Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 242, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

BLOW UP OF SOLUTIONS FOR VISCOELASTIC WAVE EQUATIONS OF KIRCHHOFF TYPE WITH ARBITRARY POSITIVE INITIAL ENERGY

ERHAN PIŞKIN, AYŞE FIDAN

Communicated by Mokhtar Kirane

Abstract. In this article we consider the nonlinear Viscoelastic wave equations of Kirchhoff type

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + u_t = (p+1)|v|^{q+1}|u|^{p-1}u,$$
$$v_{tt} - M(\|\nabla v\|^2)\Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + v_t = (q+1)|u|^{p+1}|v|^{q-1}v$$

with initial conditions and Dirichlet boundary conditions. We proved the global nonexistence of solutions by applying a lemma by Levine, and the concavity method.

1. Introduction

In this article we consider the initial boundary value problem

$$\begin{gathered}
u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + u_t = (p+1)|v|^{q+1}|u|^{p-1}u, \\
(x,t) \in \Omega \times (0,T), \\
v_{tt} - M(\|\nabla v\|^2)\Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + v_t = (q+1)|u|^{p+1}|v|^{q-1}v, \\
(x,t) \in \Omega \times (0,T), \\
u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded domain with a smooth boundary $\partial\Omega$ in R^n $(n = 1,2,3)$, $p > 1$, $q > 1$ and $M(s)$ is a nonnegative C^1 function such as

$$M(s) = a + bs^\gamma, \quad s \geq 0$$

for $s \geq 0$, $a > 0$, $b \geq 0$, $a + b \geq 0$, $\gamma > 0$. The function $g_i : R^+ \to R^+$ represents the kernel of the memory term and is a given positive function to be specified later.

2010 *Mathematics Subject Classification.* 35B44, 35L05, 35L53.
Key words and phrases. Blow up; viscoelastic wave equation; arbitrary positive initial energy.

Submitted March 11, 2017. Published October 4, 2017.

The single viscoelastic wave equation of Kirchhoff type of the form

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(u_t) = |u|^{q-1}u, \tag{1.2}$$

has been extensively studied and many results concerning nonexistence have been proved. See in this regard [9, 5]. When $M \equiv 1$, the equation (1.2) reduces to

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + h(u_t) = |u|^{q-1}u. \tag{1.3}$$

The existence, and blow up in finite time of solution for (1.3) were established (see [10, 12, 13, 18] and references therein).

For the case $M \equiv 1$, system (1.1) reduces to

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + v_t = f_1(u,v). \end{aligned} \tag{1.4}$$

Han and Wang [2] obtained the existence and nonexistence of the solution of problem (1.4). Messaoudi and Said Houari [14] considered problem (1.4) and improved the blow up result in [2], for positive initial energy, using the same techniques as in [3]. Ma et al. [11] studied the blow up of the solution of the problem (1.4) with arbitrary positive initial energy. For more information about (1.4), see references [4, 6, 7, 16, 17].

In this article, we consider problem (1.4) and prove the blow up result by a concavity method with arbitrary positive initial energy.

This paper is organized as follows. In section 2, we present some lemmas. In section 3, we show the blow up of solutions.

2. Preliminaries

In this section, we introduce some notation, assumptions and lemmas which will be needed in this paper. Let $\|\cdot\|$ and $\|\cdot\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

To state and prove our main results, we make the following assumptions:

(A1) $g_i \in C^1[0,\infty]$ $(i=1,2)$ is a non-negative and non-increasing differentiable function satisfying

$$1 - \int_0^\infty g_i(s)ds = l_i > 0, \quad i = 1,2.$$

(A2) $g_i(t) \geq 0$, $g_i'(t) \leq 0$, for all $t \geq 0$, $i = 1,2$.

(A3) The function $e^{1/2}g(t)$ is of positive type in the following sense:

$$\int_0^t v(s)\int_0^s e^{(s-\tau)/2}g_i(s-\tau)v(\tau)d\tau ds \geq 0, \quad \forall v \in C^1[0,\infty) \text{ and } \forall t > 0.$$

To obtain the blow up result, we need the following lemma which repeats the same one of [9] with slight modification, we will omit it.

