta > 0$ such that

$$\begin{aligned} \|S(t)Cx - S(s)Cx\| &\leq 2M\|Cx_{n_0} - x\| + \|S(t)C^2x_{n_0} - S(s)C^2x_{n_0}\| \\ &\leq 2\epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

provided that $t, s \geq 0$ and $|t-s| < \delta$, so that the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is bounded and uniformly continuous for all $x \in X$. This simply yields (iii). □

Before proceeding further, let us recall that it is still an open problem in the theory of non-degenerate C-regularized semigroups ($C \in L(X)$ injective) whether there exists a bounded C-regularized semigroup $(S(t))_{t\geq 0}$ that is not strongly uniformly continuous; cf. [18, Remark 5.19] for more details. Concerning the asymptotical Stepanov almost periodicity of degenerate C-regularized semigroups, we can clarify the following result.

Theorem 2.19. *Suppose that $1 \leq p < \infty$, $(S(t))_{t\geq 0}$ is a C-regularized semigroup with the integral generator $\mathcal{A}$, and the mapping $t \mapsto S(t)x$, $t \geq 0$ is S^p-bounded for all $x \in X$. Then we have the following:*

- (ii) *The asymptotical S^p-almost periodicity of mapping $t \mapsto S(t)x$, $t \geq 0$ for some $x \in X$ implies that the mapping $t \mapsto S(t)C^4x$, $t \geq 0$ is asymptotically almost periodic.*
- (iii) *If $R(C)$ is dense in X and $(S(t))_{t\geq 0}$ is strongly asymptotically S^p-almost periodic, then the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is asymptotically almost periodic for all $x \in X$.*

Proof. We will only outline the most relevant details of proof, which is very similar to that of [28, Theorem 2]. By the foregoing, we have that there exists a finite constant $M \geq 0$ such that $\|S(t)C\| \leq M$, $t \geq 0$. Let $x \in X$ satisfy that the mapping $t \mapsto S(t)x$, $t \geq 0$ is asymptotically S^p-almost periodic. Then for any sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals there exists its subsequence $(s_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} S\widehat{(s_n + \cdot)}x$ exists in the space $C_b([0,\infty) : L^p([0,1] : X))$. Lemma 1.1 and the proof of afore-mentioned theorem together imply that there exist two functions $g_x(\cdot)$ and $q_x(\cdot)$ such that the conclusions of this lemma hold with the function $f(\cdot)$ replaced therein by $S(\cdot)x$, and that there exists a subsequence $(r_n)_{n\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ such that $g_x(t) = \lim_{n\to\infty} S(r_n + t)x$ a.e. $t \geq 0$. Arguing as in [28], we get that the limit $\lim_{n\to\infty} S(r_n + t)C^2x$ exists in X for all $t \geq 0$. This implies that the sequence $(S(r_n + \cdot)C^4x)_{n\in\mathbb{N}}$ is Cauchy in the space $C_b([0,\infty) : X)$ and therefore convergent (observe that $\|S(r_n+t)C^4x-S(r_m+t)C^4x\| \leq M\|S(r_n)C^2x-S(r_m)C^2x\|$, $m,\ n \in \mathbb{N}$, $t \geq 0$), finishing the proof of (i). The proof (ii) is simple and therefore omitted. □

By Example 2.4, for each number $\alpha \in (0,2)\setminus\{1\}$ there exist examples of bounded, non-degenerate, asymptotically Stepanov almost periodic (g_α, I)-resolvent families that are not (Stepanov) almost periodic. Now we will focus our attention to the case $\alpha = 2$: To the best knowledge of the author, the assertion of [28, Theorem 3] has not yet been reconsidered for C-regularized cosine operator functions. To do that, we need first to extend the well-known result by Cioranescu and Ubilla [14, Theorem 1] concerning the generation of uniformly bounded cosine operator functions in terms of boundedness and analyticity of subordinated strongly continuous semigroups as well as denseness of subspace consisted of exponential vectors (see Radyno [45] and [14, Lemma, pp. 2-3]):

