Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 265, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NONLOCAL PROBLEM WITH MOMENT CONDITIONS FOR HYPERBOLIC EQUATIONS

VOLODYMYR S. IL'KIV, ZINOVII M. NYTREBYCH, PETRO YA. PUKACH

Communicated by Ludmila S. Pulkina

Abstract. We investigate a problem with nonlocal integral moment conditions with respect to the time variable for partial differential equation with constant coefficients. We obtain necessary and sufficient conditions for the existence of solutions in the class of periodic functions with respect to the spatial variables. For studying the asymptotic properties of this problem, we use only the partial integration formula and the length of the interval of integration.

1. Introduction

Nonlocal conditions for partial differential equations are widely used in mathematical models of many physical, biological and other natural processes. In particular, measuring the weighted average values of the solution is interpreted by integral conditions, while the values at certain points are interpreted by local ones.

In general, the problems with nonlocal conditions are ill-posed in the sense of Hadamard, and are related to the small denominators, in which the Diophantine properties of the problem parameters show up. The metric approach to investigating nonlocal problems in the Sobolev scales and other scales of periodic, almost periodic functions, was used in [10, 16].

The results of studying the problems with integral conditions for partial differential equations are published in various papers (see e.g. [1, 2, 3, 6, 7, 8, 9, 11, 12, 13, 15, 17, 18]).

In this paper, we consider a problem with nonlocal integral moment conditions with respect to the time variable. We study the dependence of the problem solvability on the length of the integration interval in the Sobolev scale of periodic functions with respect to spatial variables.

The problem is formulated in the second section. The existence of a generalized solution is proved in the third section and the characteristic matrix structure with the solution existence is investigated in the fourth section. In the fifth section we examine some examples and indicate some conclusions.

2010 *Mathematics Subject Classification.* 35L35, 35B15, 56B30.
Key words and phrases. Boundary value problem; partial differential equation; integral conditions; moment-type conditions; small denominators.

Submitted January 28, 2017. Published October 20, 2017.

2. Formulation of the problem

In this section, we introduce the domain in which we consider the problem, the partial differential equation and the nonlocal moment conditions, the spaces of periodic functions, and give the definition of a solution as well.

Let Ω^p be a p-dimentional torus $(\mathbb{R}/2\pi\mathbb{Z})^p$, $Q_p = [0,T] \times \Omega^p$ be a cylinder, where $p \in \mathbb{N}$, $0 < T_0 \le T \le T_1 < +\infty$, and let $t \in [0,T]$, $x = (x_1, \dots, x_p) \in \Omega^p$, $\partial_t = \partial/\partial t$, $\partial_{x_j} = \partial/\partial x_j$, $\partial_x = (\partial_{x_1}, \dots, \partial_{x_p})$ and $\partial_x^s = \partial_{x_1}^{s_1} \cdots \partial_{x_p}^{s_p}$ for $s = (s_1, \dots, s_p) \in \mathbb{Z}_+^p$.

In the domain Q_p, we consider the following nonlocal problem in t variable:

$$L(\partial_t, \partial_x)u \equiv \partial_t^n u + \sum_{j=1}^n A_j(\partial_x)\partial_t^{n-j} u = 0, \quad n \ge 2, \tag{2.1}$$

$$\mathcal{M}(r_j; u) \equiv \int_0^T \overset{r_j}{t}\, u(t, \cdot)\, dt = \varphi_j, \quad j = 1, \dots, n, \tag{2.2}$$

where $A_j(\partial_x) = \sum_{|s| \le j} a_{js}\partial_x^s$ are differential expressions with complex coefficients a_{js}, the orders r_j of the moments $\mathcal{M}(r_j; u)$ of the solution $u = u(t,x)$ are non-negative real numbers, sorted ascending by $r_1 < \cdots < r_n$, and $\overset{r_j}{t} = t^{r_j}/r_j!$ for $j = 1, \dots, n$, and moreover, $r_j! = \Gamma(r_j + 1)$ is a factorial. The right-hand sides $\varphi_1, \dots, \varphi_n$ in conditions (2.2) are given and 2π-periodic functions.

Assume that there exist such positive numbers K, R, m, m_0, m_1, where $R \ge 1$, that the roots $\mu_j = \mu_j(k)$ of the algebraic equation

$$L(-\lambda, ik) \equiv L(i\tilde{k}\mu, ik) = 0, \quad \tilde{k} = \sqrt{1 + k_1^2 + \cdots + k_p^2}, \tag{2.3}$$

in the case $\tilde{k} \ge K$ have the following properties:

$$\left|e^{-i\tilde{k}\mu_j(k)T}\right| \le R, \quad 0 < m_0 \le |\mu_j(k)| \le m_1 < +\infty, \quad |\mu_\alpha(k) - \mu_\beta(k)| \ge m > 0. \tag{2.4}$$

Also we denote

$$\lambda_j = \lambda_j(k) = -i\tilde{k}\mu_j(k), \quad j = 1, \dots, n. \tag{2.5}$$

The numbers $m = m(\vec{a})$, $R = R(\vec{a})$ and $m_1 = m_1(\vec{a})$ exist for a strictly hyperbolic equation (m is the hyperbolicity constant), and for an arbitrary vector of coefficients $\vec{a} = (a_{js}; j = 1, \dots, n, |s| \le j)$ respectively. Additionally, in conditions (2.4), we only assume the existence of a positive number $m_0 = m_0(\vec{a})$ (the last condition could be weakened by multiplying the constant m_0 by the function $\tilde{k}^{-\gamma}$ of the vector k, where $0 < \gamma < 1$).