Lemma 2.1. *There exists positive constants m_i and $s \geq 0$, $a > 0$, $b \geq 0$, $\gamma > 0$ such that*

$$\frac{p+q+2}{2}\overline{M}(s) - \Big[M(s) + \frac{p+q+2}{2}\int_0^\infty g_i(\tau)d\tau\Big]s \geq m_i s, \quad \forall s \geq 0, \tag{2.1}$$

where

$$\overline{M}(s) = \int_0^s M(\tau)d\tau.$$

Lemma 2.2 ([15])**.** *For any $g \in C^1$ and $\phi \in H^1(0,T)$ we have*

$$\begin{aligned} &\int_\Omega \int_0^t g(t-\tau)\Delta\phi(\tau)\phi'(t)d\tau\, dx \\ &= -\frac{1}{2}(g' \circ \nabla\phi)(t) + \frac{1}{2}g(t)\|\nabla\phi\|^2 + \frac{1}{2}\frac{d}{dt}[(g\circ\nabla\phi)(t) - \int_0^t g(\tau)\|\nabla\phi\|^2 d\tau]. \end{aligned} \tag{2.2}$$

Lemma 2.3 (Sobolev-Poincaré inequality [1])**.** *Let p be a number with $2 \le p < \infty$ $(n = 1, 2)$ or $2 \le p \le 2n/(n-2)$ $(n \ge 3)$, then there is a constant $C_* = C_*(\Omega,\ p)$ such that*

$$\|u\|_p \le C_*\|\nabla u\|, \quad \forall u \in H_0^1(\Omega).$$

Lemma 2.4 ([8])**.** *Suppose that $F(t)$ is a twice continuously differentiable positive function satisfying*

$$F''(t)F(t) - (1+\alpha)[F'(t)]^2 \ge 0, \quad \forall t \ge 0$$

where $\alpha > 0$. If $F(0) > 0$ and $F'(0) > 0$. Then there exists a positive constant $T^* \le \frac{F(0)}{\alpha F'(0)}$ such that $\lim_{t\to T^*} F(t) = \infty$.

3. Blow up of solution

In this section, we shall discuss the global nonexistence of the problem (1.1). Let us first introduce the functionals

$$\begin{aligned} J(t) = &\frac{1}{2}\int_0^t g_1(\tau)d\tau\|\nabla u\|^2 + \frac{1}{2}\int_0^t g_2(\tau)d\tau\|\nabla v\|^2 \\ &+ \frac{1}{2}[(g_1\circ\nabla u)(t) + (g_2\circ\nabla v)(t)] - \int_\Omega |u|^{p+1}|v|^{q+1}\, dx, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} I(t) = &M(\|\nabla u(t)\|^2)\|\nabla u\|^2 + M(\|\nabla v(t)\|^2)\|\nabla v\|^2 \\ &- (p+q+2)\int_\Omega |u|^{p+1}|v|^{q+1}\, dx. \end{aligned} \tag{3.2}$$

We also define the energy function

$$\begin{aligned} E(t) = &\frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{1}{2}[\overline{M}(\|\nabla u(t)\|_2^2) + \frac{1}{2}\overline{M}(\|\nabla v(t)\|_2^2)] \\ &- \frac{1}{2}\int_0^t g_1(\tau)d\tau\|\nabla u\|^2 - \frac{1}{2}\int_0^t g_2(\tau)d\tau\|\nabla v\|^2 \\ &+ \frac{1}{2}[(g_1\circ\nabla u)(t) + (g_2\circ\nabla v)(t)] - \int_\Omega |u|^{p+1}|v|^{q+1}\, dx, \end{aligned} \tag{3.3}$$

where

$$(\phi\circ\psi)(t) = \int_0^t \phi(t-\tau)\int_\Omega |\psi(t)-\psi(\tau)|^2\, dx\, d\tau = \int_0^t \phi(t-\tau)\|[\psi(t)-\psi(\tau)]\|^2 d\tau.$$

Finally, we define

$$W = \big\{(u,v) : (u,v) \in H_0^1(\Omega)\times H_0^1(\Omega),\ I(u,v) > 0\} \cup \{(0,0)\big\}. \tag{3.4}$$

The next lemma shows that our energy functional (3.3) is a nonincreasing function along the solution of the problem (1.1).

