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ANTIPERIODIC SOLUTIONS TO VAN DER POL EQUATIONS WITH STATE-DEPENDENT IMPULSES

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Abstract. In this article we give sufficient conditions for the existence of an antiperiodic solution to the van der Pol equation

$$x'(t) = y(t), \quad y'(t) = \mu\Big(x(t) - \frac{x^3(t)}{3}\Big)' - x(t) + f(t) \text{for a. e. } t \in \mathbb{R},$$

subject to a finite number of state-dependent impulses

$$\Delta y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \dots, m\,.$$

Our approach is based on the reformulation of the problem as a distributional differential equation and on the Schauder fixed point theorem. The functionals τ_i and $\mathcal{J}_i$ need not be Lipschitz continuous nor bounded. As a direct consequence, we obtain an existence result for problem with fixed-time impulses.

1. Introduction

The study of anti-periodic solutions is closely related to the study of periodic solutions and their existence plays an important role in characterizing the behaviour of nonlinear differential equations. On the other hand impulsive problems are characterized by the occurrence of abrupt changes of their solutions which implies that such solution does not preserve the basic properties which are associated with non-impulsive problems. In real world problems, the impulses often do not occur at fixed times, but moments of their appearance depend on the state and situation of a differential model. Then the corresponding impulse conditions are called *state-dependent* in contrast to *fixed-time* impulse conditions where the moments of discontinuity are prescribed.

First order differential systems with fixed-time impulses can be found for example in [11, 2]. They mostly appear as models of neural networks and their anti-periodic solutions are investigated in many papers [1, 7, 15, 17, 16, 18, 19, 21]. For state-dependent impulses in such models see [14], where Lipschitz nonlinearities are assumed.

Second order differential equations can serve as physical models, for example: Rayleigh equation (acoustics), Duffing, Liénard or van der Pol equations (oscillation theory). Anti-periodic solutions of these equations without impulses are discussed

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in [6, 9, 12, 20] and of Rayleigh equation with fixed-time impulses in [10]. The first result about the existence and uniqueness of anti-periodic solutions of the distributional Liénard equation with state-dependent impulses has been reached by Belley and Bondo [3] under the assumption that functionals describing moments and values of impulses are globally Lipschitz continuous and bounded. Close results for periodic problems can be found in [4, 5]. Here, we focus our considerations on anti-periodic solutions of the van der Pol equation with state-dependent impulses both in "classical" and distributional formulations.

Namely, we investigate the existence of solutions to the van der Pol differential equation

$$x'(t) = y(t), \quad y'(t) = \mu\Big(x(t) - \frac{x^3(t)}{3}\Big)' - x(t) + f(t) \quad \text{for a. e. } t \in \mathbb{R}, \tag{1.1}$$

with a parameter $\mu \in (0, \infty)$ and a function f which is Lebesgue integrable on $[0, T]$ and satisfies

$$f(t + T) = -f(t) \quad \text{for a. e. } t \in \mathbb{R}. \tag{1.2}$$

We are interested in the existence of a solution fulfilling the antiperiodic conditions

$$x(0) = -x(T), \quad y(0) = -y(T). \tag{1.3}$$

It is natural to search for a solution (x, y) such that

$$x(t + T) = -x(t), \quad t \in \mathbb{R}. \tag{1.4}$$

In addition, (1.1) is subject to the state-dependent impulse conditions

$$\Delta y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \dots, m, \tag{1.5}$$

where $\tau_i, \mathcal{J}_i$ $i = 1, \dots, m$, are real-valued functionals, τ_i have values in $(0, T)$ and $\Delta y(\tau) = y(\tau+) - y(\tau-)$ for $\tau \in \mathbb{R}$. Then (1.1), (1.4) and (1.5) lead to

$$y(t + T) = -y(t), \quad t \in \mathbb{R},\ t \neq \tau_i(x),\ i = 1, \dots, m. \tag{1.6}$$

Since x satisfying (1.4) is $2T$-periodic and has zero mean value, i.e.,

$$\bar{x} = \frac{1}{2T} \int_0^{2T} x(t)\, dt = 0,$$

the functionals τ_i and $\mathcal{J}_i$ are defined on the set of $2T$-periodic functions of bounded variation with zero mean value. We will consider such solutions (x, y) for which y is piecewise absolutely continuous with the only instants of discontinuity at $t = \tau_i(x)$, $i = 1, \dots, m$. Then the assumption that τ_i, $i = 1, \dots, m$, have values in $(0, T)$ guarantees the continuity of y at the points nT, $n \in \mathbb{Z}$, and consequently the second equality in (1.3).

Our main result is contained in the next theorem, which is a direct consequence of Theorem 5.1 from Section 5.

Theorem 1.1. *Assume that $T \in (0, \sqrt{3})$, $\tau_1, \dots, \tau_m$ are continuous with values in $(0, T)$ and if $i \neq j$, then $\tau_i(x) \neq \tau_j(x)$ for each $2T$-periodic absolutely continuous function x with zero mean value. Further assume that $\mathcal{J}_1, \dots, \mathcal{J}_m$ are continous and bounded. Then there exists $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0]$ the problem (1.1), (1.3), (1.5) has a solution.*

The novelty of this paper is the following:

- Our existence result for problem (1.1), (1.3), (1.5) is the first in the literature.

- We need not the Lipschitz continuity of functionals τ_i and $\mathcal{J}_i$ in problem (1.1), (1.3), (1.5) as well as in (3.1) in contrast to [3].
- We also get the solvability provided these functionals are unbounded.
- Our solvability conditions can be very easily checked, which we illustrate on two nontrivial examples.

2. Preliminaries

Motivated by the paper [3] we construct a distributional differential equation equivalent to the problem (1.1), (1.3), (1.5). This enables to work in more advantageous space $\widetilde{\mathrm{NBV}}$ and to use properties of Fourier series of distributions. To this aim, by $\mathcal{P}_{2T}$ we denote the complex vector space of all complex-valued $2T$-periodic functions of one real variable having continuous derivatives of all orders on $\mathbb{R}$. The elements of $\mathcal{P}_{2T}$ are called *test functions* and $\mathcal{P}_{2T}$ is equipped with a locally convex topological space structure (see [8]). Its topological dual is denoted by $(\mathcal{P}_{2T})'$. The elements of $(\mathcal{P}_{2T})'$ are called *$2T$-periodic distributions* or only *distributions*, i.e., these elements are complex-valued continuous linear functionals on $\mathcal{P}_{2T}$.

For a distribution $u \in (\mathcal{P}_{2T})'$ and a test function $\varphi \in \mathcal{P}_{2T}$, the symbol $\langle u, \varphi\rangle$ stands for a value of the distribution u at φ. The distributional derivative Du of a distribution u is a distribution which is defined by

$$\langle Du, \varphi\rangle = -\langle u, \varphi'\rangle \quad \text{for each } \varphi \in \mathcal{P}_{2T}.$$

Let us take $n \in \mathbb{Z}$ and introduce a complex-valued function $e_n \in \mathcal{P}_{2T}$ by

$$e_n(t) := \mathrm{e}^{\mathrm{i}n\omega t}, \quad t \in \mathbb{R},$$

where $\omega = \pi/T$. Then every distribution u can be expressed uniquely by the *Fourier series*

$$u = \sum_{n\in\mathbb{Z}} \hat{u}(n) e_n, \tag{2.1}$$

where $\hat{u}(n) \in \mathbb{C}$ are *Fourier coefficients* of u,

$$\hat{u}(n) = \langle u, e_{-n}\rangle, \quad n \in \mathbb{Z}.$$

For a distribution u we define the *mean value* $\bar{u}$ as

$$\bar{u} := \hat{u}(0) = \langle u, e_0\rangle = \langle u, 1\rangle, \tag{2.2}$$

and, for simplicity of notation, we write $\widetilde{u} := u - \bar{u}$.

