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INEQUALITIES AMONG EIGENVALUES OF STURM LIOUVILLE EQUATIONS WITH PERIODIC COEFFICIENTS

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ABSTRACT. It is well known that for h-periodic coefficients, every periodic eigenvalue on every interval $[a, a+kh]$, $k = 2, 3, 4, \dots$, is also an eigenvalue on the interval $[a, a+h]$ of a periodic, semi-periodic or complex self-adjoint boundary condition. Here we give an explicit 1-1 correspondence between these eigenvalues.

1. Introduction

Consider the equation

$$-(py')' + qy = \lambda wy, \quad \lambda \in \mathbb{C}, \text{ on } \mathbb{R} \tag{1.1}$$

with coefficients satisfying:

$$\begin{gathered} \frac{1}{p}, q, w \in L_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}), \quad p > 0,\ w > 0 \text{ a.e. on } \mathbb{R}, \\ p(t+h) = p(t), \quad q(t+h) = q(t), \quad w(t+h) = w(t), \quad \text{a.e. } t \in \mathbb{R}, \end{gathered} \tag{1.2}$$

and for $K = I$, or $K = -I$ and $0 \le \gamma \le \pi$, the boundary conditions

$$Y(a + k\,h) = e^{i\gamma} K\, Y(a + (k-1)h), \quad Y = \begin{bmatrix} y \\ (py') \end{bmatrix} \quad k \in \mathbb{N}. \tag{1.3}$$

Here $\mathbb{R}$, $\mathbb{C}$ denote the real and complex numbers, respectively, I the identity matrix, $\mathbb{N} = \{1, 2, 3, \dots\}$, and $L_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ the real valued functions which are Lebesgue integrable on every compact subinterval of $\mathbb{R}$, in particular on the k-intervals $[a + k\,h]$, $k \in \mathbb{N}$. Note that $L_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ contains the piecewise continuous functions on any compact subinterval. Also note that for $\gamma = 0$ and $K = I$ the conditions (1.3) are periodic, for $\gamma = \pi$, $K = I$ as well as for $\gamma = 0$ and $K = -I$ the conditions (1.3) are semi-periodic. For $0 < \gamma < \pi$ (1.3) are complex valued. It is well known that for all these cases the conditions (1.3) are self-adjoint and for each of these self-adjoint conditions the spectrum is real, discrete, bounded below, not bounded above, has no finite cluster point, and the eigenvalues can be ordered to satisfy

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \tag{1.4}$$

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with no consecutive equalities. This ordering determines λ_n uniquely. In case of multiplicity 2 the eigenfunctions are not determined uniquely.

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and, for $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, we define

$$P(k) = \cup_{n=0}^{\infty} \lambda_n^P(k), \quad S(k) = \cup_{n=0}^{\infty} \lambda_n^S(k), \quad \Gamma(\gamma) = \cup_{n=0}^{\infty} \lambda_n(\gamma), \tag{1.5}$$

where $\lambda_n^P(k)$, $\lambda_n^S(k)$, denote the periodic and semi-periodic eigenvalues on the k-interval $[a, a+kh]$, respectively, and $\lambda_n(\gamma)$ denote the eigenvalues on the 1-interval $[a, a+h]$ for $0 < \gamma < \pi$; we also use the notation $\lambda_n^P(1) = \lambda_n^P = \lambda_n^P(0)$, $\lambda_n^S(1) = \lambda_n^S = \lambda_n^S(\pi)$, $n \in \mathbb{N}_0$, since the periodic eigenvalues correspond to the endpoint 0 and the semi-periodic eigenvalues to the endpoint π of the interval $(0, \pi)$ in a natural sense as we will see below.

For reference below we specialize (1.4) to these three cases

$$-\infty < \lambda_0^P(k) \le \lambda_1^P(k) \le \lambda_2^P(k) \le \lambda_3^P(k) \le \dots, \tag{1.6}$$

$$-\infty < \lambda_0^S(k) \le \lambda_1^S(k) \le \lambda_2^S(k) \le \lambda_3^S(k) \le \dots, \tag{1.7}$$

$$-\infty < \lambda_0(\gamma) < \lambda_1(\gamma) < \lambda_2(\gamma) < \lambda_3(\gamma) < \dots, \tag{1.8}$$

which are of special interest here. Note that in (1.8) the inequalities are all strict [7]. See the book [7] for a general discussion of basic results about Sturm-Liouville problems and as a reference for results, definitions, and notation used here.

Remark 1.1. The eigenvalues (1.6), (1.7), (1.8) can be computed with the Bailey-Everitt- Zettl Fortran code SLEIGN2 [2], [1] which can be downloaded free and comes with a user friendly interface.

This paper is a follow up of [6] where we proved, under the general hypothesis (1.2), that for every $n \in \mathbb{N}_0$, and every $k \in \mathbb{N}$, every eigenvalue $\lambda_n^P(k)$, $\lambda_n^S(k)$ on the k-interval for $k > 1$ is also an eigenvalue on the $k = 1$ interval. In this paper we identify which values of $\gamma \in (0, \pi)$ generate periodic and semi-periodic eigenvalues on the intervals $[a + k\,h]$, for $k \in \mathbb{N}$ and construct an explicit 1-1 correspondence between these eigenvalues.

Although we are influenced by some of the methods in Eastham's well known book [3] there are some significant differences in our approach. The boundary conditions (1.3) are defined in terms of the quasi-derivative (py') rather than the classical derivative y' used in [3]. This not only allows the use of the much more general hypothesis (1.2) but has numerous other advantages. Our focus is on the eigenvalues of the boundary conditions (1.3) and their relationships to each other. Also we use the parameterization $\gamma \in (0, \pi)$, rather $t \in (0, 1)$ as in [3], directly. This makes our presentation clearer and more transparent. In particular the 1-1 correspondence.

The organization of the paper is as follows. This Introduction is followed by general eigenvalue characterizations and inequalities in Section 2, eigenvalue inequalities for different intervals in Section 3, the 1-1 correspondence between these in Section 4. Examples to illustrate the inequalities and the 1-1 correspondence between the eigenvalues for different intervals are given in Section 5.

2. Eigenvalue inequalities and characterizations

Russel Bertrand (1872-1970): A good notation has a subtlety and suggestiveness which at times make it almost seem like a live teacher.

In [6] we proved the following two theorems.