Let $C \in L(X)$ be injective, and let A be a closed single-valued linear operator commuting with C. Denote by D_A^μ the vector space consisting of all vectors $x \in D_\infty(A)$ such that there exists $c > 0$ satisfying that $\|A^kx\| \leq c\mu^k$, $k \in \mathbb{N}_0$ $(\mu > 0)$. Equipped with the norm $\|x\|_\mu := \sup_{k\geq 0} \mu^{-k}\|A^kx\|$, this space becomes one of Banach's. The space of exponential vectors of A, Exp_A for short, is defined by $\mathrm{Exp}_A := \cup_{\mu>0} D_A^\mu$; then it is clear that Exp_A consists as a subspace the linear span of all eigenfunctions corresponding to some eigenvalue of A. Let $A_\mu := A_{|D_A^\mu}$ and $C_\mu := C_{|D_A^\mu}$ $(\mu > 0)$. Then $A_\mu,\ C_\mu \in L(D_A^\mu)$ mutually commute, C_μ is injective and $(\lambda - A_\mu)^{-1}C_\mu = ((\lambda - A)^{-1}C)_{|D_A^\mu}$ for any $\lambda \in \rho_C(A)$, with the estimate $\|(\lambda - A_\mu)^{-1}C_\mu\|_\mu \leq \|(\lambda - A)^{-1}C\|$, $\lambda \in \rho_C(A)$, $\mu > 0$; see e.g. the proof of [45, Theorem 11]. Furthermore, if the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \Omega$ is analytic on some open domain $\Omega \subseteq \mathbb{C}$, then we know that $R(C) \subseteq R((\lambda - A)^n)$, $n \in \mathbb{N}$ and

$$\frac{d^{n-1}}{d\lambda^{n-1}}\big(\lambda - A\big)^{-1}C = (-1)^{n-1}(n-1)!\big(\lambda - A\big)^{-n}C \in L(X), \quad n \in \mathbb{N}.$$

In this case, we can inductively prove that $\big(\lambda - A_\mu\big)^{-n}C_\mu \in L(D_A^\mu)$, $n \in \mathbb{N}$, $\mu > 0$ and $\|(\lambda - A_\mu)^{-n}C_\mu\|_\mu \leq \|(\lambda - A)^{-n}C\|$, $n \in \mathbb{N}$, $\lambda \in \Omega$, $\mu > 0$; furthermore, the estimate

$$\begin{aligned}
&\Big\|\frac{(\lambda - A_\mu)^{-1}C_\mu x - (z - A_\mu)^{-1}C_\mu x}{\lambda - z} + \big(\lambda - A_\mu\big)^{-2}C_\mu x\Big\|_\mu \\
&\leq \Big\|\frac{(\lambda - A)^{-1}Cx - (z - A)^{-1}Cx}{\lambda - z} + \big(\lambda - A\big)^{-2}Cx\Big\| \, \|x\|_\mu,
\end{aligned}$$

for $\lambda \in \Omega$, $\mu > 0$, $x \in D_A^\mu$ enables to see that the mapping $\lambda \mapsto (\lambda - A_\mu)^{-1}C_\mu \in L(D_A^\mu)$, $\lambda \in \Omega$ is analytic, as well ($\mu > 0$). Keeping in mind these facts and the well known structural results from the theory of C-regularized semigroups and C-regularized cosine functions ([31]-[32]), a careful inspection of the proof of [14, Theorem 1] enables one to deduce the following result of independent interest (for (ii), define $u(t)$ in the proof of above-mentioned theorem by $u(t) := \sum_{\mu=1}^{\infty}(t^{2n}/(2n)!)A_\mu^n C_\mu x_\mu$, $t \geq 0$, $x = \sum_{\mu=1}^{\infty} x_\mu \in D(A)$, $x_\mu \in D_A^\mu$ for $1 \leq \mu < \infty$):

Theorem 2.20. *Let $C \in L(X)$ be injective, and let A be a closed single-valued linear operator commuting with C. Then the following holds:*

- (i) *If A generates a bounded C-regularized cosine function, then A generates a bounded analytic C-regularized semigroup of angle $\pi/2$ and $\overline{R(C)} \subseteq \overline{\mathrm{Exp}_A}$.*
- (ii) *If A generates a bounded analytic C-regularized semigroup of angle $\pi/2$ and $\overline{\mathrm{Exp}_A} = X$, then A generates a bounded C-regularized cosine function.*
- (iii) *Suppose that $\overline{R(C)} = X$. Then A generates a bounded C-regularized cosine function if and only if A generates a bounded analytic C-regularized semigroup of angle $\pi/2$ and $\overline{\mathrm{Exp}_A} = X$.*

Now we are ready to prove the following slight extension of [28, Theorem 3].