In general, the Fourier series in (2.1) need not be pointwise convergent and the equality in (2.1) is understood in the sense of distributions written as

$$\lim_{N\to\infty} \langle s_N, \varphi\rangle = \langle u, \varphi\rangle \in \mathbb{C} \quad \text{for each } \varphi \in \mathcal{P}_{2T}, \quad \text{where } s_N = \sum_{|n|\le N} \hat{u}(n) e_n.$$

In particular, the Dirac $2T$-periodic distribution δ is defined by

$$\langle \delta, \varphi\rangle = \varphi(0) \quad \text{for each } \varphi \in \mathcal{P}_{2T},$$

and it has the Fourier series

$$\delta = \sum_{n\in\mathbb{Z}} e_n. \tag{2.3}$$

The convolution $u * v$ of two distributions has the Fourier series

$$u * v = \sum_{n\in\mathbb{Z}} \hat{u}(n)\hat{v}(n) e_n, \tag{2.4}$$

and the Fourier series for *distributional derivatives* Du and D^2u reads

$$Du = \sum_{n\in\mathbb{Z}, n\neq 0} \mathrm{i}n\omega\, \hat{u}(n)e_n \quad \text{and} \quad D^2u = \sum_{n\in\mathbb{Z}, n\neq 0} (\mathrm{i}n\omega)^2\hat{u}(n)e_n. \tag{2.5}$$

This immediately implies that

$$u * \delta = u, \quad \overline{Du} = \overline{D^2u} = 0, \quad D\widetilde{u} = Du, \quad D^2\widetilde{u} = D^2u. \tag{2.6}$$

Let us introduce distributions E_1 and E_2 by

$$E_1 := \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{\mathrm{i}n\omega} e_n, \quad E_2 := E_1 * E_1 = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{(\mathrm{i}n\omega)^2} e_n, \tag{2.7}$$

and define linear operators I and I^2 by

$$\begin{gathered} Iu := E_1 * u = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{\mathrm{i}n\omega}\hat{u}(n)e_n, \\ I^2u := I(Iu) = E_1 * (E_1 * u) = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{(\mathrm{i}n\omega)^2}\hat{u}(n)e_n = E_2 * u. \end{gathered} \tag{2.8}$$

Using (2.4) and (2.5), for every distribution u, we obtain

$$\begin{gathered} D(Iu) = I(Du) = \widetilde{u}, \quad D^2(I^2u) = I^2(D^2u) = \widetilde{u}, \\ I^2(Du) = Iu = I\widetilde{u}, \quad D^2(Iu) = Du = D\widetilde{u}. \end{gathered} \tag{2.9}$$

From these identities we see that I is an inverse to D on the set of all distributions with zero mean value and therefore we call I an *antiderivative operator*.

Consider $\tau \in \mathbb{R}$. Let us remind the translation operator $\mathbb{T}_\tau$ on test functions and distributions. For a function $\varphi \in \mathcal{P}_{2T}$ we define $\mathbb{T}_\tau\varphi \in \mathcal{P}_{2T}$ by

$$(\mathbb{T}_\tau\varphi)(t) := \varphi(t-\tau), \quad t \in \mathbb{R},$$

and for a distribution $u \in (\mathcal{P}_{2T})'$ we define $\mathbb{T}_\tau u \in (\mathcal{P}_{2T})'$ by

$$\langle \mathbb{T}_\tau u, \varphi\rangle := \langle u, \mathbb{T}_{-\tau}\varphi\rangle, \quad \varphi \in \mathcal{P}_{2T}.$$

Since

$$\widehat{(\mathbb{T}_\tau u)}(n) = \langle \mathbb{T}_\tau u, e_{-n}\rangle = \langle u, \mathbb{T}_{-\tau}e_{-n}\rangle = \mathrm{e}^{-\mathrm{i}n\omega\tau}\langle u, e_{-n}\rangle = \mathrm{e}^{-\mathrm{i}n\omega\tau}\hat{u}(n), \tag{2.10}$$

for $n \in \mathbb{Z}$, the Fourier series of $\mathbb{T}_\tau u$ reads

$$\mathbb{T}_\tau u = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n\omega\tau}\hat{u}(n)e_n, \quad u \in (\mathcal{P}_{2T})', \tag{2.11}$$

in particular, for $\tau = T$

$$\mathbb{T}_T u = \sum_{n\in\mathbb{Z}} (-1)^n\hat{u}(n)e_n, \quad u \in (\mathcal{P}_{2T})'. \tag{2.12}$$

Further, by (2.3) and (2.11), the *Dirac $2T$-periodic distribution δ_τ at the point* $\tau \in \mathbb{R}$ which is defined as

$$\delta_\tau := \mathbb{T}_\tau\delta,$$

and satisfies

$$\begin{gathered} \delta_\tau = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n\omega\tau}e_n, \quad \overline{\delta_\tau} = 1, \\ u * \delta_\tau = \mathbb{T}_\tau u, \quad u \in (\mathcal{P}_{2T})'. \end{gathered} \tag{2.13}$$

Hence

$$I\delta_\tau = E_1 * \delta_\tau = \mathbb{T}_\tau E_1, \quad I^2\delta_\tau = E_2 * \delta_\tau = \mathbb{T}_\tau E_2. \tag{2.14}$$

We are interested in solutions of (1.1) satisfying the antiperiodic conditions (1.3) and so we work here with *antiperiodic distributions*. Exactly, we say that a distribution $u \in (\mathcal{P}_{2T})'$ is called *antiperiodic* provided u satisfies

$$\mathbb{T}_T u = -u. \tag{2.15}$$

By (2.12) we see that $u \in (\mathcal{P}_{2T})'$ is antiperiodic if and only if $\hat{u}(n) = 0$ for each even $n \in \mathbb{Z}$. Consequently, if $u \in (\mathcal{P}_{2T})'$ is antiperiodic, then $\hat{u}(0) = \overline{u} = 0$ and Du, Iu are antiperiodic, as well. On the other hand, (2.13) yields that the Dirac $2T$-periodic distribution δ_τ, which could characterize impulses from (1.5), is not antiperiodic. Therefore, motivated by [3], we introduce the distribution

$$\varepsilon_\tau := \delta_\tau - \mathbb{T}_T\delta_\tau \tag{2.16}$$

which is antiperiodic for any $\tau \in \mathbb{R}$.

Now, we turn our attention to real-valued functions and distributions which we use in next sections. To this aim the functional spaces defined below consist of *real-valued $2T$-periodic functions.* Clearly it suffices to prescribe their values on a semiclosed interval with the length $2T$:

- L^1 is the Banach space of Lebesgue integrable functions equipped with the norm $\|x\|_{L^1} := \frac{1}{2T}\int_0^{2T} |x(t)|\,dt$,
- BV is the space of functions of bounded variation; the total variation of $x \in \mathrm{BV}$ is denoted by $\mathrm{var}(x)$; for $x \in \mathrm{BV}$ we also define $\|x\|_\infty := \sup\{|x(t)| : t \in [0, 2T]\}$,
- NBV is the space of functions from BV normalized in the sense that $x(t) = \frac{1}{2}(x(t+) + x(t-))$,
- $\widetilde{\mathrm{NBV}}$ represents the Banach space of functions from NBV having zero mean value ($\bar{x} := \frac{1}{2T}\int_0^{2T} x(t)\,dt = 0$), which is equipped with the norm equal to the total variation $\mathrm{var}(x)$,
- for an interval $J \subset [0, 2T]$ we denote by $\mathrm{AC}(J)$ the set of absolutely continuous functions on J, and if $J = [0, 2T]$ we simply write AC,
- $C^\infty \subset \mathcal{P}_{2T}$ is the classical Fréchet space of functions having derivative of an arbitrary order,
- for finite $\Sigma \subset [0, 2T)$ we denote by PAC_Σ the set of all functions $x \in \mathrm{NBV}$ such that $x \in \mathrm{AC}(J)$ for each interval $J \subset [0, 2T]$ for which $\Sigma \cap J = \emptyset$. For $\tau \in [0, 2T)$, we write $\mathrm{PAC}_\tau := \mathrm{PAC}_{\{\tau\}}$,
- $\widetilde{\mathrm{AC}} = \mathrm{AC} \cap \widetilde{\mathrm{NBV}}$; for finite $\Sigma \subset [0, 2T)$ we denote $\widetilde{\mathrm{PAC}}_\Sigma = \mathrm{PAC}_\Sigma \cap \widetilde{\mathrm{NBV}}$,
- $\Delta y(\tau) := y(\tau+) - y(\tau-)$ for $y \in \widetilde{\mathrm{NBV}}$, $\tau \in \mathbb{R}$.