Theorem 2.1. *Let* (1.1) *to* (1.5) *hold. Then, for* $k = 2s$, $s \geq 1$ *and for* $k = 2s+1$, $s \geq 0$, *we have*

$$P(k) = \cup_{l=0}^{s} \Gamma(\frac{2l\pi}{k}). \tag{2.1}$$

Furthermore, if $k > 2$ *then every eigenvalue in* $S(k)$ *has multiplicity* 2. *In particular, for* $k = 1$ *we have* $P(1) = \Gamma(0) = \{\lambda_n^P(1) = \lambda_n^P : n \in \mathbb{N}_0\}$.

For a proof of the above theorem see [6]. The case $k = 2$ in Theorem 2.1 is 'special' in the sense that there is no γ in the open $(0, \pi)$ which generates a periodic eigenvalue in the interval $k = 2$. For every $k > 2$ there is at least one such γ. It is clear that if λ is a periodic eigenvalue for $k = 1$ then it is also a periodic eigenvalue for $k = 2$. Also if λ is a semi-periodic eigenvalue for $k = 1$ then λ is a periodic eigenvalue for $k = 2$. The next corollary shows that the converse is true: If λ is a periodic eigenvalue for $k = 2$ then it is either a periodic or semi-periodic eigenvalue for $k = 1$.

Corollary 2.2. *Let the hypotheses and notation of Theorem 2.1 hold. Then*

$$P(2) = \Gamma(0) \cup \Gamma(\pi) = P(1) \cup S(1).$$

The above corollary follows directly from (2.1).

Theorem 2.3. *Let* (1.1) *to* (1.5) *hold. Then, for* $k = 2s$, $s \geq 1$ *and for* $k = 2s+1$, $s \geq 0$, *we have*

$$S(k) = \cup_{l=0}^{s} \Gamma(\frac{(2l+1)\pi}{k}). \tag{2.2}$$

Furthermore, if $k > 2$, *then every eigenvalue in* $P(k)$ *has multiplicity* 2. *In particular, for* $k = 1$ *we have* $S(1) = \Gamma(\pi) = \{\lambda_n^S(1) = \lambda_n^S : n \in \mathbb{N}_0\}$.

For a proof of the above theorem see [6]. The next theorem plays an important role below and is stated here for the benefit of the reader.

Fix $a \in \mathbb{R}$, and $\lambda \in \mathbb{C}$ define solutions $u(\cdot, \lambda)$, $v = v(\cdot, \lambda)$ of equation (1.1) with the initial conditions

$$u(a, \lambda) = 1 = (pv')(a, \lambda), \quad v(a, \lambda) = 0 = (pu')(a, \lambda). \tag{2.3}$$

When a and λ are fixed we abbreviate this notation to $u = u(\cdot, \lambda)$, $v = v(\cdot, \lambda)$ and sometimes to just u, v.

Theorem 2.4. *Let* (1.1)–(1.5) *hold. Let* $a \in \mathbb{R}$, $k \in \mathbb{N}$, $b = a + k\,h$ *and let* $K = I$. *With* u, v *determined by* (2.3) *define* $D(\lambda)$ *by*

$$D(\lambda) = u(b, \lambda) + v^{[1]}(b, \lambda), \quad \lambda \in \mathbb{R} \tag{2.4}$$

Then

(1) *The real number* $\lambda = \lambda_n(\gamma)$ *for some* $n \in \mathbb{N}_0$ *and some* $\gamma \in (0, \pi)$ *if and only if*

$$D(\lambda) = 2\cos\gamma, quad - \pi < \gamma < \pi. \tag{2.5}$$

In this case

$$-2 < D(\lambda) < 2. \tag{2.6}$$

(2) *Let* $0 < \gamma < \pi$. *Then* $\lambda_n(\gamma)$ *is simple and* $\lambda_n(\gamma) = \lambda_n(-\gamma)$, $n \in \mathbb{N}_0$. *If* u_n *is an eigenfunction of* $\lambda_n(\gamma)$, *then it is unique up to constant multiples and its complex conjugate* $\overline{u}_n$ *is an eigenfunction of* $\lambda_n(-\gamma)$, $n \in \mathbb{N}_0$.

(3) *$\lambda = \lambda_n^P$ for some $n \in \mathbb{N}_0$ if and only if*
$$D(\lambda) = 2. \tag{2.7}$$
(4) *$\lambda = \lambda_n^S$ for some $n \in \mathbb{N}_0$ if and only if*
$$D(\lambda) = -2. \tag{2.8}$$
(5) *The following inequalities hold for $0 < \gamma < \pi$,*
$$\begin{aligned} -\infty < \lambda_0^P < \lambda_0(\gamma) < \lambda_0^S \le \lambda_1^S < \lambda_1(\gamma) < \lambda_1^P \le \lambda_2^P < \lambda_2(\gamma) < \lambda_2^S \\ \le \lambda_3^S < \lambda_3(\gamma) < \lambda_3^P \le \lambda_4^P < \lambda_4(\gamma) < \lambda_4^S \le \lambda_5^S < \dots . \end{aligned} \tag{2.9}$$
(6) *$\lambda_n \le \lambda_n^D \le \lambda_{n+2}$, $n \in \mathbb{N}_0$ where λ_n is the n -th eigenvalue for any self-adjoint boundary condition* (1.3)*; there is no lower bound for λ_0 and λ_1 as functions of the self-adjoint boundary conditions.*
(7) *λ_0^P and each $\lambda_n(\gamma)$, $n \in \mathbb{N}_0$ is simple.*
(8) *For $0 < \alpha < \beta < \pi$ we have*
$$\begin{aligned} \lambda_0(\beta) < \lambda_0(\alpha) < \lambda_1(\alpha) < \lambda_1(\beta) < \lambda_2(\beta) < \lambda_2(\alpha) \\ < \lambda_3(\alpha) < \lambda_3(\beta) < \lambda_4(\beta) < \lambda_4(\alpha) < \dots \end{aligned} \tag{2.10}$$
In other words, $\lambda_0(\gamma)$ is decreasing, $\lambda_1(\gamma)$ is increasing, $\lambda_2(\gamma)$ decreasing, $\lambda_3(\gamma)$ increasing, $\dots$, for $\gamma \in (0, \pi)$.
(9) *$D(\lambda)$ is strictly decreasing in the intervals $(\lambda_{2n}^P, \lambda_{2n}^S)$, $n \in \mathbb{N}_0$ and strictly increasing in the intervals $(\lambda_{2n+1}^S, \lambda_{2n+1}^P)$, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$.*
(10) *$D'(\lambda) \neq 0$ for $\lambda \in (0, \pi)$.*

The above theorem is a special case of [7, Theorem 4.8.1]. We omit its proof.