Lemma 3.1. *$E(t)$ is a non-creasing function for $t \geq 0$, that is*

$$E'(t) \leq -(\|u_t\|^2 + \|v_t\|^2) + \frac{1}{2}[(g_1' \circ \nabla u)(t) + (g_2' \circ \nabla v)(t)] \leq 0, \tag{3.5}$$

and

$$E(t) \leq E(0) - \int_0^t (\|u_\tau\|^2 + \|v_\tau\|^2) d\tau. \tag{3.6}$$

Proof. Multiplying the first equation of (1.1) by u_t and the second equation by v_t, integrating over Ω, and using (2.2) and assumption (A1)–(A2), we obtain (3.5). □

Lemma 3.2 ([18])**.** *Assume that g_i satisfies assumptions* (A1), (A2) *and $H(t)$ is a function that is twice continuously differentiable, satisfying*

$$\begin{gathered} H''(t) + H'(t) \\ > 2\int_0^t g(t-\tau)\int_\Omega [\nabla u(\tau,x)\nabla u(t,x) + \nabla u(\tau,x)\nabla u(t,x)]\,dx\,d\tau \\ H(0) > 0, \quad H'(0) > 0 \end{gathered} \tag{3.7}$$

for every $t \in [0, T_0)$ and $(u(x,t), v(x,t))$ is the solution of problem (1.1)*. Then the function $H(t)$ is strictly increasing on $[0, T_0)$.*

Lemma 3.3. *Assume $(u_0, v_0) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega))$, $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and satisfy*

$$\int_\Omega (u_0 u_1 + v_0 v_1)\,dx \geq 0. \tag{3.8}$$

If the local solution $(u(t), v(t))$ of (1.1) *satisfies*

$$I(u(t), v(t)) < 0,$$

then $H(t) = \|u(t,\cdot)\|_2^2 + \|v(t)\|_2^2$ is strictly increasing on $[0, T)$.

Proof. Since

$$\begin{aligned} I(t) &= M(\|\nabla u(t)\|^2)\|\nabla u\|^2 + M(\|\nabla v(t)\|^2)\|\nabla v\|^2 \\ &\quad - (p+q+2)\int_\Omega |u|^{p+1}|v|^{q+1}\,dx < 0, \end{aligned}$$

and $(u(t), v(t))$ is the local solution of problem (1.1), by a simple computation, we have

$$H(t) = \|u(t,\cdot)\|_2^2 + \|v(t,\cdot)\|_2^2 = \int_\Omega |u(t)|^2\,dx + \int_\Omega |v(t)|^2\,dx, \tag{3.9}$$

$$\frac{1}{2}\frac{d}{dt}H(t) = \int_\Omega u u_t\,dx + \int_\Omega v v_t\,dx \tag{3.10}$$