Further, Car designates the set of real functions $f(t, x)$ such that $f(\cdot, x) \in L^1$ for each $x \in \mathbb{R}$ and satisfy the Carathéodory conditions on $[0, 2T] \times \mathbb{R}$.

We say that $u \in (\mathcal{P}_{2T})'$ is a *real-valued distribution* if

$$\langle u, \varphi\rangle \in \mathbb{R} \quad \text{for each } \varphi \in C^\infty.$$

A real-valued distribution u is characterized by the fact that its Fourier coefficients $\hat{u}(n)$ and $\hat{u}(-n)$ are complex conjugate for each $n \in \mathbb{Z}$. Obviously, if $\tau \in \mathbb{R}$ and u and v are real-valued distributions, then $u * v$, $\widetilde{u}$, Du, D^2u, Iu, I^2u, $\mathbb{T}_\tau u$, δ_τ and ε_τ are real-valued distributions, as well.

We say that $u \in (\mathcal{P}_{2T})'$ is a *regular distribution* if u is a real-valued distribution and there exists $y \in L^1$ such that

$$\langle u, \varphi\rangle = \frac{1}{2T}\int_0^{2T} y(s)\varphi(s)\,ds \quad \text{for each } \varphi \in C^\infty. \tag{2.17}$$

Then we say that $u = y$ in the sense of distributions and write y in place of u in (2.17). Hence all functions from L^1 can be understood as regular distributions. For $x \in \mathrm{BV}$, we write x' as a classical derivative, which is defined a.e. on $\mathbb{R}$ and which is an element of L^1 and consequently a regular distribution. If $x \in \mathrm{AC}$, then $x' = Dx$ in the sense of distributions.

Since the first series in (2.7) converges pointwise to the $2T$-periodic function

$$E_1(t) = \begin{cases} T - t & \text{for } t \in (0, 2T),\\ 0 & \text{for } t = 0,\end{cases}$$

we see that E_1 is a regular distribution and it can be considered as a function from $\widetilde{\mathrm{PAC}}_0$. The second series in (2.7) uniformly converges to the $2T$-periodic function

$$E_2(t) = \frac{t(2T - t)}{2} - \frac{T^2}{3}, \quad t \in [0, 2T],$$

and so E_2 is a regular distribution which can be considered as a function from $\widetilde{\mathrm{AC}}$. Similarly for $\tau \in \mathbb{R}$,

$$\mathbb{T}_\tau E_1 \in \widetilde{\mathrm{PAC}}_\tau, \quad \mathbb{T}_\tau E_2 \in \widetilde{\mathrm{AC}}. \tag{2.18}$$

Obviously, $E_2' = E_1$, $E_1' = -1$ a.e. on $[0, 2T)$ and

$$\mathrm{var}(E_1) = 4T, \quad \|E_1\|_\infty = T, \quad \mathrm{var}(E_2) = T^2, \quad \|E_2\|_\infty = \frac{T^2}{3}. \tag{2.19}$$

Since

$$(x * y)(t) := \frac{1}{2T}\int_0^{2T} x(t - s)y(s)\,ds, \quad t \in [0, 2T] \text{ for } x, y \in L^1,$$

for $h \in L^1$, we have

$$(E_1 * h)(t) = \frac{1}{2T}\int_0^{2T}(s - t)h(s)\,ds + \frac{1}{2}\Big(\int_0^t h(s)\,ds - \int_t^{2T} h(s)\,ds\Big),$$

for $t \in [0, 2T]$. Therefore Ih is a regular distribution which is equal to the function $E_1 * h \in \mathrm{AC}$, and we conclude by (2.8),

$$h \in L^1 \implies Ih,\ I^2 h \in \widetilde{\mathrm{AC}}, \quad (Ih)'(t) = h(t) - \bar{h} \text{ a.e. } t \in [0, 2T]\,. \tag{2.20}$$

Further, for $x \in \mathrm{BV}$ and $t \in \mathbb{R}$ we have $(\mathbb{T}_\tau x)(t) = x(t - \tau)$ which implies

$$\mathrm{var}\,(\mathbb{T}_\tau x) = \mathrm{var}\, x \quad \text{and} \quad \|\mathbb{T}_\tau x\|_\infty = \|x\|_\infty, \quad x \in \mathrm{BV}. \tag{2.21}$$

Let us recall the following inequalities

$$\mathrm{var}(x * y) \le \mathrm{var}(x)\|y\|_\infty, \quad x, y \in \mathrm{NBV}\,, \tag{2.22}$$

$$\mathrm{var}(x * f) \le \mathrm{var}(x)\|f\|_{L^1}, \quad x \in \mathrm{NBV},\ f \in L^1\,, \tag{2.23}$$

$$\|x\|_{L^1} \le \|x\|_\infty \le \mathrm{var}(x), \quad x \in \widetilde{\mathrm{NBV}}\,. \tag{2.24}$$

Remark 2.1. Let x be antiperiodic. If $x \in \mathrm{NBV}$, then $x(t+T) = -x(t)$ for $t \in \mathbb{R}$,

$$\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|,$$

and $\operatorname{var}(x)$ is double the total variation of x over the interval $[0,T]$ (or any semi-closed interval of the length T). If $x \in L^1$, then $x(t+T) = -x(t)$ for a.e. $t \in \mathbb{R}$ and

$$\|x\|_{L^1} = \frac{1}{T}\int_0^T |x(t)|\,dt.$$

Therefore it is sufficient to define an antiperiodic function on any interval of the length T.

By (2.14) and (2.16) and (2.18), it holds for $\tau \in \mathbb{R}$,

$$I\varepsilon_\tau = I\delta_\tau - I\mathbb{T}_T\delta_\tau = T_\tau E_1 - T_{\tau+T}E_1 \in \widetilde{\mathrm{PAC}}_{\{\tau,\tau+T\}}, \tag{2.25}$$

$$I^2\varepsilon_\tau = I^2\delta_\tau - I^2\mathbb{T}_T\delta_\tau = T_\tau E_2 - T_{\tau+T}E_2 \in \widetilde{\mathrm{AC}}. \tag{2.26}$$

So, for $\tau = 0$, we have

$$I\varepsilon_0 = E_1 - \mathbb{T}_T E_1, \quad I^2\varepsilon_0 = E_2 - \mathbb{T}_T E_2,$$

and in detail

$$I\varepsilon_0(t) = \begin{cases} 0 & t = 0, \\ T & t \in (0,T), \end{cases} \qquad I^2\varepsilon_0(t) = \frac{T(2t-T)}{2}, \quad t \in [0,T], \tag{2.27}$$

Since $I\varepsilon_\tau = \mathbb{T}_\tau I\varepsilon_0$, by (2.21) and (2.27) and according to Remark 2.1, we obtain

$$\operatorname{var}(I\varepsilon_\tau) = 4T, \quad \|I\varepsilon_\tau\|_\infty = T, \quad \operatorname{var}(I^2\varepsilon_\tau) = 2T^2, \quad \|I^2\varepsilon_\tau\|_\infty = \frac{T^2}{2}, \tag{2.28}$$

for $\tau \in \mathbb{R}$. Choosing $\tau_1, \tau_2 \in \mathbb{R}$, where $|\tau_1 - \tau_2| < T$, from (2.8), (2.19), (2.23) and (2.27), we deduce the estimate

$$\begin{aligned} \operatorname{var}(I^2\varepsilon_{\tau_1} - I^2\varepsilon_{\tau_2}) &= \operatorname{var}(I(I\varepsilon_{\tau_1} - I\varepsilon_{\tau_2})) = \operatorname{var}(E_1 * (I\varepsilon_{\tau_1} - I\varepsilon_{\tau_2})) \\ &\le \operatorname{var}(E_1)\|I\varepsilon_{\tau_1} - I\varepsilon_{\tau_2}\|_{L^1} \le 8T|\tau_1 - \tau_2|. \end{aligned} \tag{2.29}$$