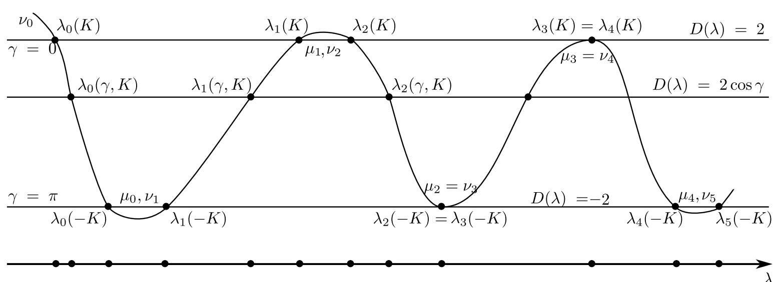

FIGURE 1. $D(\lambda)$

The special case of Figure 1 when $K = I$, $\lambda_n(K) = \lambda_n^P$, $\lambda_n(-K) = \lambda_n^S$, $\lambda_n(\gamma, K) = \lambda_n(\gamma)$, and ν_n, υ_n denote the Neumann and Dirichlet eigenvalues illustrates the results below. (We make no direct use of Neumann and Dirichlet eigenvalues in this paper.)

It is clear that $\lambda_0^P(1)$ is also a periodic eigenvalue on interval k for $k > 1$ but, given the ordering (1.6), is it the first eigenvalue determined by this ordering? The next Corollary answers this question.

Corollary 2.5. *Let the hypotheses and notation of Theorem 2.4 hold. Then*
$$\lambda_0^P(k) = \lambda_0^P(1) = \lambda_0^P, \quad k \in \mathbb{N} \tag{2.11}$$

Proof. Clearly $\lambda_0^P \in P(k)$. By definition $\lambda_0^P(k)$ is the lowest eigenvalue determined by the ordering (1.6). It follows from (2.1) and (2.9) that this is λ_0^P. $\square$

3. Inequalities among eigenvalues of different intervals

Note that for both Theorems 2.1 and 2.3 the eigenvalues on the right side are all from the interval $k = 1$ while the eigenvalues on the left are from intervals for $k > 1$. Corollary 2.5 shows that $\lambda_0^P(k)$ stays constant as k changes but how do the other eigenvalues change? More specifically:

- Given an eigenvalue λ in $P(k)$ for some $k > 1$, by Theorem 2.1 λ is also an eigenvalue for $k = 1$, which eigenvalue?
- Given an eigenvalue λ in $S(k)$ for some $k > 1$, by Theorem 2.3 λ is also an eigenvalue for $k = 1$, which eigenvalue?

These questions are answered in this section. Our proof is based on Theorems 2.1, 2.3, 2.4 and develops a method for finding a 1-1 correspondence between these eigenvalues for each fixed $k > 1$. This method is used in Section 4 to explicitly construct this 1-1 correspondence.

For each k we identify the values of γ which generate periodic and semi-periodic eigenvalues on k-interval. Note that the set $\cup_{k=1}^{\infty} P(k)$ is a countable union of countable sets and is therefore countable, whereas the set $\Gamma(\gamma) = \{\cup_{n=0}^{\infty} \lambda_n(\gamma) : \gamma \in (0, \pi)\}$ is not countable so there can be no 1-1 correspondence between these two sets.

As mentioned above, the inequalities (1.6), (1.7), (1.8) determine $\lambda_n^P(k)$, $\lambda_n^S(k)$ and $\lambda_n(\gamma)$ for each $\gamma \in (0, \pi)$ and each $n \in \mathbb{N}_0$. This is the 'natural' ordering which defines λ_n for any self-adjoint boundary condition when the eigenvalues are bounded below. In [3] the assumption that p is positive seems to have been omitted. Möller [5] has shown that if p is positive and negative each on a set of positive Lebesgue measure then the eigenvalues are unbounded above and below. In this case λ_n is not well defined. Using Theorems 2.1, 2.3, and 2.4 we will find a different ordering and a 1-1 correspondence between these two orderings. This new correspondence will be illustrated with some examples for both the periodic and the semi-periodic case. We start with a remark.

Remark 3.1. Although (1.5) defines $\Gamma(\gamma)$ only for γ in the open interval $(0, \pi)$ Theorems 2.1 and 2.3 show that the 'boundary sets' $\Gamma(0)$, $\Gamma(\pi)$ represent the periodic eigenvalues and semi-periodic eigenvalues on the interval $[a, a+h]$, respectively. However, it is important to keep in mind that the eigenvalues when $\gamma \in (0, \pi)$ are all simple but the eigenvalues in $\Gamma(0)$, $\Gamma(\pi)$ may be simple or double, except for λ_0^P which is always simple. It follows from Theorem 2.3 that $\Gamma(0) = \Gamma(2l\pi)$ and $\Gamma(\pi) = \Gamma((2l+1)\pi)$ for any $l \in \mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$.

In the next two theorems we establish inequalities between the eigenvalues of $P(k) = \cup_{n=0}^{\infty} \lambda_n^P(k)$, $S(k) = \cup_{n=0}^{\infty} \lambda_n^S(k)$, and $\Gamma(\gamma) = \cup_{n=0}^{\infty} \lambda_n(\gamma)$.

Theorem 3.2. *Let* (1.1)–(1.5) *hold. Fix $k > 2$, let $P(k)$, $S(k)$, $\Gamma(\gamma)$ be defined by* (1.5) *and let*

$$\begin{aligned} P(1) &= \{\lambda_n^P(1) : n \in \mathbb{N}_0\} = \Gamma(0) = \{\lambda_n(0) : n \in \mathbb{N}_0\}, \\ S(1) &= \{\lambda_n^S(1) : n \in \mathbb{N}_0\} = \Gamma(\pi) = \{\lambda_n(\pi) : n \in \mathbb{N}_0\}. \end{aligned} \tag{3.1}$$

(1) If $k = 2s$, $s > 1$, then

$$\begin{aligned}
&\lambda_0^P(0) \\
&= \lambda_0(0) < \lambda_0(2\pi/k) < \lambda_0(4\pi/k) < \dots < \lambda_0(2(s-1)\pi)/k) < \lambda_0(\pi) \\
&\le \lambda_1(\pi) < \lambda_1(2(s-1)\pi/k) < \lambda_1(2(s-2)\pi/k) < \dots < \lambda_1(2\pi/k) < \lambda_1(0) \\
&\le \lambda_2(0) < \lambda_2(2\pi/k) < \lambda_2(4\pi/k) < \dots < \lambda_2(2(s-1)\pi/k) < \lambda_2(\pi) \\
&\le \lambda_3(\pi) < \lambda_3(2(s-1)\pi/k) < \lambda_3(2(s-2)\pi/k) \dots < \lambda_3(2\pi/k) < \lambda_3(0) \\
&\le \lambda_4(0) < \lambda_4(2\pi/k) < \dots
\end{aligned} \tag{3.2}$$