Theorem 2.21. *Let $C \in L(X)$ be injective, let A be a closed single-valued linear operator, and let $\overline{R(C)} = X$. Suppose that A generates a asymptotically Stepanov almost periodic C-regularized cosine function $(C(t))_{t\geq 0}$. Then $(C(t))_{t\geq 0}$ is almost periodic.*

Proof. The proof of theorem is very similar to that of [28, Theorem 3] and, because of that, we will only outline the main differences. It is well known that $R(C) \subseteq \overline{D(A)}$, so that A is densely defined. Moreover, the mapping $B : X \to L_S^p([0,\infty) : X)$ given by $Bx := C(\cdot)x$ ($x \in X$) is linear and closed, therefore continuous. Using the abstract Weierstrass formula [32, Theorem 2.4.18] and the proof of [28, Theorem 3], we can simply verify that A generates a bounded analytic C-regularized semigroup of angle $\pi/2$. Denote by $S(\cdot)$ the induced C-regularized sine function generated by A. By Theorem 2.20(iii) and [67, Theorem 4.1], it suffices to show that the set D consisted of all eigenvectors of operator A which correspond to the real non-positive eigenvalues of A is total in X. For this, define $P_\lambda x$ and $Q_\lambda x$ as well as the functions

$g_x(\cdot)$ and $q_x(\cdot)$ as in the proof of afore-mentioned theorem ($\lambda \in \mathbb{R}$, $x \in X$). Then the arguments contained in the proofs of [28, Theorem 3] and [67, Theorem 4.1] enable one to see that $C(t)P_\lambda x = \cos(\lambda t)CP_\lambda x$, $C(t)Q_\lambda x = \cos(\lambda t)CQ_\lambda x$ as well as that $\{P_\lambda x,\ Q_\lambda x\} \subseteq N(\lambda^2 + A)$ for all $\lambda \in \mathbb{R}$, $x \in X$. Let $x^* \in X^*$, and let x^* annulate D. By d'Alambert's formula, for every $a \in [0,1]$, $t \geq 0$ and $x \in X$, we have that

$$2|\langle x^*, C(t)S(a)x\rangle| \leq \int_0^1 |\langle x^*, q_{Cx}(t+s)\rangle|\,ds + \int_0^1 |\langle x^*, q_{Cx}(t-s)\rangle|\,ds.$$

Set $y_a^* := S(a)^* x^*$, $a \in [0,1]$. Then the above yields $\lim_{t\to\infty}\langle C(t)^* y_a^*, x\rangle = 0$, $x \in X$, $a \in [0,1]$ and the boundedness of the set $\{C(t)^* y_a^* : t \geq 0\}$ in X^*, for every fixed $a \in [0,1]$. If we define the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ as in the proof of Henríquez, then the d'Alambert functional equation $C(t_n)Cx + C(t)Cx = 2C(u_n)C(v_n)x$, $x \in X$, $n \in \mathbb{N}$ and the arguments contained on page 431 of the proof show that $\langle y_a^*, C(t)x\rangle = 0$, $t \geq 0$, $a \in [0,1]$. Hence, $\langle x^*, S(a)C(t)x\rangle = 0$, $t \geq 0$, $a \in [0,1]$, $x \in X$ and therefore $\langle x^*, S(a)Cx\rangle = 0$, $a \in [0,1]$, $x \in X$. Since $R(C^2)$ is dense in X and $C^2x = \lim_{a\to 0+} a^{-1}S(a)Cx$, $x \in X$, we get that the set $\{S(a)Cx : 0 \leq a \leq 1,\ x \in X\}$ is total in X, so that $x^* = 0$. This completes the proof. $\square$