3. Auxiliary distributional equation

Here we consider the distributional differential equation

$$D^2 z = \mu D\big(z - \frac{z^3}{3}\big) - z + f + \frac{1}{2T}\sum_{i=1}^m \mathcal{J}_i(z)\varepsilon_{\tau_i(z)} \tag{3.1}$$

with a parameter $\mu \in (0,\infty)$, where $f \in L^1$ fulfils (1.2), $\tau_i : \widetilde{\mathrm{NBV}} \to (0,T)$, $\mathcal{J}_i : \widetilde{\mathrm{NBV}} \to \mathbb{R}$, and $\varepsilon_{\tau_i(z)}$ is defined in (2.16) for $i = 1,\dots,m$.

Definition 3.1. A function $z \in \widetilde{\mathrm{NBV}}$ is called a *solution* of the distributional equation (3.1) if

$$\langle D^2 z, \varphi\rangle = \big\langle \mu D\big(z - \frac{z^3}{3}\big) - z + f + \frac{1}{2T}\sum_{i=1}^m \mathcal{J}_i(z)\varepsilon_{\tau_i(z)}, \varphi\big\rangle \tag{3.2}$$

for every $\varphi \in C^\infty$.

Remark 3.2. Definition 3.1 is justified by the following considerations.

- If $z \in \mathrm{NBV}$ satisfies (3.2), then for $\varphi = 1$ in (3.2) we have by (2.2)
$$\overline{D^2 z} = \mu \overline{D\big(z - \frac{z^3}{3}\big)} - \overline{z} + \overline{f} + \frac{1}{2T}\sum_{i=1}^{m} \mathcal{J}_i(z)\overline{\varepsilon_{\tau_i(z)}}.$$
Antiperiodicity of f and $\varepsilon_{\tau_i(z)}$ together with (2.6) imply $\overline{z} = 0$, i.e. $z \in \widetilde{\mathrm{NBV}}$.
- For $z \in \widetilde{\mathrm{NBV}}$, Eq. (3.1) has two equivalent forms
$$Dz = \mu\big(z - \frac{z^3}{3}\big) - \mu\overline{\big(z - \frac{z^3}{3}\big)} + I\Big(-z + f + \frac{1}{2T}\sum_{i=1}^{m} \mathcal{J}_i(z)\varepsilon_{\tau_i(z)}\Big), \tag{3.3}$$
$$z = \mu I\big(z - \frac{z^3}{3}\big) + I^2\Big(-z + f + \frac{1}{2T}\sum_{i=1}^{m} \mathcal{J}_i(z)\varepsilon_{\tau_i(z)}\Big), \tag{3.4}$$
which are obtained from (3.1) by means of the antiderivative operator I and identities (2.9). Vice versa, differentiating (3.4) and using the facts $\widetilde{z} = z$, $\widetilde{f} = f$ and $\widetilde{\varepsilon_{\tau_i(z)}} = \varepsilon_{\tau_i(z)}$ we arrive at (3.1).
- A solution z of (3.1) is a solution of (3.4) and, due to (2.20) and (2.26), we see that $z \in \widetilde{\mathrm{AC}} \subset \widetilde{\mathrm{NBV}}$.

We are ready to compare equation (3.1) with our original problem (1.1), (1.3), (1.5). To do it consider $x \in \widetilde{\mathrm{AC}}$ and denote the set
$$\Sigma_x := \{\tau_1(x), \dots, \tau_m(x), \tau_1(x) + T, \dots, \tau_m(x) + T\}. \tag{3.5}$$

Definition 3.3. Assume that the condition
$$\tau_i(x) \neq \tau_j(x) \quad \text{for all } i, j = 1, \dots, m,\ i \neq j,\ x \in \widetilde{\mathrm{AC}} \tag{3.6}$$
is fulfilled. The couple $(x, y) \in \widetilde{\mathrm{AC}} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ is called a *solution* of the impulsive problem (1.1), (1.5) if it satisfies the differential equation (1.1) and the impulse conditions (1.5). A solution (x, y) of (1.1), (1.5) is called *antiperiodic* if it satisfies the antiperiodic conditions (1.3).

Lemma 3.4. *Let* (3.6) *hold. If $z \in \widetilde{\mathrm{NBV}}$ is a solution of the distributional equation* (3.1)*, then the couple $(x, y) \in \widetilde{\mathrm{AC}} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ with $x = z$ on $\mathbb{R}$ and $y = Dz$ a.e. on $\mathbb{R}$ satisfies* (1.1) *and*
$$\Delta y(\tau_i(x)) = \mathcal{J}_i(x), \quad \Delta y(\tau_i(x) + T) = -\mathcal{J}_i(x), \quad i = 1, \dots, m. \tag{3.7}$$
Conversely, if the couple $(x, y) \in \widetilde{\mathrm{AC}} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ satisfies (1.1) *and* (3.7)*, then $z = x$ is a solution of* (3.1).

Proof. (i) Assume that $z \in \widetilde{\mathrm{NBV}}$ is a solution of (3.1) and put
$$\begin{aligned} x(t) &= \mu I\big(z - \frac{z^3}{3}\big)(t) + I^2(-z + f)(t) \\ &\quad + \frac{1}{2T}\sum_{i=1}^{m} \mathcal{J}_i(z)\left(\mathbb{T}_{\tau_i(z)}E_2(t) - \mathbb{T}_{\tau_i(z)+T}E_2(t)\right), \quad t \in \mathbb{R}, \end{aligned}$$
and
$$y(t) = \mu\big(z(t) - \frac{z^3(t)}{3}\big) - \mu\overline{\big(z - \frac{z^3}{3}\big)} + I(-z + f)(t)$$

$$+\frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(z)\left(\mathbb{T}_{\tau_i(z)}E_1(t)-\mathbb{T}_{\tau_i(z)+T}E_1(t)\right), \quad t\in\mathbb{R}.$$

According to Remark 3.2, by (3.4), (2.26) and (2.20), we see that $x\in\widetilde{\mathrm{AC}}$ and $z=x$ on $\mathbb{R}$. Similarly, using in addition (3.3), (2.25), we get $y\in\widetilde{\mathrm{PAC}}_{\Sigma_x}$ and $Dz=z'=y$ a.e. on $\mathbb{R}$. Due to $z=x$ the first equation in (1.1) is fulfilled. Since $E_1'=-1$ a.e. on $\mathbb{R}$, we get for each $\tau\in\mathbb{R}$ the equality $\mathbb{T}_\tau E_1'=E_1'$ a.e. on $\mathbb{R}$. Having in mind that z and If are absolutely continuous and $z=\widetilde{z}$, $f=\widetilde{f}$, we use (2.9) and find that the second equation in (1.1) is satisfied, as well. Finally, since for $\tau\in\mathbb{R}$,

$$\mathbb{T}_\tau E_1(t)=\begin{cases}T-(t-\tau) & \text{for } t\in(\tau,\tau+2T),\\ 0 & \text{for } t=\tau,\end{cases}$$

we see that if $\tau\in(0,T)$, the function $\mathbb{T}_\tau E_1$ has in the interval $[0,2T]$ exactly one jump at τ, in particular

$$\Delta\mathbb{T}_\tau E_1(\tau)=T-(-T)=2T,$$

and the function $-\mathbb{T}_{\tau+T}E_1$ has in the interval $[0,2T]$ exactly one jump at $\tau+T$, in particular

$$-\Delta\mathbb{T}_{\tau+T}E_1(\tau+T)=-2T.$$

Therefore,

$$\Delta y(\tau_i(x))=\frac{1}{2T}\mathcal{J}_i(x)2T=\mathcal{J}_i(x), \quad i=1,\dots,m,$$

and similarly,

$$\Delta y(\tau_i(x)+T)=\frac{1}{2T}\mathcal{J}_i(x)(-2T)=-\mathcal{J}_i(x), \quad i=1,\dots,m.$$

Hence, the impulse condition (3.7) is fulfilled.