Therefore

$$\begin{gathered}
\lambda_0^P(k) = \lambda_0^P, \\
\lambda_s^P(k) = \lambda_0(2s\pi/k) = \lambda_0^S, \\
\lambda_{s+1}^P(k) = \lambda_1(2s\pi/k) = \lambda_1^S, \\
\lambda_{s+2}^P(k) = \lambda_1((2s-2)\pi/k) \\
\dots
\end{gathered} \tag{3.3}$$

(2) If $k = 2s+1$, $s > 1$, then

$$\begin{aligned}
\lambda_0^P &= \lambda_0(0) < \lambda_0(2\pi/k) < \lambda_0(4\pi/k) < \lambda_0(6\pi/k) \dots < \lambda_0(2s\pi/k) \\
&< \lambda_1(2s\pi/k) < \lambda_1(2(s-1)\pi/k) < \dots < \lambda_1(2\pi/k) < \lambda_1(0) \\
&\le \lambda_2(0) < \lambda_2((2\pi/k) < \lambda_2(4\pi/k) < \dots < \lambda_2(2s\pi/k) \\
&< \lambda_3(2s\pi/k) < \lambda_3(2(s-1)\pi/k) < \dots < \lambda_3(2\pi/k) < \lambda_3(0) \\
&\le \lambda_4(0) < \lambda_4(2\pi/k) \dots .
\end{aligned} \tag{3.4}$$

Therefore

$$\begin{gathered}
\lambda_0^P(k) = \lambda_0^P, \\
\lambda_s^P(k) = \lambda_0(2s\pi/k), \\
\lambda_{s+1}^P(k) = \lambda_1(2s\pi/k), \\
\lambda_{s+2}^P(k) = \lambda_1((2s-2)\pi/k) \\
\dots
\end{gathered} \tag{3.5}$$

Proof. These inequalities follow from Theorems 2.1, 2.3 and 2.4, particularly (2.8) and (2.9). The fact $\lambda_0(\gamma)$ is decreasing, $\lambda_1(\gamma)$ is increasing, $\lambda_2(\gamma)$ decreasing, $\lambda_3(\gamma)$ increasing, $\dots$, for $\gamma \in (0,\pi)$ is reflected in the pattern for the alternating rows in (3.2), (3.4). This pattern is clearly seen in the examples below. □

Theorem 3.3. *Let the hypotheses and notation of Theorem 3.2 hold.*
(1) If $k = 2s$, $s > 1$, then

$$\begin{aligned}
\lambda_0(\pi/k) &< \lambda_0(3\pi/k) < \dots < \lambda_0((2s-1)\pi/k) \\
&\quad < \lambda_1((2s-1)\pi/k) < \lambda_1((2s-3)\pi/k) < \dots < \lambda_1(\pi/k) \\
&\quad < \lambda_2(\pi/k) < \dots < \lambda_2(3\pi/k) < \dots < \lambda_3((2s-1)\pi/k) \\
&\quad < \lambda_3((2s-1)\pi/k) < \lambda_3((2s-3)\pi/k) < \dots < \lambda_3(\pi/k) \\
&\quad < \lambda_4(\pi/k) < \dots < \lambda_4(3\pi/k) < \dots < \lambda_4((2s-1)\pi/k) \dots
\end{aligned} \tag{3.6}$$

Therefore

$$
\begin{aligned}
\lambda_0^S(k) &= \lambda_0(\pi/k),\\
\lambda_{s-1}^S(k) &= \lambda_0((2s-1)\pi/k),\\
\lambda_s(k) &= \lambda_1((2s-1)\pi/k),\\
\lambda_{s+1}^S(k) &= \lambda_1((2s-3)\pi/k)\\
&\dots
\end{aligned}
\tag{3.7}
$$

(2) *If $k = 2s+1$, $s > 1$, then*

$$
\begin{aligned}
\lambda_0(\pi/k) &< \lambda_0(3\pi/k) < \dots < \lambda_0((2s+1)\pi/k) = \lambda_0^S\\
&\le \lambda_1^S = \lambda_1(\pi) < \lambda_1((2s-1)\pi/k) < \dots < \lambda_1(\pi/k)\\
&< \lambda_2(\pi/k) < \lambda_2(3\pi/k) < \dots < \lambda_2((2s+1)\pi/k) = \lambda_2^S\\
&\le \lambda_3^S = \lambda_3(\pi) < \lambda_3((2s-1)\pi/k) < \dots < \lambda_3(\pi/k) < \dots
\end{aligned}
\tag{3.8}
$$

Therefore

$$
\begin{aligned}
\lambda_0^S(k) &= \lambda_0(\pi/k),\\
\lambda_s^S(k) &= \lambda_0^S,\\
\lambda_{s+1}^S(k) &= \lambda_1^S,\\
\lambda_{s+2}^S(k) &= \lambda_1((2s-1)\pi/k),\\
&\dots
\end{aligned}
\tag{3.9}
$$

Proof. These inequalities follow from Theorems 2.1, 2.3 and 2.4. Particularly (2.8) and (2.9). The fact $\lambda_0(\gamma)$ is decreasing, $\lambda_1(\gamma)$ is increasing, $\lambda_2(\gamma)$ decreasing, $\lambda_3(\gamma)$ increasing, ..., for $\gamma \in (0,\pi)$ is reflected in the pattern for the alternating rows in (3.6), (3.8). This pattern is used in the proofs of Theorems below and illustrated in the examples below. □

Now we list some examples to illustrate Theorem 3.2 and clarify its proof. We start with the periodic case for $k = 2$. This case is special and does not illustrate the general pattern because it does not involve γ.

As k gets large the eigenvalues $\lambda_n^P(k)$ and $\lambda_n^S(k)$ approach $\lambda_0^P(1) = \lambda_0^P$ from the right. More precisely we have the following result.

Theorem 3.4. *For any $n \in \mathbb{N}$ we have*

$$
\lim_{k\to\infty} \lambda_n^P(k) = \lambda_0^P, \quad \lim_{k\to\infty} \lambda_n^S(k) = \lambda_0^P. \tag{3.10}
$$

Proof. Let $n \in \mathbb{N}$. For $k = 2(n+1) = 2s$. From (3.2) we have $\lambda_n^P(k) = \lambda_0(2s\pi)/k)$ and therefore

$$
\lim_{k\to\infty} \lambda_n^P(k) = \lambda_0^P. \tag{3.11}
$$

For $k = 2n+1 = 2s+1$ from (3.6) we have $\lambda_n^P(k) = \lambda_0(2s\pi/k)$ and (3.10) follows. By Theorem 3.2 $\lambda_n^P(k) > \lambda_0^P$ for k even or odd; hence the limit in (3.11) is from the right.