We can similarly prove an analogue of [28, Proposition 1] for C-regularized cosine functions in weakly sequentially complete Banach spaces:

Proposition 2.22. *Let $C \in L(X)$ be injective, let A be a closed single-valued linear operator, and let $\overline{R(C)} = X$. Suppose that A generates a scalarly Stepanov almost periodic C-regularized cosine function $(C(t))_{t\geq 0}$. Then $(C(t))_{t\geq 0}$ is almost periodic, provided that X is weakly sequentially complete.*

The following proposition is motivated by Casarino's result [13, Proposition 3.1; 2)] (we feel duty bound to say that, in the formulation of this statement, one has to impose the condition on uniform boundedness of considered cosine operator function); in our approach, the operator $C \neq I$ need not be injective and $(C(t))_{t\geq 0}$ can be degenerate in time.

Proposition 2.23. *Let $\mathcal{A}$ be the integral generator of a uniformly bounded C-cosine function $(C(t))_{t\geq 0}$. Suppose that $x \in X$ satisfies that the mapping $t \mapsto C(t)x$, $t \geq 0$ is asymptotically almost periodic. Then the mapping $t \mapsto C(t)Cx$, $t \geq 0$ is almost periodic.*

Proof. The prescribed assumption implies that there exists a finite constant $M > 0$ such that $\|C(t)\| \leq M$, $t \geq 0$. Let $\epsilon > 0$ be given in advance. Then we can find numbers $l = l(\epsilon) > 0$ and $K = K(\epsilon) > 0$ such that every subinterval I of $[0,\infty)$ of length l contains, at least, one number τ such that $\|C(t+\tau)x - C(t)x\| \leq \epsilon/(2M + \|C\|)$ for all $t \geq K$. Let $s \geq K$. Using the d'Alambert functional equality $C(t)Cx = 2C(s)C(t+s)x - C(t+2s)Cx$, $t \geq 0$, we obtain

$$\begin{aligned}
&\|C(t+\tau)Cx - C(t)Cx\| \\
&\leq 2M\|C(t+s+\tau)x - C(t+s)x\| + \|C\|\|C(t+2s+\tau)x - C(t+2s)x\| \\
&\leq (2M + \|C\|)\epsilon/(2M + \|C\|) = \epsilon, \quad t \geq 0,
\end{aligned}$$

where $\tau \in I$ is chosen as above. This completes the proof. $\square$

Now we would like to raise the following question concerning Theorem 2.21 and Proposition 2.23.

Open problem. Let $\mathcal{A}$ be the integral generator of a bounded C-cosine function $(C(t))_{t\geq 0}$. Suppose that $x \in X$ satisfies that the mapping $t \mapsto C(t)x$, $t \geq 0$ is asymptotically Stepanov almost periodic. Is it true that the mapping $t \mapsto C(t)Cx$, $t \geq 0$ is almost periodic?

The following extension of Vesentini's result [58, Proposition 4] for degenerate C-groups is deduced similarly (cf. [33]-[34] for more details on the subject):

Proposition 2.24. *Suppose that $(S(t))_{t\in\mathbb{R}} \subseteq L(X)$ is a bounded, strongly continuous operator family commuting with C, and $S(t)S(s) = S(t+s)C$, $t, s \in \mathbb{R}$. If $x \in X$ satisfies that the mapping $t \mapsto S(t)x$, $t \geq 0$ is asymptotically almost periodic, then the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is almost periodic.*

In a series of his research papers, Rao has investigated the conditions under which the Stepanov almost periodic (bounded) solutions of certain abstract differential equations are almost periodic (see e.g. [46] and [47]). We close this subsection by explaining how we can prove a slight extension of the main result of paper [46], Theorem, for infinitesimal generators of almost periodic C-regularized groups. Before do that, let us agree on the following notion: Suppose that A and B are two closed, not necessarily densely defined, single-valued linear operators in X and a function $f : \mathbb{R} \to X$ is continuous. By a solution of the second-order differential equation

$$u''(t) = Au'(t) + Bu(t) + f(t) \quad \text{a.e. on } \mathbb{R}$$

we mean any two times differentiable function $u(t)$ with $u'(t) \in D(A)$, $u(t) \in D(B)$ for all $t \in \mathbb{R}$ and satisfying the above equation a.e. on $\mathbb{R}$.