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{dt^2}H(t) \\ &= \int_\Omega |u_t|^2\,dx + \int_\Omega u u_{tt}\,dx + \int_\Omega |v_t|^2\,dx + \int_\Omega v v_{tt}\,dx \\ &= \int_\Omega |u_t|^2\,dx + \int_\Omega |v_t|^2\,dx + \int_\Omega u M(\|\nabla u\|^2)\Delta u\,dx \\ &\quad - \int_\Omega u \int_0^t g_1(t-\tau)\Delta u(\tau) d\tau\,dx - \int_\Omega u u_t\,dx + \int_\Omega u(p+1)|v|^{q+1}|u|^{p-1}u\,dx \end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega vM(\|\nabla v\|^2)\Delta v\,dx-\int_\Omega v\int_0^t g_2(t-\tau)\Delta v(\tau)d\tau\,dx\\
&-\int_\Omega vv_t\,dx+\int_\Omega v(q+1)|u|^{p+1}|v|^{q-1}v\,dx\\
\ge&\int_\Omega M(\|\nabla u\|^2)u\Delta u\,dx-\int_\Omega\int_0^t g_1(t-\tau)u\Delta u(\tau)d\tau\,dx\\
&-\int_\Omega uu_t\,dx+\int_\Omega(p+1)|v|^{q+1}|u|^{p-1}u\,dx\\
&+\int_\Omega M(\|\nabla v\|^2)v\Delta v\,dx-\int_\Omega\int_0^t g_2(t-\tau)v\Delta v(\tau)d\tau\,dx\\
&-\int_\Omega vv_t\,dx+\int_\Omega(q+1)|u|^{p+1}|v|^{q-1}v\,dx\\
>&-\int_\Omega(uu_t+vv_t)\,dx+\int_0^t g_1(t-\tau)\int_\Omega\nabla u(\tau)\nabla u(t)\,dx\,d\tau\\
&+\int_0^t g_2(t-\tau)\int_\Omega\nabla v(\tau)\nabla v(t)\,dx\,d\tau\\
=&-\frac12\frac{dH}{dt}+\int_0^t g_1(t-\tau)\int_\Omega\nabla u(\tau)\nabla u(t)\,dx\,d\tau\\
&+\int_0^t g_2(t-\tau)\int_\Omega\nabla v(\tau)\nabla v(t)\,dx\,d\tau
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac12\frac{d^2H}{dt^2}+\frac12\frac{dH}{dt}\\
&>\int_0^t g_1(t-\tau)\int_\Omega\nabla u(\tau)\nabla u(t)\,dx\,d\tau+\int_0^t g_2(t-\tau)\int_\Omega\nabla v(\tau)\nabla v(t)\,dx\,d\tau.
\end{aligned}
$$

Therefore, by (3.7), the proof is complete. □

Theorem 3.4. *Under* (A1)–(A3) *hold, and the initial data*

$$
\begin{gathered}
(u_0,v_0)\in(H_0^1(\Omega)\cap H^2(\Omega))\times(H_0^1(\Omega)\cap H^2(\Omega)),\\
(u_1,v_1)\in H_0^1(\Omega)\times H_0^1(\Omega)
\end{gathered}
$$

satisfy

$$E(0)>0, \tag{3.11}$$

$$I(u_0,v_0)<0, \tag{3.12}$$

$$\int_\Omega(u_0u_1+v_0v_1)\,dx\ge0, \tag{3.13}$$

$$\|u_0\|^2+\|v_0\|^2\ge\frac{(p+q+2)\eta}{\min\{m_1,m_2\}}E(0). \tag{3.14}$$

Then the solution of problem (1.1) *blows up in finite* $T<\infty$*.*

Lemma 3.5. *If* $(u_0,v_0)\in(H_0^1(\Omega)\cap H^2(\Omega))\times(H_0^1(\Omega)\cap H^2(\Omega))$ *and* $(u_1,v_1)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *satisfy the assumptions in Theorem 3.4, then the solution* (u,v) *of* *the problem* (1.1) *satisfies*

$$I(u(t,x),v(t,x))<0, \tag{3.15}$$

$$\|u(t)\|^2+\|v(t)\|^2\geq \frac{(p+q+2)\eta}{\min\{m_1,m_2\}}E(0), \tag{3.16}$$

for every $t\in[0,T)$.

Proof. We will prove this lemma by a contradiction argument. First we assume that (3.15) is not true over $[0,T)$, so, that there exists a time $t_1>0$ such that

$$t_1=\min\{t\in(0,T):I(u,v)=0\}. \tag{3.17}$$

Since $I(u,v)<0$ on $[0,t_1)$, by Lemma 3.3, we see that $H(t)=\|u(t,\cdot)\|_2^2+\|v(t,\cdot)\|_2^2$ is strictly increasing over $[0,t_1)$, which implies

$$H(t)=\|u(t,\cdot)\|_2^2+\|v(t,\cdot)\|_2^2>\|u_0\|^2+\|v_0\|^2>\frac{(p+q+2)\eta}{\min\{m_1,m_2\}}E(0).$$

It is obvious that $H(t)=\|u(t,\cdot)\|_2^2+\|v(t,\cdot)\|_2^2$ is continuous on $[0,t_1)$. Thus we obtain the inequality

$$H(t_1)=\|u(t_1,.)\|_2^2+\|v(t_1,.)\|_2^2\geq\frac{(p+q+2)\eta}{\min\{m_1,m_2\}}E(0)\,. \tag{3.18}$$