The proof of $\lim_{k\to\infty} \lambda_n^S(k) = \lambda_0^P$ is similar using (3.4), (3.8) and the limit is also from the right. □

It is well known that equation (1.1) is oscillatory on $\mathbb{R}$ when $\lambda > \lambda_0^P$ and non-oscillatory when $\lambda \le \lambda_0^P$. In the next theorem we give an elementary proof of this using Theorem 3.4 valid under our general hypotheses (1.2).

Theorem 3.5. *Let the hypotheses and notation of Theorem 3.2 hold. Then* (1.1) *is oscillatory on* $\mathbb{R}$ *when* $\lambda > \lambda_0^P$ *and non-oscillatory when* $\lambda \le \lambda_0^P$.

Proof. Suppose that $\lambda = \lambda_0^P$ and u is an eigenfuntion of λ. Then by [6, Theorem 8] u has no zero in the closed interval $[a, a+h]$. Hence the extension of u to $\mathbb{R}$ has no zero on $\mathbb{R}$. By the Sturm Comparison Theorem equation (1.1) is non-oscillatory for $\lambda \le \lambda_0^P$. Let $\lambda > \lambda_0^P$. By Theorems 2.1, 3.4, $\lambda_0^P < \lambda_n^P(k) < \lambda$ for all sufficiently large n and k. Since $\lambda_n^P(k)$ has zeros in the interval $[a, kh]$, its extension to $\mathbb{R}$ has infinitely many zeros, i.e. it is oscillatory. □

4. Construction of the 1-1 correspondence

The next two theorems give the explicit 1-1 correspondence between the periodic and semi-periodic eigenvalues on the k interval $k > 1$ and the corresponding eigenvalues from the interval $k = 1$.

Theorem 4.1. *Let the hypotheses and notation of Theorem 3.2 hold and let the eigenvalues* $\lambda_n^P(k)$ *be ordered according to* (1.6).

- *If* $k = 2s$, $s \in \mathbb{N}$, *then:*
  (1) *for* m *even we have*
  $$\lambda_{ms+n}^P(k) = \lambda(2(n-m)\pi)/k), \quad n = m, m+1, \dots, m+s. \tag{4.1}$$
  (2) *for* m *odd we have*
  $$\lambda_{ms+n}^P(k) = \lambda(2(m+s-n)\pi)/k), \quad n = m,\ m+1, \dots, m+s. \tag{4.2}$$
- *If* $k = 2s+1$, $s > 0$, *then:*
  (1) *for* m *even and we have*
  $$\lambda_{ms+n}^P(k) = \lambda(2(n-m)\pi)/k), \quad n = m, m+1, \dots, m+s. \tag{4.3}$$
  (2) *for* m *odd we have*
  $$\lambda_{ms+n}^P(k) = \lambda(2(m+s-n)\pi)/k), \quad n = m, m+1, \dots, m+s. \tag{4.4}$$

Proof. For clarity of presentation we use the notation discussed in Theorem 3.2. Suppose $k = 2s$, $s \in \mathbb{N}$. From (3.2) and the natural ordering (1.6) it follows that

$$\lambda_0^P = \lambda_0^P, \quad \lambda_1^P(k) = \lambda_0(2\pi/k), \ \dots, \quad \lambda_{s-1}^P(k) = \lambda_0(2(s-1)\pi)/k), \quad \lambda_s^P(k) = \lambda_0^S,$$
$$\lambda_{s+1}^P(k) = \lambda_1^S, \quad \lambda_{s+2}^P(k) = \lambda_1(2(s-1)\pi/k), \ \dots,$$
$$\lambda_{2s}^P(k) = \lambda_1(2\pi/k), \quad \lambda_{2s+1}^P(k) = \lambda_1^P,$$
$$\lambda_{2s+2}^P(k) = \lambda_2^P, \quad \lambda_{2s+3}^P(k) = \lambda_2(2\pi/k), \ \dots,$$
$$\lambda_{3s+1}^P(k) = \lambda_2(2(s-1)\pi/k), \quad \lambda_{3s+2}^P(k) = \lambda_2^S,$$
$$\lambda_{3s+3}^P(k) = \lambda_3^S, \quad \lambda_{s+4}^P(k) = \lambda_3(2(s-1)\pi/k), \ \dots,$$
$$\lambda_{4s+2}^P(k) = \lambda_3(2\pi/k), \quad \lambda_{4s+3}^P(k) = \lambda_3^P$$

and so on.

Note that for $\lambda_{ms+n}^P(k)$ the values of γ increase $0, 2\pi/k, \dots, 2(s-1)\pi/k, 2s\pi/k = \pi$ as the index n goes from m to $m+s$ when m is even and decreases $2s\pi/k = \pi$, $2(s-1)\pi/k, \dots, 2\pi/k, 0$ when m is odd. This establishes (4.1) and (4.2).

Suppose $k = 2s+1$, $s > 0$. From (3.4) and the natural ordering (1.6) it follows that

$$\lambda_0^P(k) = \lambda_0^P, \quad \lambda_1^P(k) = \lambda_0(2\pi/k), \ \dots,$$

$$
\begin{gathered}
\lambda^P_{s-1}(k)=\lambda_0(2(s-1)\pi/k), \quad \lambda^P_s(k)=\lambda_0(2s\pi/k),\\
\lambda^P_{s+1}(k)=\lambda_1(2s\pi/k), \quad \lambda^P_{s+2}(k)=\lambda_1(2(s-1)\pi/k),\ \dots,\\
\lambda^P_{2s}(k)=\lambda_1(2\pi/k), \quad \lambda^P_{2s+1}(k)=\lambda^P_1,\\
\lambda^P_{2s+2}(k)=\lambda^P_2, \quad \lambda^P_{2s+3}(k)=\lambda_2(2\pi/k),\ \dots,\\
\lambda^P_{3s+1}(k)=\lambda_2(2(s-1)\pi/k), \quad \lambda^P_{3s+2}(k)=\lambda_2(2s\pi/k),\\
\lambda^P_{3s+3}(k)=\lambda_3(2s\pi/k), \quad \lambda^P_{3s+4}(k)=\lambda_3(2(s-1)\pi/k),\ \dots,\\
\lambda^P_{4s+2}(k)=\lambda_3(2\pi/k), \quad \lambda^P_{4s+3}(k)=\lambda^P_3
\end{gathered}
$$

and so on.