Theorem 2.25. *Suppose that X is a reflexive Banach space, $f : \mathbb{R} \to X$ is an S^1-almost periodic continuous function, and A is the infinitesimal generator of an almost periodic non-degenerate C-regularized group $(T(t))_{t\in\mathbb{R}}$, where $C \in L(X)$ is injective. Let $u : \mathbb{R} \to X$, with its derivative $u'(t) \in D(A)$ for all $t \in \mathbb{R}$ be a solution of the differential equation*

$$u''(t) = Au'(t) + B(t)u(t) + f(t) \ \text{a.e. on } \mathbb{R}$$

where $B : \mathbb{R} \to L(X)$ is almost periodic. If $u(\cdot)$ is S^1-almost periodic and $u(\cdot)$ is S^1-bounded on $\mathbb{R}$, then $C^2u'(\cdot)$ and $C^2u(\cdot)$ are both almost periodic from $\mathbb{R}$ to X.

Proof. The proof of this result can be obtained by an insignificant modification of the proof of Theorem in [46]. First of all, note that for each $x \in D(A)$ we have that the mapping $t \mapsto T(t)x$, $t \in \mathbb{R}$ is continuously differentiable with $(d/dt)T(t)x = T(t)Ax$, $t \in \mathbb{R}$. Then the computation given in the proof of [46, Lemma 1] shows that

$$Cu'(t) = T(t)u'(0) + \int_0^t T(t-s)[B(s)u(s) + f(s)]\,ds, \quad t \in \mathbb{R},$$

which implies

$$T(-t)u'(t) = Cu'(0) + \int_0^t T(-s)[B(s)u(s) + f(s)]\,ds, \quad t \in \mathbb{R}.$$

Furthermore, we can simply prove by definition that $(T(-t))_{t\in\mathbb{R}}$ is an almost periodic C-regularized group with the infinitesimal generator $-A$. The only thing that should be noted else is that the assertion of [46, Lemma 2] holds for C-regularized groups. Speaking-matter-of-fact, suppose that $h(\cdot)$ is an almost periodic function from $\mathbb{R}$ to X. Then there exists a sequence of X-valued trigonometric polynomials $h_n(\cdot)$ converging uniformly to $h(\cdot)$ as $n \to \infty$. Using the fact that, if $g \in AP(\mathbb{R} : X)$ and $p \in AP(\mathbb{R} : \mathbb{C})$, then $gp \in AP(\mathbb{R} : X)$, we get that the mapping $t \mapsto T(-t)h_n(t)$, $t \in \mathbb{R}$ is almost periodic. Since $(T(-t))_{t\in\mathbb{R}}$ is almost periodic, it is uniformly bounded and therefore $T(-\cdot)h_n(\cdot)$ converges uniformly to $T(-\cdot)(\cdot)$ as $n \to \infty$, so that $T(-\cdot)(\cdot)$ is almost periodic, as well. $\square$

Arguing similarly, we can prove some results on almost periodic solutions of the first-order infinitesimal generator differential equation

$$u'(t) \in [A + B(t)]u(t) + f(t) \text{ a.e. on } \mathbb{R},$$

in reflexive Banach spaces. If A is the infinitesimal generator of an almost periodic non-degenerate C-regularized group $(T(t))_{t\in\mathbb{R}}$, where $C \in L(X)$ is injective, $u : \mathbb{R} \to D(A)$ is an S^1-almost periodic solution of the above differential equation (for the topology of X), $f : \mathbb{R} \to X$ is an S^1-almost periodic continuous function and $B : \mathbb{R} \to L(X)$ is almost periodic, then the mapping $C^2u(\cdot)$ is almost periodic, as well.