On the other hand, by (3.17) we have

$$\begin{aligned}
E(0)&\geq E(t_1)+\int_0^t[\|u_\tau\|^2+\|v_\tau\|^2]d\tau\\
&=\frac12\Big(\|u_t\|^2+\|v_t\|^2\Big)+\frac12[\overline{M}(\|\nabla u\|^2)+\overline{M}(\|\nabla v\|^2)]\\
&\quad-\frac12\Big(\int_0^t g_1(\tau)\|\nabla u(t_1)\|^2d\tau+\int_0^t g_2(\tau)\|\nabla v(t_1)\|^2d\tau\Big)\\
&\quad+\frac12((g_1\circ\nabla u)(t_1)+(g_2\circ\nabla v)(t_1))\\
&\quad-\int_\Omega|u|^{p+1}|v|^{q+1}\,dx+\int_0^t[\|u_\tau\|^2+\|v_\tau\|^2]d\tau\\
&\geq\frac12[\overline{M}(\|\nabla u\|^2)+\overline{M}(\|\nabla v\|^2)]\\
&\quad-\frac12\Big(\int_0^t g_1(\tau)\|\nabla u(t_1)\|^2d\tau+\int_0^t g_2(\tau)\|\nabla v(t_1)\|^2d\tau\Big)-\int_\Omega|u|^{p+1}|v|^{q+1}\,dx.
\end{aligned}$$

Combining this inequality and (3.18), we have

$$\begin{aligned}
&(p+q+2)E(0)\\
&\geq\frac{p+q+2}{2}\overline{M}(\|\nabla u(t_1)\|^2)+\frac{p+q+2}{2}\overline{M}(\|\nabla v(t_1)\|^2)\\
&\quad-\frac{p+q+2}{2}(\int_0^t g_1(\tau)\|\nabla u(t_1)\|^2d\tau+\int_0^t g_2(\tau)\|\nabla v(t_1)\|^2d\tau)\\
&\quad-M(\|\nabla u(t_1)\|^2)\|\nabla u(t_1)\|^2-M(\|\nabla v(t_1)\|^2)\|\nabla v(t_1)\|^2
\end{aligned}$$

By (2.1), we get

$$(p+q+2)E(0)$$

$$\begin{aligned}
&\geq \frac{p+q+2}{2}\overline{M}(\|\nabla u(t_1)\|^2) \\
&\quad - [M(\|\nabla u(t_1)\|^2) + \frac{p+q+2}{2}\int_0^t g_1(\tau)d\tau]\|\nabla u(t_1)\|^2 \\
&\quad + \frac{p+q+2}{2}\overline{M}(\|\nabla v(t_1)\|^2) \\
&\quad - [M(\|\nabla v(t_1)\|^2) + \frac{p+q+2}{2}\int_0^t g_2(\tau)d\tau]\|\nabla v(t_1)\|^2 \\
&\geq m_1\|\nabla u(t_1)\|^2 + m_2\|\nabla v(t_1)\|^2 \\
&\geq \min\{m_1, m_2\}[\|\nabla u(t_1)\|^2 + \|\nabla v(t_1)\|^2].
\end{aligned}$$

Thus, by the Poincaré inequality, we have

$$(p+q+2)E(0) \geq \min\{m_1, m_2\}\frac{1}{\eta}[\|u(t_1)\|^2 + \|v(t_1)\|^2],$$

$$H(t_1) = \|u(t_1)\|^2 + \|v(t_1)\|^2 \leq \frac{(p+q+2)\eta}{\min\{m_1, m_2\}}E(0)$$

for every $t \in [0, T)$. The proof is complete. $\square$

4. Proof of Theorem 3.4

To prove our main result, we adopt the concavity method introduced by Levine and define the auxiliary function

$$\begin{aligned}
F(t) = \|u(t)\|^2 + \|v(t)\|^2 + \int_0^t (\|u(\tau)\|^2 + \|v(\tau)\|^2)d\tau \\
+ (t_2 - t)(\|u_0\|^2 + \|v_0\|^2) + \beta(t_3 + t)^2
\end{aligned} \tag{4.1}$$

where t_2, t_3 and β are positive constants, which will be determined later.