(ii) Now, conversely assume that $(x,y)\in\widetilde{\mathrm{AC}}\times\widetilde{\mathrm{PAC}}_{\Sigma_x}$ satisfy (1.1) and (3.7) and put $z=x$. Then $Dz=Dx=x'=y$ a.e. on $\mathbb{R}$. According to (3.5), (3.6) and the assumption that $\tau_i(x)\in(0,T)$, $i=1,\dots,m$, we can write $\Sigma_x=\{s_1,\dots,s_{2m}\}$, where

$$0=:s_0<s_1<\ldots<s_{2m}<s_{2m+1}:=2T.$$

Then for $\varphi\in C^\infty$ we have

$$\begin{aligned}
&\langle D^2z,\varphi\rangle\\
&=-\langle Dz,\varphi'\rangle=-\langle y,\varphi'\rangle\\
&=-\frac{1}{2T}\int_0^{2T}y(t)\varphi'(t)\,dt=-\frac{1}{2T}\sum_{i=1}^{2m+1}\int_{s_{i-1}}^{s_i}y(t)\varphi'(t)\,dt\\
&=-\frac{1}{2T}\sum_{i=1}^{2m+1}\Big([y(t)\varphi(t)]_{s_{i-1}}^{s_i}-\int_{s_{i-1}}^{s_i}y'(t)\varphi(t)\,dt\Big)\\
&=\frac{1}{2T}\sum_{i=1}^{2m+1}\left(y(s_{i-1}+)\varphi(s_{i-1})-y(s_i-)\varphi(s_i)\right)+\frac{1}{2T}\int_0^{2T}y'(t)\varphi(t)\,dt\\
&=\frac{1}{2T}\sum_{i=1}^{2m}\Delta y(s_i)\varphi(s_i)+\langle y',\varphi\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2T}\sum_{i=1}^{m}\Delta y(\tau_i(x))\varphi(\tau_i(x)) + \frac{1}{2T}\sum_{i=1}^{m}\Delta y(\tau_i(x)+T)\varphi(\tau_i(x)+T) + \langle y', \varphi\rangle \\
&= \sum_{i=1}^{m}\frac{1}{2T}\mathcal{J}_i(x)\delta_{\tau_i(x)} - \sum_{i=1}^{m}\frac{1}{2T}\mathcal{J}_i(x)\delta_{\tau_i(x)+T} + \langle y', \varphi\rangle \\
&= \Big\langle \frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(x)\varepsilon_{\tau_i(x)} + \mu\big(x - \frac{x^3}{3}\big)' - x + f, \varphi\Big\rangle \\
&= \Big\langle \frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(z)\varepsilon_{\tau_i(z)} + \mu D\big(z - \frac{z^3}{3}\big) - z + f, \varphi\Big\rangle.
\end{aligned}
$$

Therefore z is a solution of (3.1). □

Remark 3.5. Condition (3.7) contains the impulse condition (1.5). On the other hand, if x and y are antiperiodic and satisfy (1.5), then they fulfil (3.7).

Remark 3.6. If we drop the assumption (3.6) in Lemma 3.4, the couple (x, y) is a solution of differential equation (1.1), but the condition (1.5) is not correctly formulated. For example if $\tau_1(x) = \tau_2(x)$ and $\mathcal{J}_1(x) \neq \mathcal{J}_2(x)$. Therefore, in this case, the condition (1.5) must be replaced by

$$
\Delta y\,(\tau_i(x)) = \sum_{\substack{1\le j\le m:\\ \tau_j(x)=\tau_i(x)}} \mathcal{J}_j(x), \quad i = 1, \dots, m.
$$

Theorem 3.7. *Let* (3.6) *be satisfied. Assume that* $z \in \widetilde{\mathrm{NBV}}$ *is a solution of the distributional equation* (3.1) *and* z *satisfies* (1.4). *Then the couple* (z, Dz) *is an antiperiodic solution of problem* (1.1),(1.5).

Proof. By Lemma 3.4 and Remark 3.5, the couple $(z, Dz) \in \widetilde{\mathrm{AC}} \times \widetilde{\mathrm{PAC}}_{\Sigma_z}$ is a solution of problem (1.1), (1.5). Since $z(t+T) = -z(t)$ for $t \in \mathbb{R}$, we have $Dz(t + T) = -Dz(t)$ for $t \in [0, T]$. Consequently

$$
z(0) = -z(T) \quad \text{and} \quad Dz(0) = -Dz(T),
$$

i.e. $(x, y) = (z, Dz)$ satisfies condition (1.3). □

4. Fixed point problem

According to Theorem 3.7, to get an antiperiodic solution of problem (1.1), (1.5), it suffices to prove the existence of a solution $z \in \widetilde{\mathrm{NBV}}$ of the distributional equation (3.1) which in addition satisfies (1.4). Motivated by the equivalent form (3.4) of (3.1), we define an operator $\mathcal{F} : \widetilde{\mathrm{NBV}} \to \widetilde{\mathrm{NBV}}$ by

$$
\mathcal{F}z = \mu I\big(z - \frac{z^3}{3}\big) + I^2\Big(-z + f + \frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(z)\varepsilon_{\tau_i(z)}\Big). \tag{4.1}
$$

If we summarize the assertions of Theorem 3.7 with those in Remark 3.5, we have the following assertion.

Lemma 4.1. *Each fixed point* z *of the operator* $\mathcal{F}$ *is a solution of the distributional equation* (3.1). *Moreover, if* (3.6) *is fulfilled and* z *is antiperiodic, then* (z, Dz) *is an antiperiodic solution of problem* (1.1), (1.5).

Together with the basic assumptions from Sections 1 and 3 – that μ is a positive parameter and $f \in L^1$ fulfils (1.2) – we now consider boundedness and continuity of functionals τ_i, $\mathcal{J}_i$. Exactly we moreover assume

$$\tau_i : \widetilde{\mathrm{NBV}} \to [a,b] \subset (0,T), \quad i = 1,\dots,m, \quad \text{are continuous,} \tag{4.2}$$

$$\mathcal{J}_i : \widetilde{\mathrm{NBV}} \to [-a_i, a_i], \quad i = 1,\dots,m, \quad \text{are continuous,} \tag{4.3}$$

where $a_i \in (0,\infty)$, $i = 1,\dots,m$.

Lemma 4.2. *Let the assumptions* (4.2) *and* (4.3) *be satisfied. Then the operator* $\mathcal{F}$ *is completely continuous.*

Proof. Let us divide our proof into two steps.