2.3. **Subspace asymptotical almost periodicity of C-distribution semigroups and C-distribution cosine functions.** In this subsection, we reconsider some structural results of Xie, Li, Huang [63] and Pilipović, Velinov, Kostić [36]. Here, we will always assume that the operator $C \in L(X)$ is injective and that any operator family under our consideration is non-degenerate. For the notion and notation of various types of C-distribution semigroups and C-distribution cosine functions used henceforth, we refer the reader to [36].

Let $\mathcal{G}$ be a C-distribution semigroup, (C-DS) for short. Define

$$G(T) := \big\{(x, y) \in X \times X : \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0\big\}.$$

Then it can be easily seen that $G(T)$ is a closed linear operator commuting with C. We define the (infinitesimal) generator A of $\mathcal{G}$ by $A := G(-\delta')$. By $D(\mathcal{G})$ we denote the set consisting of those elements $x \in X$ for which $x \in D(G(\delta_t))$, $t \geq 0$ and the mapping $t \mapsto G(\delta_t)x$, $t \geq 0$ is continuous. We have

$$D\big(G(\delta_s)G(\delta_t)\big) = D\big(G(\delta_s * \delta_t)\big) \cap D\big(G(\delta_t)\big) = D\big(G(\delta_{t+s})\big) \cap D\big(G(\delta_t)\big), \tag{2.13}$$

for $t, s \geq 0$, which clearly implies $G(\delta_t)(D(\mathcal{G})) \subseteq D(\mathcal{G})$, $t \geq 0$.

The solution space for a closed linear operator A, denoted by $Z(A)$, is defined as the set of all $x \in X$ for which there exists a continuous mapping $u(\cdot, x) \in C([0,\infty) : X)$ satisfying $\int_0^t u(s, x)\,ds \in D(A)$ and $A\int_0^t u(s, x)\,ds = u(t, x) - x$, $t \geq 0$. If A generates a (C-DS) $\mathcal{G}$, then it is well known that $Z(A) = D(\mathcal{G})$ and that for each $x \in Z(A)$, we have $u(t, x) = G(\delta_t)x$, $t \geq 0$ and $\mathcal{G}(\psi)x = \int_0^\infty \psi(t)Cu(t, x)\,dt$, $\psi \in \mathcal{D}_0$.

We say that a function $u(\cdot; x, y)$ is a mild solution of the abstract Cauchy problem

$$\begin{gathered} u \in C([0,\infty) : [D(A)]) \cap C^2([0,\infty) : X), \\ u''(t) = Au(t), \quad t \geq 0, \\ u(0) = x, \quad u'(0) = y \end{gathered} \tag{2.14}$$

if the mapping $t \mapsto u(t; x, y)$, $t \geq 0$ is continuous, $\int_0^t (t - s)u(s; x, y)ds \in D(A)$ and $A\int_0^t (t - s)u(s; x, y)ds = u(t; x, y) - x - ty$, $t \geq 0$; henceforward we primarily consider the mild solutions of (2.14) with $y = 0$. Denote by $Z_2(A)$ the set consisting of all $x \in X$ for which there exists such a solution. We refer the reader to [36] for the notion of integral generator of a C-distribution cosine function and operator $G(\delta_t)$, $t \geq 0$ ($x \in Z_2(A)$). Let us recall that, for every $x \in Z_2(A)$, one has $G(\delta_t)(Z_2(A)) \subseteq Z_2(A)$, $t \geq 0$, $2G(\delta_s)G(\delta_t)x = G(\delta_{t+s})x + G(\delta_{|t-s|})x$, $t,\ s \geq 0$ and $\mathbf{G}(\varphi)x = \int_0^\infty \varphi(t)CG(\delta_t)x\,dt$, $\varphi \in \mathcal{D}_0$.

Now we are ready to introduce the following definition (cf. [36, Definition 2.1] for the corresponding notion introduced in the case of almost periodicity; this can be done for the general class of (a,k)-regularized C-resolvent families, as well).