Note that for $\lambda^P_{ms+n}(k)$ the values of γ increase $0, 2\pi/k, \dots, 2(s-1)\pi/k, 2s\pi/k=\pi$ as the index n goes from m to $m+s$ when m is even and decreases $2s\pi/k=\pi$, $2(s-1)\pi/k, \dots, 2\pi/k, 0$ when m is odd. This establishes (4.3) and (4.4). □

Theorem 4.2. *Let the hypotheses and notation of Theorem 3.2 hold and let the eigenvalues $\lambda^S_n(k)$ be ordered according to* (1.7).

- *If $k=2s$, $s>1$, then:*
  1. *for m even we have*
  $$\lambda^S_{ms+n}(k)=\lambda(2n+1)\pi)/k), \quad n=0,1,\dots,s-1. \tag{4.5}$$
  2. *for m odd we have*
  $$\lambda^S_{ms+n}(k)=\lambda(2(s-1-n)\pi)/k), \quad n=0,1,\dots,s-1. \tag{4.6}$$
- *If $k=2s+1$, $s>0$, then:*
  1. *for m even and $n\in[m,m+s]$ we have*
  $$\lambda^S_{ms+n}(k)=\lambda(2(n-m)\pi+1)/k), \quad n=m,\ m+1,\dots,m+s. \tag{4.7}$$
  2. *for m odd and $n\in[m,m+s]$ we have*
  $$\lambda^S_{ms+n}(k)=\lambda(2(m+s-n)+1\pi)/k), \quad n=m,\ m+1,\dots,m+s. \tag{4.8}$$

Proof. For clarity of presentation we use the notation discussed in Theorem 3.3. Suppose $k=2s$, $s\in\mathbb{N}$. From (3.6) and the natural ordering (1.7) it follows that

$$
\begin{gathered}
\lambda^S_0(k)=\lambda_0(\pi/k),\ \dots,\ \lambda^S_{s-2}(k)=\lambda_0((2s-3)\pi/k), \quad \lambda^S_{s-1}(k)=\lambda_0((2s-1)\pi/k),\\
\lambda^S_s(k)=\lambda_1((2s-1)\pi/k),\ \dots,\ \lambda^S_{2s-2}(k)=\lambda_1(3\pi/k), \quad \lambda^S_{2s-1}(k)=\lambda_1(\pi/k),\\
\lambda^S_{2s}(k)=\lambda_2(\pi/k),\ \dots,\ \lambda^S_{3s-2}(k)=\lambda_2((2s-3)\pi/k), \quad \lambda^S_{3s-1}(k)=\lambda_2((2s-1)\pi/k),\\
\lambda^S_{3s}(k)=\lambda_3((2s-1)\pi/k),\ \dots,\ \lambda^S_{4s-2}(k)=\lambda_3(3\pi/k), \quad \lambda^S_{4s-1}(k)=\lambda_3(\pi/k),
\end{gathered}
$$

and so on.

Note that for $\lambda^S_{ms+n}(k)$ the values of γ increase $\pi/k, \dots, (2s-1)\pi/k$, as the index n goes from 0 to $s-1$ when m is even, and decreases $(2s-1)\pi/k, \dots, \pi/k$, when m is odd. This establishes (4.5) and (4.6).

Suppose $k=2s+1$, $s>0$. From (3.8) and the natural ordering (1.7) it follows that

$$
\begin{gathered}
\lambda^S_0(k)=\lambda_0(\pi/k),\ \dots,\ \lambda^S_{s-1}(k)=\lambda_0((2s-1)\pi/k),\\
\lambda^S_s(k)=\lambda_0((2s+1)\pi/k)=\lambda^S_0, \quad \lambda^S_{s+1}(k)=\lambda^S_1,\\
\lambda^S_{s+2}(k)=\lambda_1((2s-1)\pi/k),\ \dots,\ \lambda^S_{2s}(k)=\lambda_1(3\pi/k),
\end{gathered}
$$

$$\lambda^S_{2s+1}(k) = \lambda_1(\pi/k), \quad \lambda^S_{2s+2}(k) = \lambda_2(\pi/k), \ \dots, \ \lambda^S_{3s+1}(k) = \lambda_2((2s-1)\pi/k),$$
$$\lambda^S_{3s+2}(k) = \lambda_2((2s+1)\pi/k) = \lambda^S_2, \quad \lambda^S_{3s+3}(k) = \lambda^S_3,$$
$$\lambda^S_{3s+4}(k) = \lambda_3((2s-1)\pi/k), \ \dots, \ \lambda^S_{4s+2}(k) = \lambda_3(3\pi/k), \quad \lambda^S_{4s+3}(k) = \lambda_3(\pi/k),$$

and so on.

Note that for $\lambda^S_{ms+n}(k)$ the values of γ increase $\pi/k, \dots, (2s-1)\pi/k$, as the index n goes from m to $m+s$ when m is even, and decreases $(2s+1)\pi)/k = \pi$, $\dots, \pi/k$, when m is odd. This establishes (4.7) and (4.8). □

5. Examples

In this section we give some examples. First for the cases $k = 2, 3, 4$, then for some higher order cases. There are some key differences between k even and k odd. For the periodic even order case any periodic eigenvalue for $k = 1$ is also a periodic eigenvalue for $k > 1$. Also a semi-periodic eigenvalue for $k = 1$ is a periodic eigenvalue for even k. A more subtle difference is the effect of the inequalities of Theorem 3.2 on the 1-1 correspondence. This has to do with the alternating increasing and decreasing values of γ for the even and odd order cases. These will be illustrated in the examples below.

Example 5.1. $k = 2$. As mentioned above the case $k = 2$ is special. By Corollary 2.2: $P(2) = P(1) \cup S(1) = \Gamma(0) \cup \Gamma(\pi)$. From this and 2.9) we get

$$\lambda^P_0 < \lambda^S_0 \le \lambda^S_1 < \lambda^P_1 \le \lambda^P_2 < \lambda^S_2 \le \lambda^S_3 < \lambda^P_3 \le \lambda^P_4 < \dots .$$

Hence the 1-1 correspondence is:

$$\lambda^P_0(2) = \lambda^P_0(1) = \lambda^P_0, \quad \lambda^P_1(2) = \lambda^S_0, \quad \lambda^P_2(2) = \lambda^S_1,$$
$$\lambda^P_3(2) = \lambda^P_1, \quad \lambda^P_4(2) = \lambda^P_2, \dots .$$