Step 1. We prove that $\mathcal{F}$ is continuous. Let us consider a sequence $\{z_n\}_{n=1}^{\infty} \subset \widetilde{\mathrm{NBV}}$ converging in $\widetilde{\mathrm{NBV}}$ to $z \in \widetilde{\mathrm{NBV}}$. Denote

$$v_n := \mathcal{F}(z_n), \quad v := \mathcal{F}(z).$$

Then, by (4.1),

$$\begin{aligned} v_n - v &= \mu I\left(z_n - z\right) - \frac{\mu}{3} I\left(z_n^3 - z^3\right) - I^2(z_n - z) \\ &\quad + \frac{1}{2T}\sum_{i=1}^{m}\left(\mathcal{J}_i(z_n) I^2 \varepsilon_{\tau_i(z_n)} - \mathcal{J}_i(z) I^2 \varepsilon_{\tau_i(z)}\right). \end{aligned} \tag{4.4}$$

By (2.24) we see that $\|z_n - z\|_\infty \to 0$ as $n \to \infty$. Hence, by (2.22) and (2.8), for $n \to \infty$, we have

$$\begin{gathered} \operatorname{var}(I^i(z_n - z)) = \operatorname{var}(E_i * (z_n - z)) \le \operatorname{var}(E_i)\|z_n - z\|_\infty \to 0, \quad i = 1,2, \\ \operatorname{var}\left(I\left(z_n^3 - z^3\right)\right) \le \operatorname{var}(E_1)\|z_n^3 - z^3\|_\infty \to 0. \end{gathered}$$

Further,

$$\begin{aligned} &\operatorname{var}\left(\mathcal{J}_i(z_n) I^2 \varepsilon_{\tau_i(z_n)} - \mathcal{J}_i(z) I^2 \varepsilon_{\tau_i(z)}\right) \\ &= \operatorname{var}\left(\mathcal{J}_i(z_n) I^2 \varepsilon_{\tau_i(z_n)} - \mathcal{J}_i(z) I^2 \varepsilon_{\tau_i(z_n)}\right) + \operatorname{var}\left(\mathcal{J}_i(z) I^2 \varepsilon_{\tau_i(z_n)} - \mathcal{J}_i(z) I^2 \varepsilon_{\tau_i(z)}\right) \\ &\le |\mathcal{J}_i(z_n) - \mathcal{J}_i(z)| \operatorname{var}(I^2 \varepsilon_{\tau_i(z_n)}) + |\mathcal{J}_i(z)| \operatorname{var}(I^2 \varepsilon_{\tau_i(z_n)} - I^2 \varepsilon_{\tau_i(z)}), \end{aligned}$$

and using (2.28), (2.29) and (4.3), for $i \in \{1,\dots,m\}$, we obtain

$$\operatorname{var}\left(\mathcal{J}_i(z_n) I^2 \varepsilon_{\tau_i(z_n)} - \mathcal{J}_i(z) I^2 \varepsilon_{\tau_i(z)}\right) \le 2T^2 |\mathcal{J}_i(z_n) - \mathcal{J}_i(z)| + 8T a_i |\tau_i(z_n) - \tau_i(z)|.$$

It follows from (4.2) and (4.3) that $\mathcal{J}_i(z_n) \to \mathcal{J}_i(z)$ and $\tau_i(z_n) \to \tau_i(z)$ as $n \to \infty$. We infer from (4.4) that $\operatorname{var}(v_n - v) \to 0$ as $n \to \infty$, which means that $\mathcal{F}$ is continuous.

Step 2. Let us choose a bounded set $B \subset \widetilde{\mathrm{NBV}}$ and prove that the set $\mathcal{F}(B)$ is relatively compact in $\widetilde{\mathrm{NBV}}$. To this aim we take an arbitrary sequence $\{v_n\}_{n=1}^{\infty} \subset \mathcal{F}(B)$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset B$ such that

$$v_n = \mathcal{F}(z_n), \quad n \in \mathbb{N}.$$

Since B is bounded, there exists $\kappa > 0$ such that

$$\operatorname{var}(z_n) \le \kappa, \ n \in \mathbb{N}. \tag{4.5}$$

By (4.2) and (4.3) we have

$$\tau_i(z_n) \in [a,b], \quad |\mathcal{J}_i(z_n)| \le a_i, \quad i = 1,\dots,m, \ n \in \mathbb{N},$$

and we can choose a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \tau_i(z_{n_k}) = \tau_{0,i}, \quad \lim_{k\to\infty} \mathcal{J}_i(z_{n_k}) = J_{0,i}, \tag{4.6}$$

where $\tau_{0,i} \in (0,T)$, $J_{0,i} \in [-a_i, a_i]$ for $i = 1,\dots,m$. By (4.5) and the Helly's selection theorem (see e.g. [13, p. 222]) there exists a subsequence $\{z_{n_\ell}\}_{\ell=1}^{\infty} \subset \{z_{n_k}\}_{k=1}^{\infty}$ which is pointwise converging to a function $z^* \in \mathrm{BV}$ and moreover $\overline{z^*} = 0$. Normalizing z^* in the sense of $z(t) = (z^*(t-) + z^*(t+))/2$ we obtain $z \in \widetilde{\mathrm{NBV}}$ and a subsequence $\{z_{n_\ell}\}_{\ell=1}^{\infty}$ converging to z a.e. on $[0,2T]$. Using (2.24), (4.5) and the Lebesgue convergence theorem, we see that $\|z_{n_\ell} - z\|_{L^1} \to 0$ as $n_\ell \to \infty$. Denote

$$v := \mu I\big(z - \frac{1}{3}z^3\big) + I^2\Big(-z + f + \frac{1}{2T}\sum_{i=1}^{m} J_{0,i}\varepsilon_{\tau_{0,i}}\Big).$$

In the same way as in step 1 we get

$$\begin{aligned}
&\operatorname{var}(v_{n_\ell} - v)\\
&\le \mu \operatorname{var}(E_1)\|z_{n_\ell} - z\|_{L^1} + \frac{\mu}{3}\operatorname{var}(E_1)\|z_{n_\ell}^3 - z^3\|_{L^1} + \operatorname{var}(E_2)\|z_{n_\ell} - z\|_{L^1}\\
&\quad + \frac{1}{2T}\sum_{i=1}^{m}\Big(|\mathcal{J}_i(z_{n_\ell}) - J_{0,i}|\operatorname{var}(I^2\varepsilon_{\tau_i(z_{n_\ell})}) + |J_{0,i}|\operatorname{var}(I^2\varepsilon_{\tau_i(z_{n_\ell})} - I^2\varepsilon_{\tau_{0,i}})\Big),
\end{aligned}$$

and derive that the sequence $\{v_{n_\ell}\}_{\ell=1}^{\infty}$ is convergent to v in $\widetilde{\mathrm{NBV}}$. This yields that $\mathcal{F}(B)$ is relatively compact in $\widetilde{\mathrm{NBV}}$. □

We are ready to prove the existence of a fixed point of the operator $\mathcal{F}$ in $\widetilde{\mathrm{NBV}}$. To do it we denote

$$c_1 := T\|f\|_{L^1} + \sum_{i=1}^{m} a_i, \quad T_0 := 1 - \mu T - \frac{T^2}{3}, \quad c_2 := \frac{1}{2}\sqrt{\frac{T_0}{\mu T}}, \tag{4.7}$$

assume that μ and T satisfy

$$Tc_1 \le \frac{T_0}{3}\sqrt{\frac{T_0}{\mu T}}, \tag{4.8}$$

and define the set

$$\Omega := \big\{z \in \widetilde{\mathrm{NBV}}\text{such that } \operatorname{var}(z) \le c_2, \quad z \text{ is antiperiodic}\big\}. \tag{4.9}$$

Remark 4.3. The construction of the set Ω is based on these observations:

- The parameter c_2 is well defined for $T_0 \ge 0$ which requires the inequality $1 - T^2/3 > 0$. Therefore we have to assume $T \in (0, \sqrt{3})$. Further, $T_0 \ge 0$ implies $\mu \le \frac{1}{T} - \frac{T}{3}$.
- If $c_1 > 0$, then $c_2 > 0$ and Ω is nonempty, convex, bounded and closed set in $\widetilde{\mathrm{NBV}}$.
- If $T \in (0, \sqrt{3})$, then

$$\sqrt{\frac{1 - \mu T - \frac{T^2}{3}}{\mu T}} \to \infty \quad \text{as } \mu \to 0+,$$

and therefore (4.8) is always valid for each sufficiently small μ. If $c_1 > 0$, then the optimal (maximal) value of the parameter μ is determined by

$$Tc_1 = \frac{T_0}{3}\sqrt{\frac{T_0}{\mu T}}. \tag{4.10}$$

- If $c_1 = 0$, then $f = 0$ a. e. on $\mathbb{R}$ and the impulses (1.5) disappear.