By direct computations, we obtain

$$\begin{aligned}
F'(t) &= 2\int_\Omega (uu_t + vv_t)d\tau + 2\int_0^t \int_\Omega (uu_\tau + vv_\tau)\,dx\,d\tau - \|u_0\|^2 - \|v_0\|^2 \\
&\quad - (\|u_0\|^2 + \|v_0\|^2) + 2\beta(t_3 + t) \\
&= 2\int_\Omega (uu_t + vv_t)\,dx + 2\int_0^t \int_\Omega (uu_\tau + vv_\tau)\,dx\,d\tau + 2\beta(t_3 + t)
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&F''(t) \\
&= 2\int_\Omega (u_t^2 + v_t^2)\,dx + 2\int_\Omega (uu_{tt} + vv_{tt})\,dx + 2\int_\Omega (uu_t + vv_t)\,dx + 2\beta \\
&= 2\|u_t\|^2 + 2\|v_t\|^2 + 2\int_\Omega M(\|\nabla u\|^2)u\Delta u\,dx - 2\int_0^t g_1(t-\tau)\int_\Omega u\Delta u(\tau)\,dx\,d\tau \\
&\quad - 2\int_\Omega (uu_t + vv_t)\,dx + 2(p+1)\int_\Omega |v|^{q+1}|u|^{p+1}\,dx + 2\int_\Omega M(\|\nabla v\|^2)v\Delta v\,dx \\
&\quad - 2\int_0^t g_2(t-\tau)\int_\Omega v\Delta v(\tau)\,dx\,d\tau + 2(q+1)\int_\Omega |u|^{p+1}|v|^{q+1}\,dx \\
&\quad + 2\int_\Omega (uu_t + vv_t)\,dx + 2\beta
\end{aligned}$$

By Young and Poincare inequalities, (3.6), (3.14), Lemma 3.3, we obtain

$$
\begin{aligned}
&F''(t)\\
&\geq (p+q+4)(\|u_t\|^2+\|v_t\|^2)+2\min\{m_1,m_2\}[\|\nabla u(t)\|^2+\|\nabla v(t)\|^2]\\
&\quad +2(p+q+2)\Big(-E(0)+\int_0^t[\|u_\tau\|^2+\|v_\tau\|^2]d\tau\Big)+2\beta\\
&=(p+q+4)(\|u_t\|^2+\|v_t\|^2)+2\min\{m_1,m_2\}[\|\nabla u(t)\|^2+\|\nabla v(t)\|^2]\\
&\quad -2(p+q+2)E(0)+2(p+q+2)\int_0^t[\|u_\tau\|^2+\|v_\tau\|^2]d\tau+2\beta\\
&\geq (p+q+4)(\|u_t\|^2+\|v_t\|^2)+2\min\{m_1,m_2\}\frac{1}{\eta}(\|\nabla u(t)\|^2+\|\nabla v(t)\|^2)\\
&\quad -2(p+q+2)E(0)+2(p+q+2)\int_0^t[\|u_\tau\|^2+\|v_\tau\|^2]d\tau+2\beta\\
&\geq (p+q+4)(\|u_t\|^2+\|v_t\|^2)+2\min\{m_1,m_2\}\frac{1}{\eta}(\|\nabla u(t)\|^2+\|\nabla v(t)\|^2)\\
&\quad -2(p+q+2)E(0)+2(p+q+2)\int_0^t[\|u_\tau\|^2+\|v_\tau\|^2]d\tau+2\beta\geq 0
\end{aligned}
$$

which means that $F''(t)>0$ for every $t\in(0,T)$. Since $F'(t)\geq 0$ and $F(t)\geq 0$, thus we obtain that $F'(t)$ and $F(t)$ are strictly increasing on $[0,T)$.