Example 5.2. $k = 3$. This case is similar to 5.1. In this case there is one $\gamma = 2\pi/3$ generates the additional eigenvalues rather than the semi-periodic ones which can be identified with $\gamma = \pi$. Thus we have

$$\lambda^P_0 < \lambda_0(2\pi/3) < \lambda^P_1 \le \lambda^P_2 < \lambda_2(2\pi/3) < \lambda^P_3 \le \lambda^P_4 < \lambda_4(2\pi/3) < \dots .$$

Hence the 1-1 correspondence is:

$$\lambda^P_0(2) = \lambda^P_0(1) = \lambda^P_0, \quad \lambda^P_1(2) = \lambda_0(2\pi/3), \quad \lambda^P_2(2) = \lambda^P_2,$$
$$\lambda^P_3(2) = \lambda_3(2\pi/3), \quad \lambda^P_4(2) = \lambda^P_4, \dots .$$

Example 5.3. $k = 2s$, $s = 4$.This and the next example illustrates the fact that the values of γ increase $\pi/k, \dots, (2s-1)\pi/k$, as the index n goes from m to $m+s$ when m is even and decrease $(2s+1)\pi)/k = \pi, \dots, \pi/k$, when m is odd. By Theorem 3.3 we have

$$\begin{aligned}
\lambda^P_0(0) = \lambda_0(0) &< \lambda_0(2\pi/8) < \lambda_0(4\pi/8) < \lambda_0(6\pi/8) < \lambda_0(\pi) \\
&\le \lambda_1(\pi) < \lambda_1(6\pi/8) < \lambda_1(4\pi/8) < \lambda_1(2\pi/8) < \lambda_1(0) \\
&\le \lambda_2(0) < \lambda_2(2\pi/8) < \lambda_2(4\pi/8) < \lambda_2(6\pi/8) < \lambda_2(\pi) \\
&\le \lambda_3(\pi) < \lambda_3(6\pi/8) < \lambda_3(4\pi/8) < \lambda_3(2\pi/8) < \lambda_3(0) \\
&\le \lambda_4(0) < \lambda_4(2\pi/8) < \dots
\end{aligned}$$

Therefore

(1) for $m = 0$ we have

$$\lambda_0^P(8) = \lambda_0^P, \quad \lambda_1^P(8) = \lambda_0(2\pi/8), \quad \lambda_2^P(8) = \lambda_0(4\pi/8),$$
$$\lambda_3^P(8) = \lambda_0(6\pi/8), \quad \lambda_4^P(8) = \lambda_0(8\pi/8) = \lambda_0^S;$$

(2) for $m = 1$ we have

$$\lambda_5^P(8) = \lambda_1^S, \quad \lambda_6^P(8) = \lambda_1(6\pi/8), \quad \lambda_7^P(8) = \lambda_1(4\pi/k),$$
$$\lambda_8^P(8) = \lambda_1(2\pi/8), \quad \lambda_9^P(8) = \lambda_1(0) = \lambda_1^P;$$

(3) for $m = 2$ we have

$$\lambda_{10}^P(8) = \lambda_2^P, \quad \lambda_{11}^P(8) = \lambda_2(2\pi/8), \quad \lambda_{12}^P(8) = \lambda_2(4\pi/8),$$
$$\lambda_{13}^P(8) = \lambda_2(6\pi/8), \quad \lambda_{14}^P(8) = \lambda_2(\pi) = \lambda_2^S;$$

(4) for $m = 3$ we have

$$\lambda_{15}^P(8) = \lambda_3^S, \quad \lambda_{16}^P(8) = \lambda_3(6\pi/8), \quad \lambda_{17}^P(8) = \lambda_3(4\pi/8),$$
$$\lambda_{18}^P(8) = \lambda_3(2\pi/8), \quad \lambda_{19}^P(8) = \lambda_3^P.$$

Example 5.4. $k = 2s + 1$, $s = 4$. By Theorem 3.3 we have

$$\begin{aligned}
\lambda_0^P &< \lambda_0(2\pi/9) < \lambda_0(4\pi/9) < \lambda_0(6\pi/9) < \lambda_0(8\pi/9) < \\
&< \lambda_1^S = \lambda_1(\pi) < \lambda_1((2s-1)\pi/9) < \dots < \lambda_1(\pi/9) \\
&< \lambda_2(\pi/9) < \lambda_2(3\pi/9) < \dots < \lambda_2((2s+1)\pi/9) = \lambda_2^S \\
&\le \lambda_3^S = \lambda_3(\pi) < \lambda_3((2s-1)\pi/9) < \dots < \lambda_3(\pi/9) < \dots
\end{aligned}$$

Therefore

(1) for $m = 0$ we have

$$\lambda_0^P(9) = \lambda_0^P, \quad \lambda_1^P(9) = \lambda_0(\pi/9), \quad \lambda_2^P(9) = \lambda_0(3\pi/9),$$
$$\lambda_3^P(9) = \lambda_0(5\pi/9), \quad \lambda_4^P(9) = \lambda_0(7\pi/9);$$

(2) for $m = 1$ we have

$$\lambda_5^P(9) = \lambda_1(7\pi/9), \quad \lambda_6^P(9) = \lambda_1(5\pi/9),$$
$$\lambda_7^P(9) = \lambda_1(3\pi/9), \quad \lambda_8^P(9) = \lambda_1(\pi/9) < \lambda_9^P(9) = \lambda_1^P;$$

(3) for $m = 2$ we have

$$\lambda_{10}^P(9) = \lambda_2^P, \quad \lambda_{11}^P(9) = \lambda_2(\pi/9), \quad \lambda_{12}^P(9) = \lambda_2(3\pi/9),$$
$$\lambda_{13}^P(9) = \lambda_2(5\pi/9), \quad \lambda_{14}^P(9) = \lambda_2(7\pi/9);$$

(4) for $m = 3$ we have

$$\lambda_{15}^P(9) = \lambda_3(7\pi/9), \quad \lambda_{16}^P(9) = \lambda_3(5\pi/9),$$
$$\lambda_{17}^P(9) = \lambda_3(3\pi/9), \quad \lambda_{18}^P(9) = \lambda_3(\pi/9) < \lambda_{19}^P(9) = \lambda_3^P.$$

The next examples illustrate the semi-periodic case. For $S(2) = \Gamma(\frac{\pi}{2})$ the 1-1 correspondence is just the identity so we start with $S(3)$.