Theorem 4.4. *Consider* (4.7) *and* (4.9) *and assume that* $T \in (0, \sqrt{3})$ *and* $c_1 > 0$. *Let* (4.2) *and* (4.3) *hold. Then there exists a solution* $\mu_0 > 0$ *of* (4.10) *such that for each* $\mu \in (0, \mu_0]$ *the operator* $\mathcal{F}$ *maps* Ω *into* Ω.

Proof. By Remark 4.3, there exists $\mu_0 > 0$ satisfying (4.10). Consider $\mu \in (0, \mu_0]$. Clearly μ fulfils (4.8). As we mentioned in Section 2, if $z \in \widetilde{\mathrm{NBV}}$ is antiperiodic, that is z fulfils (1.4), then Iz is antiperiodic as well. Since f is supposed to satisfy (1.2) and the distribution ε_τ is antiperiodic for any $\tau \in \mathbb{R}$, we can conclude that if $z \in \widetilde{\mathrm{NBV}}$ is antiperiodic, then $\mathcal{F}z \in \widetilde{\mathrm{NBV}}$ is antiperiodic, as well. Therefore, if we have the set Ω from (4.9), we only need to prove

$$\operatorname{var}(\mathcal{F}z) \le c_2 \quad \text{for each } z \in \Omega\,. \tag{4.11}$$

So, let us choose $z \in \Omega$. By (4.1) and (2.8),

$$\begin{aligned}
\operatorname{var}(\mathcal{F}z) &\le \mu \operatorname{var}(Iz) + \frac{\mu}{3}\operatorname{var}\big(I(z^3)\big) + \operatorname{var}(I^2 z) + \operatorname{var}(I^2 f) \\
&\quad + \frac{1}{2T}\sum_{i=1}^{m} a_i \operatorname{var}(I^2\varepsilon_{\tau_i(z)}) \\
&= \mu \operatorname{var}(E_1 * z) + \frac{\mu}{3}\operatorname{var}\big(E_1 * z^3\big) + \operatorname{var}(E_2 * z) + \operatorname{var}(E_2 * f) \\
&\quad + \frac{1}{2T}\sum_{i=1}^{m} a_i \operatorname{var}(I^2\varepsilon_{\tau_i(z)}).
\end{aligned}$$

Consequently, using (2.22), (2.23) and (2.28), we derive

$$\begin{aligned}
\operatorname{var}(\mathcal{F}z) \le \mu\|E_1\|_\infty \operatorname{var}(z) + \frac{\mu}{3}\operatorname{var}(E_1)\|z^3\|_\infty + \|E_2\|_\infty \operatorname{var}(z) \\
+ \operatorname{var}(E_2)\|f\|_{L^1} + T\sum_{i=1}^{m} a_i.
\end{aligned}$$

Therefore, by (2.19), (2.24), (4.7), we get

$$\begin{aligned}
\operatorname{var}(\mathcal{F}z) &\le \mu T \operatorname{var}(z) + \frac{4\mu T}{3}\left(\|z\|_\infty\right)^3 + \frac{T^2}{3}\operatorname{var}(z) + T^2\|f\|_{L^1} + T\sum_{i=1}^{m} a_i \\
&\le \big(\mu T + \frac{T^2}{3}\big)\operatorname{var}(z) + \frac{4\mu T}{3}\left(\operatorname{var}(z)\right)^3 + Tc_1\,.
\end{aligned}$$

Hence, to derive (4.11), it suffices to prove the inequality

$$\big(\mu T + \frac{T^2}{3}\big)c_2 + \frac{4\mu T}{3}c_2^3 + Tc_1 \le c_2\,. \tag{4.12}$$

Subtracting the first term on the left-hand side we get

$$\frac{4\mu T}{3}c_2^3 + Tc_1 \le \big(1 - \mu T - \frac{T^2}{3}\big)c_2\,,$$

and using (4.7) we obtain

$$\frac{4\mu T}{3}\Big(\frac{1}{2}\sqrt{\frac{T_0}{\mu T}}\Big)^3 + Tc_1 \le \frac{T_0}{2}\sqrt{\frac{T_0}{\mu T}}\,,$$

which is equivalent to (4.8). Therefore (4.12) is proved. □

5. Main results

Theorem 5.1. *Consider* (4.7) *and assume that* $T \in (0, \sqrt{3})$ *and* $c_1 > 0$. *Let* (4.2) *and* (4.3) *hold. Then there exists a solution* $\mu_0 > 0$ *of* (4.10) *such that for each* $\mu \in (0, \mu_0]$ *the distributional equation* (3.1) *has at least one antiperiodic solution* z *such that* $\operatorname{var}(z) \le c_2$. *If in addition* (3.6) *holds, then problem* (1.1),(1.5) *has an antiperiodic solution* $(x, y) = (z, Dz)$.

Proof. By Remark 4.3, there exists $\mu_0 > 0$ satisfying (4.10). Let us consider the operator $\mathcal{F} : \widetilde{\mathrm{NBV}} \to \widetilde{\mathrm{NBV}}$ defined in (4.1), and the set Ω defined in (4.9), where $\mu \in (0, \mu_0]$. According to Theorem 4.4 the operator $\mathcal{F}$ maps Ω to Ω. Due to Lemma 4.2 the operator $\mathcal{F}$ is completely continuous. Therefore, by the Schauder fixed point theorem $\mathcal{F}$ has a fixed point $z \in \Omega$. Finally, by Lemma 4.1 we see that z is an antiperiodic solution of the distributional equation (3.1) and that under the assumption (3.6) the couple (z, Dz) is an antiperiodic solution of problem (1.1),(1.5). □

Theorem 5.2. *Let* $T \in (0, \sqrt{3})$. *Let* (3.6), (4.2) *and* (4.3) *hold. Then the equation*

$$x'(t) = y(t), \quad y'(t) = -x(t) + f(t), \quad \textit{for a. e. } t \in \mathbb{R},$$

subject to the state-dependent impulse conditions

$$\Delta y(\tau_i(x)) = \mathcal{J}_i(x), \quad i = 1, \dots, m,$$

has at least one antiperiodic solution (x, y) *such that*

$$\operatorname{var}(x) \le \frac{T^2\|f\|_{L^1} + T\sum_{i=1}^m a_i}{1 - \frac{T^2}{3}}.$$

Proof. Let us put

$$c_1 = T\|f\|_{L^1} + \sum_{i=1}^m a_i, \quad c_2 = \frac{Tc_1}{1 - \frac{T^2}{3}}. \tag{5.1}$$

Consider the operator $\mathcal{F}$ from (4.1), where $\mu = 0$ and the set Ω from (4.9) with c_2 defined by (5.1). Similarly as in the proof of Theorem 4.4 we prove (4.11). Since now $\mu = 0$, we derive

$$\frac{T^2}{3}c_2 + Tc_1 \le c_2$$

(compare with (4.12)). Using (5.1), we get

$$Tc_1 \le c_2\big(1 - \frac{T^2}{3}\big) = Tc_1.$$

Hence $\mathcal{F}$ maps Ω to Ω, and arguing as in the proof of Theorem 5.1, we finish the proof. □

If τ_i, $i = 1, \dots, m$, do not depend on $x \in \widetilde{\mathrm{NBV}}$, then the state-dependent impulse conditions (1.5) have the form of the fixed-time impulse conditions

$$\Delta y(\tau_i) = \mathcal{J}_i(x), \quad i = 1, \dots, m, \tag{5.2}$$

where the points $\tau_i \in (0, T)$, $i = 1, \dots, m$, are known and fixed. It is clear that (4.2) holds and Theorem 5.1 yields the following corollary.

Corollary 5.3. *Consider* (4.7) *and assume that* $T \in (0, \sqrt{3})$ *and* $c_1 > 0$. *Let* (4.3) *hold. Then there exists a solution* $\mu_0 > 0$ *of* (4.10) *such that for each* $\mu \in (0, \mu_0]$ *the distributional equation*

$$D^2 z = \mu D\big(z - \frac{z^3}{3}\big) - z + f + \frac{1}{2T} \sum_{i=1}^{m} \mathcal{J}_i(z) \varepsilon_{\tau_i} \tag{5.3}$$

has an antiperiodic solution z.