Thus, we can choose β to satisfy

$$
\min\{m_1,m_2\}(\|u_0\|^2+\|v_0\|^2)-(p+q+2)\eta E(0)>\beta(p+q+2) \tag{4.3}
$$

consequently,

$$
\begin{aligned}
F''(t)&\geq (p+q+4)(\|u_t\|^2+\|v_t\|^2)+2(p+q+2)\int_0^t[\|u_\tau\|^2+\|v_\tau\|^2]d\tau\\
&\quad +(p+q+4)\beta.
\end{aligned} \tag{4.4}
$$

As far as β is fixed, we select t_3 large enough satisfying

$$
\frac{p+q}{2}\Big(\int_\Omega(u_0u_1+v_0v_1)\,dx+\beta t_3\Big)>\|u_0\|^2+\|v_0\|^2. \tag{4.5}
$$

From (4.1), (4.2) and (4.5), we now choose

$$
t_2>\frac{\|u_0\|^2+\|v_0\|^2}{\frac{p+q}{2}(\int_\Omega(u_0u_1+v_0v_1)\,dx+\beta t_3)},
$$

which ensures that

$$
t_2>\frac{\|u_0\|^2+\|v_0\|^2}{\frac{p+q}{2}(\int_\Omega(u_0u_1+v_0v_1)\,dx+\beta t_3)}=\frac{4}{p+q}\frac{F(0)}{F'(0)}. \tag{4.6}
$$

Now let

$$
\begin{gathered}
A=\|u(t)\|^2+\|v(t)\|^2+\int_0^t[\|u(\tau)\|^2+\|v(\tau)\|^2]\,d\tau+\beta(t_3+t)^2,\\
B=\frac{1}{2}F'(t)=\int_\Omega(uu_t+vv_t)\,dx+\int_0^t\int_\Omega(uu_\tau+vv_\tau)\,dx\,d\tau+\beta(t_3+t),\\
C=\|u_t(t)\|^2+\|v_t(t)\|^2+\int_0^t[\|u_\tau(\tau)\|^2+\|v_\tau(\tau)\|^2]d\tau+\beta.
\end{gathered}
$$

By (4.2) and a simple computation, for all $s \in R$, we have

$$\begin{aligned} As^2 - 2Bs + C &= [\|u(t)\|^2 + \|v(t)\|^2 + \int_0^t [\|u(\tau)\|^2 + \|v(\tau)\|^2] d\tau + \beta(t_3 + t)^2] s^2 \\ &\quad - 2[\int_\Omega (uu_t + vv_t)\,dx + \int_0^t \int_\Omega (uu_\tau + vv_\tau)\,dx\,d\tau + \beta(t_3 + t)] s \\ &\quad + \|u_t(t)\|^2 + \|v_t(t)\|^2 + \int_0^t [\|u_\tau(\tau)\|^2 + \|v_\tau(\tau)\|^2] d\tau + \beta \\ &= \int_\Omega (su(t) - u_t(t))^2\,dx + \int_\Omega (sv(t) - v_t(t))^2\,dx \\ &\quad + \int_0^t \int_\Omega (su(\tau) - u_\tau(\tau))^2\,dx\,d\tau + \int_0^t \int_\Omega (sv(\tau) - v_\tau(\tau))^2\,dx\,d\tau \\ &\quad + \beta(s(t_3 + t) - 1)^2 \geq 0 \end{aligned}$$

which implies $B^2 - AC \leq 0$. Since we assume that the solution (u, v) to problem (1.1) exists for every $t \in [0, T)$, we have

$$F(t)F''(t) - \frac{(p+q+4)}{4}(F'(t))^2 \geq 0.$$

Let $\alpha = \frac{p+q}{2} > 0$. As $\frac{p+q+4}{4} > 1$, we have

$$F(t)F''(t) - (1+\alpha)(F'(t))^2 \geq 0.$$

We see that

$$\begin{aligned} (F^{-\alpha}(t))' &= -\alpha F^{-\alpha-1}F' < 0, \\ (F^{-\alpha}(t))'' &= -\alpha(-\alpha-1)F^{-\alpha-2}F'F' - \alpha F^{-\alpha-1}F'' \\ &= \alpha(\alpha+1)F^{-\alpha-2}(F')^2 - \alpha F^{-\alpha-1}F'' \\ &= -\alpha F^{-\alpha-2}[F''F - (1+\alpha)(F')^2] \end{aligned} \tag{4.7}$$

for every $t \in [0, T)$, which means that the function $F^{-\alpha}$ is concave. Obviously $F(0) > 0$, then from (4.7) it follows that

$$F^{-\alpha} \to 0, \quad \text{as } t \to T < \frac{4}{p+q}\frac{F(0)}{F'(0)}.$$

Therefore, we see that there exist a finite time $T > 0$ such that

$$\lim_{t \to T^-} \Big[\|u\|^2 + \|v\|^2 + \int_0^t (\|u_\tau(\tau, x)\|^2 + \|v_\tau(\tau, x)\|^2) d\tau\Big] = \infty.$$

The proof is complete.