Example 5.5. $k=3$. For $S(3)=S(1)\cup\Gamma(\frac{\pi}{3})=\Gamma(\pi)\cup\Gamma(\frac{\pi}{3})$ and from Theorem 2.4 we get the inequalities:

$$\begin{aligned}\lambda_0(\pi/3)<\lambda_0(\pi)=\lambda_0^S\le\lambda_1^S=\lambda_1(\pi)<\lambda_1(\pi/3)<\lambda_2(\pi/3)<\lambda_2(\pi)=\lambda_2^S\\ \le\lambda_3^S=\lambda_3(\pi)<\lambda_3(\pi/3)<\lambda_4(\pi/3)<\lambda_4(\pi)=\lambda_4^S\le\lambda_5^S<\dots\end{aligned}$$

Hence $\lambda_0^S(3)=\lambda_0(\pi/3)$, $\lambda_1^S(3)=\lambda_0^S$, $\lambda_2^S(3)=\lambda_1^S$, $\lambda_4^S(3)=\lambda_2(\pi/3)$, $\dots$.

Example 5.6. $k=2s$, $s=4$. By Theorem 2.3 we have

$$S(8)=\Gamma(\pi/8)\cup\Gamma(3\pi/8)\cup\Gamma(5\pi/8)\cup\Gamma(7\pi/8).$$

By Theorem 2.4 we have the inequalities:

$$\begin{aligned}\lambda_0(\pi/8)&<\lambda_0(3\pi/8)<\lambda_0(5\pi/8)<\lambda_0(7\pi/8)\\ &<\lambda_1(7\pi/8)<\lambda_1(5\pi/8)<\lambda_1(3\pi/8)<\lambda_1(\pi/8)\\ &<\lambda_2(\pi/8)<\lambda_2(3\pi/8)<\lambda_2(5\pi/8)<\lambda_2(7\pi/8)\\ &<\lambda_3(7\pi/8)<\lambda_3(5\pi/8)<\lambda_3(3\pi/8)<\lambda_3(\pi/8)\\ &<\lambda_4(\pi/8)<\lambda_4(3\pi/8)<\lambda_4(5\pi/8)<\lambda_4(7\pi/8)<\dots\end{aligned}$$

From these inequalities and Theorem 4.2:

(1) for $m=0$ we have

$\lambda_0^S(k)=\lambda_0(\pi/k)$, $\lambda_1^S(k)=\lambda_0(3\pi/k)$, $\lambda_2^S(k)=\lambda_0(5\pi/k)$, $\lambda_3^S(k)=\lambda_0(7\pi/k)$;

(2) for $m=1$ we have

$\lambda_4^S(k)=\lambda_1(7\pi/k)$, $\lambda_5^S(k)=\lambda_1(5\pi/k)$, $\lambda_6^S(k)=\lambda_1(3\pi/k)$, $\lambda_7^S(k)=\lambda_0(\pi/k)$;

(3) for $m=2$ we have

$\lambda_8^S(k)=\lambda_2(\pi/k)$, $\lambda_9^S(k)=\lambda_2(3\pi/k)$, $\lambda_{10}^S(k)=\lambda_2(5\pi/k)$, $\lambda_{11}^S(k)=\lambda_2(7\pi/k)$;

(4) for $m=3$ we have

$\lambda_{12}^S(k)=\lambda_3(7\pi/k)$, $\lambda_{13}^S(k)=\lambda_3(5\pi/k)$, $\lambda_{14}^S(k)=\lambda_3(3\pi/k)$, $\lambda_{15}^S(k)=\lambda_3(\pi/k)$.

Example 5.7. $k=2s+1$, $s=4$. From Theorem 2.3 we have:

$$\begin{aligned}S(9)=&S(1)\cup\Gamma(\frac{\pi}{9})\cup\Gamma(\frac{3\pi}{9})\cup\Gamma(\frac{5\pi}{9})\cup\Gamma(\frac{7\pi}{9})\\ &=\Gamma(\pi)\cup\Gamma(\frac{\pi}{9})\cup\Gamma(\frac{3\pi}{9})\cup\Gamma(\frac{5\pi}{9})\cup\Gamma(\frac{7\pi}{9})\end{aligned}$$

This and Theorem 2.4 yields the inequalities:

$$\begin{aligned}\lambda_0(\pi/9)&<\lambda_0(3\pi/9)<\lambda_0(5\pi/9)<\lambda_0(7\pi/9)<\lambda_0(9\pi/9)=\lambda_0(\pi)\le\lambda_1(\pi)\\ &<\lambda_1(7\pi/9)<\lambda_1(5\pi/9)<\lambda_1(3\pi/9)<\lambda_1(1\pi/9)\\ &<\lambda_2(1\pi/9)<\lambda_2(3\pi/9)<\lambda_2(5\pi/9)<\lambda_2(7\pi/9)<\lambda_2(\pi)\le\lambda_3(\pi)\\ &<\lambda_3(7\pi/9)<\lambda_3(5\pi/9)<\lambda_3(3\pi/9)<\lambda_3(1\pi/9)\\ &<\lambda_4(1\pi/9)<\lambda_4(3\pi/9)<\lambda_4(5\pi/9)<\lambda_4(7\pi/9)<\lambda_4(\pi)\le\lambda_5(\pi)<\dots\end{aligned}$$

From these inequalities and Theorem 4.2:

(1) for $m=0$ we have

$$\begin{gathered}\lambda_0^S(9)=\lambda_0(\pi/9),\quad\lambda_1^S(9)=\lambda_0(3\pi/9),\quad\lambda_2^S(9)=\lambda_0(5\pi/9),\\ \lambda_3^S(9)=\lambda_0(7\pi/9),\quad\lambda_4^S(9)=\lambda_0^S;\end{gathered}$$

(2) $m = 1$:

$$\lambda_5^S(9) = \lambda_1^S, \quad \lambda_6^S(9) = \lambda_1(7\pi/9), \quad \lambda_7^S(9) = \lambda_1(5\pi/9),$$
$$\lambda_8^S(9) = \lambda_1(3\pi/9), \quad \lambda_9^S(9) = \lambda_1(\pi/9);$$

(3) for $m = 2$ we have

$$\lambda_{10}^S(9) = \lambda_2(\pi/9), \quad \lambda_{11}^S(9) = \lambda_2(3\pi/9), \quad \lambda_{12}^S(9) = \lambda_2(5\pi/9),$$
$$\lambda_{13}^S(9) = \lambda_2(7\pi/9), \quad \lambda_{14}^S(9) = \lambda_2^S;$$

(4) for $m = 3$ we have

$$\lambda_{15}^S(9) = \lambda_3^S, \quad \lambda_{16}^S(9) = \lambda_3(7\pi/9), \quad \lambda_{17}^S(9) = \lambda_3(5\pi/9),$$
$$\lambda_{18}^S(9) = \lambda_3(3\pi/9), \quad \lambda_{19}^S(9) = \lambda_3(\pi/9);$$

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