If in addition

$$\tau_i \neq \tau_j \quad \text{for all } i, j = 1, \dots, m, \ i \neq j,$$

then problem (1.1),(5.2) *has an antiperiodic solution* $(x, y) = (z, Dz)$.

Example 5.4. Put $m = 1$, $T = 1$, choose $0 < a < b < 1$, assume that $f \in L^1$ satisfies $\|f\|_{L^1} = 1$ and define

$$\tau_1(x) = a + (b - a)|\cos(\|x\|_\infty)|, \quad \mathcal{J}_1(x) = \arctan(\mathrm{var}(x)), \quad x \in \widetilde{\mathrm{NBV}}.$$

Then $\tau_1 : \widetilde{\mathrm{NBV}} \to [a, b]$ is continuous, so τ_1 fulfils (4.2) and $\mathcal{J}_1 : \widetilde{\mathrm{NBV}} \to [-\pi/2, \pi/2]$ is continuous, so $\mathcal{J}_1$ fulfils (4.3) with $a_1 = \pi/2$. Then, by Remark 4.3, the inequality $\mu \leq \frac{2}{3}$ has to be fulfilled, and according to (4.7),

$$c_1 = 1 + \frac{\pi}{2}, \quad T_0 = \frac{2}{3} - \mu, \quad c_2 = \frac{1}{2}\sqrt{\frac{2}{3\mu} - 1}.$$

By Theorem 5.1, for each $\mu \in (0, \mu_0]$ the distributional equation (3.1) has an antiperiodic solution $z \in \widetilde{\mathrm{NBV}}$ such that $\mathrm{var}(z) \leq c_2$. Further, the state-dependent impulsive problem (1.1), (1.3), (1.5) has a solution $(x, y) = (z, Dz)$. The value $\mu_0 \approx 0.0049$ is a solution of the equation

$$9\mu\big(1 + \frac{\pi}{2}\big)^2 = \big(\frac{2}{3} - \mu\big)^3.$$

The assumptions (4.2) and (4.3) about boundedness of the functionals τ_i and $\mathcal{J}_i$, $i = 1, \dots, m$, can be restricted on the set Ω from (4.9).

Theorem 5.5. *Consider* (4.7) *and assume that* $c_1 > 0$ *and* $T \in (0, \sqrt{3})$. *Further assume that there exist* $0 < a < b < T$, $a_i > 0$, $i = 1, \dots, m$, *such that*

$$\tau_i(\Omega) \subset [a, b], \quad \tau_i : \widetilde{\mathrm{NBV}} \to \mathbb{R}, \quad i = 1, \dots, m, \quad \text{are continuous,} \tag{5.4}$$

$$\mathcal{J}_i(\Omega) \subset [-a_i, a_i], \quad \mathcal{J}_i : \widetilde{\mathrm{NBV}} \to \mathbb{R}, \quad i = 1, \dots, m, \quad \text{are continuous.} \tag{5.5}$$

Then there exists a solution $\mu_0 > 0$ *of* (4.10) *such that for each* $\mu \in (0, \mu_0]$ *the distributional equation* (3.1) *has at least one antiperiodic solution* $z \in \Omega$.

If in addition (3.6) *holds, then problem* (1.1),(1.5) *has an antiperiodic solution* $(x, y) = (z, Dz)$.

Proof. By Remark 4.3, there exists $\mu_0 > 0$ satisfying (4.10). Let $\mu \in (0, \mu_0]$. Put

$$\chi(s) = \begin{cases} 1, & s \in [0, c_2], \\ 2 - \frac{s}{c_2}, & s \in (c_2, 2c_2), \\ 0, & s \geq 2c_2, \end{cases}$$

and for $z \in \widetilde{\mathrm{NBV}}$ define

$$\tau_i^*(z) := \chi(\mathrm{var}(z))\tau_i(z), \quad i = 1, \dots, m,$$
$$\mathcal{J}_i^*(z) := \chi(\mathrm{var}(z))\mathcal{J}_i(z), \quad i = 1, \dots, m.$$

According to Lemma 4.1, each fixed point z of the operator $\mathcal{F}^* : \widetilde{\mathrm{NBV}} \to \widetilde{\mathrm{NBV}}$,

$$\mathcal{F}^* z = \mu I\big(z - \frac{z^3}{3}\big) + I^2\Big(-z + f + \frac{1}{2T} \sum_{i=1}^{m} \mathcal{J}_i^*(z) \varepsilon_{\tau_i^*(z)} \Big)$$

is a solution of the distributional equation

$$D^2 z = \mu D\big(z - \frac{z^3}{3}\big) - z + f + \frac{1}{2T} \sum_{i=1}^{m} \mathcal{J}_i^*(z) \varepsilon_{\tau_i^*(z)}.$$

By (5.4) and (5.5), the functionals τ_i^* and $\mathcal{J}_i^*$ fulfil (4.2) and (4.3). Consequently, due to Lemma 4.2, the operator $\mathcal{F}^*$ is completely continuous. In addition, if $z \in \Omega$, then $\tau_i^*(z) = \tau_i(z)$, $\mathcal{J}_i^*(z) = \mathcal{J}_i(z)$ and hence $\mathcal{F}^* z = \mathcal{F} z$. Therefore, by Theorem 4.4, the operator $\mathcal{F}^*$ maps Ω to Ω. So, by the Schauder fixed point theorem, $\mathcal{F}^*$ has a fixed point $z \in \Omega$. Consequently z is a fixed point of $\mathcal{F}$. Now, as in the proof of Theorem 5.1, we use Lemma 4.1 to get that z is an antiperiodic solution of the distributional equation (3.1). Moreover, under the assumption (3.6) the couple (z, Dz) is an antiperiodic solution of problem (1.1),(1.5). □

Example 5.6. Put $m = 1$, $T = 1$, choose $0 < a < b < 1$ and assume that $f \in L^1$ satisfies $\|f\|_{L^1} = 1$. Then, as in Example 5.4, we have

$$\mu \leq \frac{2}{3}, \quad T_0 = \frac{2}{3} - \mu, \quad c_2 = \frac{1}{2}\sqrt{\frac{2}{3\mu} - 1}.$$

Since the set Ω depends on the parameter μ, we can define

$$\tau_1(x) = a + \mu(b - a)\sqrt{\|x\|_\infty}, \quad \mathcal{J}_1(x) = \mu \int_0^2 x^2(t)\, dt, \quad \mu \in \big(0, \frac{2}{3}\big),$$

for $x \in \widetilde{\mathrm{NBV}}$. For each $\mu \in (0, 2/3)$ the functionals τ_1 and $\mathcal{J}_1$ are continuous on $\widetilde{\mathrm{NBV}}$ and $\tau_1(\Omega) \subset [a, b]$ and $\mathcal{J}_1(\Omega) \subset [0, a_1]$, where $a_1 = \frac{4}{3} - 2\mu$. Thus, according to (4.7), $c_1 = \frac{7}{3} - 2\mu$, then equation (4.10)reads

$$9\mu\big(\frac{7}{3} - 2\mu\big)^2 = \big(\frac{2}{3} - \mu\big)^3,$$

and it has a solution $\mu_0 \approx 0.0059$. By Theorem 5.5, for each $\mu \in (0, \mu_0]$ the distributional equation (3.1) has an antiperiodic solution $z \in \widetilde{\mathrm{NBV}}$ such that $\mathrm{var}(z) \leq c_2$. Further, the state-dependent impulsive problem (1.1), (1.3), (1.5) has a solution $(x, y) = (z, Dz)$. Let us note that for each $\mu \in (0, \mu_0]$, the functionals τ_1 and $\mathcal{J}_1$ are unbounded on $\widetilde{\mathrm{NBV}}$, $\mathcal{J}_1$ is not globally Lipschitz continuous and τ_1 is not even locally Lipschitz continuous.

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