Definition 2.26. Let $\mathbf{G}$ be a $(C-DCF)$ generated by A, resp. let $\mathcal{G}$ be a (C-DS) generated by A. Suppose that $\tilde{X}$ is a linear subspace of $Z_2(A)$, resp. $x \in Z(A)$. Then it is said that $\mathbf{G}$ is $\tilde{X}$-asymptotically almost periodic if for each $x \in \tilde{X}$ the mapping $t \mapsto G(\delta_t)x$, $t \geq 0$ is asymptotically almost periodic.

Similarly as in [36, Remark 2.2], we have the following observations.

Remark 2.27. (i) The notions from Definition 2.26 can be introduced for arbitrary operator family $(F(t))_{t\geq 0}$ consisted of possibly non-linear and possibly non-continuous single valued operators.

(ii) Let $\mathbf{G}$ be a $(C-DCF)$ generated by A, resp. let $\mathcal{G}$ be a (C-DS) generated by A, and let $\tilde{X}$ be a linear subspace of $Z_2(A)$, resp. $x \in Z(A)$. Assume that $\mathbf{G}$, resp. $\mathcal{G}$, is $\tilde{X}$-asymptotically almost periodic. Let $\mathbf{G}_1$ be another (C_1-DCF) generated by A, resp. let $\mathcal{G}_1$ be another (C_1-DS) generated by A. Then $\mathbf{G}_1$, resp. $\mathcal{G}_1$, is $\tilde{X}$-asymptotically almost periodic, as well.

The following characterization of subspace asymptotical almost periodicity of C-distribution semigroups is motivated by [63, Lemma 2.1, Theorem 2.2, Corollary 2.3].

Theorem 2.28. *Let $\mathcal{G}$ be a* (C-DS) *generated by A, and let $\tilde{X}$ be a linear subspace of $Z(A)$. Then the following assertions are equivalent:*

(i) *$\mathcal{G}$ is $\tilde{X}$-asymptotically almost periodic.*

(ii) *For every $x \in Z(A) \cap \tilde{X}$, there exist elements $y, z \in Z(A)$ such that $y + z \in \tilde{X}$, $x = y + z$ and the following two conditions hold:*
   (a) *The mapping $t \mapsto G(\delta_t)y$, $t \geq 0$ belongs to the space $AP([0,\infty) : X)$.*
   (b) *The mapping $t \mapsto G(\delta_t)z$, $t \geq 0$ belongs to the space $C_0([0,\infty) : X)$.*

(iii) *For every $x \in Z(A) \cap \tilde{X}$, there exist elements $y, z \in Z(A)$ such that $y + z \in \tilde{X}$, $x = y + z$ and the following two conditions hold:*
   (c) *The mapping $F : [0,\infty) \to C_b([0,\infty) : X)$, defined by $F(t) := G(\delta_{\cdot+t})y$, $t \geq 0$, belongs to the space $AP([0,\infty) : C_b([0,\infty) : X))$.*
   (d) *The mapping $H : [0,\infty) \to C_b([0,\infty) : X)$, defined by $H(t) := G(\delta_{\cdot+t})z$, $t \geq 0$, belongs to the space $C_0([0,\infty) : C_b([0,\infty) : X))$.*

Proof. Suppose first that $x \in Z(A) \cap \tilde{X}$. Arguing as in the proof of [63, Lemma 2.1], we get the existence of a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals and a mapping $\varphi \in C_0([0,\infty) : X)$ such that $\lim_{n\to\infty} t_n = \infty$, the mapping $h : [0,\infty) \to X$ defined by $h(t) := \lim_{n\to\infty} G(\delta_{t+t_n})x$, $t \geq 0$ is almost periodic, and

$$G\big(\delta_t\big)x = h(t) + \varphi(t), \quad t \geq 0. \tag{2.15}$$

Set $y := \lim_{n\to\infty} G(\delta_{t_n})x$, $z := x - y$ and $y_n := G(\delta_{t_n})x$, $n \in \mathbb{N}$. Let $t \geq 0$ be temporarily fixed. Then $\lim_{n\to\infty} y_n = y$ and (2.13) yields $\lim_{n\to\infty} G(\delta_t)y_n =$