References

[1] R. A. Adams, J. J. F. Fournier; *Sobolev Spaces*, Academic Press, 2003.

[2] X. Han, M. Wang; *Global existence and blow-up of solutions for a system of nonlinear viscoelastic wave equations with damping and source*, Nonlinear Anal., 7 (2009), 5427–5450.

[3] B. S. Houari; *Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms*, Diff. Integral Eqns., 23 (2010), 79–92.

[4] B. S. Houari, S. A. Messaoudi, A. Guesmia; *General decay of solutions of a nonlinear system of viscoelastic wave equations*, NoDEA- Nonlinear Diff., 18 (2011), 659–684.

[5] L. Jie, L. Fei; *Blow-up of solution for an integro-differential equation with arbitrary positive initial energy*, Boundary Value Problems, 2015 (2015) 96.

[6] M. Kafini, S. A. Messaoudi; *A blow-up result in a system of nonlinear viscoelastic wave equations with arbitrary positive initial energy*, Indagationes Mathematicae, 24 (2013), 602–612.
[7] M. Kafini, S. A. Messaoudi; *A blow-up result for a viscoelastic system in* $\mathbb{R}^N$, Electron. J. Differential Equations, 2007 (2007), no. 113, 1–7.
[8] H. A. Levine; *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $Pu_{tt} = Au + F(u)$, Trans. Amer. Math. Soc., 192 (1974), 1–21.
[9] Gand Li, Linghui Hong, Wenjun Liu; *Global nonexistence of solutions for viscoelastic wave equations of Kirchhoff type with high energy*, J. Func. Spaces and Appl., 2012 Art. ID 530861, 15 pp.
[10] Y. Lu, L. Fei, G. Zhenhua; *Lower bounds for blow up time of a nonlinear viscoelstic wave equation*, Boundary Value Problems, 219 (2015), 1-6.
[11] J. Ma, C. Mu, R. Zeng; *A blow up result for viscoelastic aquations with arbitrary positive initial energy*, Boundary Value Problems, 2011 (2011) :6.
[12] S. A. Messaoudi; *Blow up and global existence in a nonlinear viscoelastic wave equation*, Math. Nachr., 260 (2003), 58–66.
[13] S. A. Messaoudi; *Blow up of solutions with positive initial energy in a nonlinear viscoelastic wave equations*, J. Math. Anal. Appl., 320 (2006), 902–915.
[14] S. A. Messaoudi, B. S. Houari; *Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms*, J. Math. Anal. Appl., 365 (2010), 277–287.
[15] J. E. Munoz Rivera, M. Naso, E. Vuk; *Asymptotic behavior of the energy for electromagnetic system with memory*, Math. Methods Appl. Sci., 25 (2004), 819-841.
[16] E. Pişkin; *Global nonexistence of solutions for a system of viscoelastic wave equations with weak damping terms*, Malaya J. Mat., 3(2), 168-174 (2015).
[17] E. Pişkin; *A lower bound for the blow up time of a system of viscoelastic wave equations with nonlinear damping and source terms*, J. Nonlinear Funct. Anal., 2017 (2017), 1-9.
[18] Y. Wang; *A global nonexistence theorem for viscoelastic equations with arbitrary positive initial energy*, Appl. Math. Lett., 22 (2009), 1394–1400.

Erhan Pişkin
Dicle University, Department of Mathematics, 21280 Diyarbakir, Turkey
E-mail address: `episkin@dicle.edu.tr`

Ayşe Fidan
Dicle University, Department of Mathematics, 21280 Diyarbakir, Turkey
E-mail address: `afidanmat@gmail.com`