$\lim_{n\to\infty} G(\delta_{t+t_n})x = h(t)$. By the closedness of $G(\delta_t)$, we get $y \in D(G(\delta_t))$ and $G(\delta_t)y = h(t)$. Therefore, we have $y, z \in Z(A)$, $y + z = x \in \tilde{E}$ and, because of (2.15),

$$G\big(\delta_t\big)x = G\big(\delta_t\big)y + G\big(\delta_t\big)z, \quad t \geq 0. \tag{2.16}$$

This implies (a)-(b) in (ii). To prove that (ii) implies (iii), choose a number $\epsilon > 0$ arbitrarily. Then the almost periodicity of mapping $h(\cdot)$, defined as in the proof of implication (i) $\Rightarrow$ (ii), yields that there exists $l > 0$ such that any interval $I \subseteq [0, \infty)$ of length l contains an ϵ-period τ for $h(\cdot)$, so that $\sup_{t\geq 0} \|h(t + \tau) - h(t)\| = \sup_{t\geq 0} \|G(\delta_{t+\tau})y - G(\delta_t)y\| \leq \epsilon$. But, this is equivalent to say that $\sup_{t\geq 0} \|F(t + \tau) - F(t)\|_{C_b([0,\infty):X)} = \sup_{t,s\geq 0} \|G(\delta_{t+s+\tau})y - G(\delta_{t+s})y\| = \sup_{t\geq 0} \|G(\delta_{t+\tau})y - G(\delta_t)y\| < \epsilon$, which immediately implies (c). On the other hand, it is clear that the mapping $H(\cdot)$ is well defined because for each $t \geq 0$ we have that $\lim_{s\to+\infty} G(\delta_{s+t})z = \lim_{s\to+\infty} \varphi(s + t) = 0$, where we define $\varphi(\cdot)$ as before. Moreover, $\lim_{t\to+\infty} \|G(\delta_{\cdot+t})z\|_{C_b([0,\infty):X)} = 0$ is equivalent to say that $\lim_{t\to+\infty}[\sup_{s\geq 0} \|G(\delta_{s+t})z\|] = 0$, which can be easily verified to be true since for any $\epsilon > 0$ we have the existence of a sufficiently large number $M > 0$ such that $\|\varphi(v)\| = \|G(\delta_v)z\| < \epsilon$, $v > M$. The converse statement (iii) $\Rightarrow$ (ii) is much easier because (c) and (d) in turn imply that the mapping $t \mapsto G(\delta_t)y \in X$, $t \geq 0$ is almost periodic and the mapping $t \mapsto G(\delta_t)z$, $t \geq 0$ belongs to the space $C_0([0, \infty) : X)$. The implication (ii) $\Rightarrow$ (i) is trivial, finishing the proof. $\square$

Remark 2.29. In [63], the authors have considered the asymptotical almost periodicity of C-regularized semigroups by assuming that their integral generators have no eigenvalues in $(0, \infty)$. It is worth noting that this is not the case in our analysis, where we allow that the point spectrum of integral generator of a $\tilde{E}$-asymptotically almost periodic C-distribution semigroup could have a non-empty intersection with any ray (ω, ∞), where $\omega > 0$. This is very important in the case that $\tilde{E} \neq E$, because then we can construct a great number of non-exponential C-distribution semigroups (C-distribution cosine functions) that are $\tilde{E}$-almost periodic [36], with the subspace $\tilde{E}$ being dense in E, by using the Desch-Schappacher-Webb criterion for chaos of strongly continuous semigroups (see [32, Chapter III] for a comprehensive survey of results on hypercyclic and chaotic abstract Volterra integro-differential equations). The point spectrum of integral generator of such a C-distribution semigroup (C-distribution cosine function) may contain ray (ω, ∞), for some $\omega > 0$; see e.g. [32, Theorem 2.2.10, Example 3.2.37(iii)] and [31, Theorem 3.1.36].

It is clear that Proposition 2.23-Proposition 2.24 cannot be reconsidered for C-distribution semigroups and C-distribution cosine functions because the operators $G(\delta_t)$ are generally unbounded in this framework ($t \geq 0$).

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