Let H be a space of 2π-periodic trigonometric polynomials (the space of test functions), and H' be its adjoint space of generalized function (formal trigonometric series).

Let $H_q = H_q(\Omega^p)$ be a Sobolev space of 2π-periodic in $x_1, \dots, x_p$ functions $v(x) = \sum_{k \in \mathbb{Z}^p} v_k e^{ik \cdot x}$, which is formed by complementing the H space by the norm

$$\|v; H_q\| = \Big(\sum_{k \in \mathbb{Z}^p} \tilde{k}^{2q}|v_k|^2\Big)^{1/2}, \quad k \cdot x = k_1x_1 + \cdots + k_px_p.$$

The embedding $H \subset H_q \subset H'$ of spaces H, H_q and H' is continuous for all $q \in \mathbb{R}$.

Denote by $H_q^n = H_q^n(Q_p)$ a Banach space of functions $u = u(t,x)$ such that $\partial_t^j u \in C([0,T]; H_{q-j})$ for $j = 1, \dots, n$ and $\|u; H_q^n\|^2 = \sum_{j=0}^n \max_{t \in [0,T]} \|\partial_t^j u(t,\cdot); H_{q-j}\|^2$.

Definition 2.1. An element $u \in C^n([0,T];H')$ is called a generalized solution of problem (2.1), (2.2) if it satisfies on time interval $(0,T)$ the differential equation (2.1) and the nonlocal conditions (2.2) in the space H'.

Definition 2.2. A generalized solution of problem (2.1), (2.2) is called a solution, if it belongs to the space H_q^n.

It follows from the definition 2.2 that the condition $\{\varphi_1,\dots,\varphi_n\} \subset H_q$ is *the necessary* condition of existence of solution u of problem (2.1), (2.2), for which the following estimation is true:

$$\|\varphi_j;H_q\| \le \overset{r_1+1}{T}\|u;H_q^n\|, \quad j=1,\dots,n.$$

The problem (2.1), (2.2) is considered in the scale $\{H_q\}_{q\in\mathbb{R}}$, i.e. φ_j and $u(t,\cdot)$ belong to this scale for all $j=1,\dots,n$ and $t\in[0,T]$, and is ill-posed in the sense of Hadamard [10, 16] in this scale (as well as in other scales).

3. Existence of a generalized solution

In this section, we are introduce notations, formulate and prove the theorem of existence of generalized solutions of problem (2.1), (2.2). Also, we give the representation of these solutions. Let us assume that

$$\Lambda_k = \operatorname{diag}\big(I_{n_1}\lambda_1 + N_{n_1},\dots,I_{n_l}\lambda_l + N_{n_l}\big),$$

where $\lambda_1\dots,\lambda_l$ are roots of equation (2.3) of respective multiplicities $n_1,\dots,n_l$ $(n_1+\dots+n_l=n)$; I_{n_j} is an n_j-th order identity matrix; N_{n_j} is a nilpotent matrix of the form $N_{n_j}=\begin{pmatrix} 0 & I_{n_j-1}\\ 0 & 0\end{pmatrix}$, $N_1=0$; $\mathbf{1}_k$ is a block row vector whose blocks are the first rows of matrices $I_{n_1},\dots,I_{n_l}$.

Then elements of vector

$$E_k(t) = T^{n-1}\mathbf{1}_k\Lambda_k^n e^{(T-t)\Lambda_k} \tag{3.1}$$

form a fundamental system of solutions of the differential equation

$$L(d/dt,ik)u_k = 0.$$

From formula (3.1), when $\tilde{k}\ge K$, we have $\mathbf{1}_k=(1,\dots,1)$, and

$$E_k(t) = \big(T^{n-1}\lambda_1^n e^{(T-t)\lambda_1},\dots,T^{n-1}\lambda_n^n e^{(T-t)\lambda_n}\big), \quad \Lambda_k = \operatorname{diag}(\lambda_1,\dots,\lambda_n).$$

Denote the characteristic matrices of problem (2.1), (2.2) by M_k, where $k\in\mathbb{Z}^p$, and M_k^- are pseudoinverse matrices to them [14, p. 428]. Then

$$M_k = \operatorname{col}\big(\mathcal{M}(r_1;E_k),\dots,\mathcal{M}(r_n;E_k)\big) \tag{3.2}$$

and $M_k^- = M_k^{-1}$ for a non-degenerate matrix M_k, where the moments $\mathcal{M}(\cdot;\cdot)$ are defined by formula (2.2).

Let the projectors P and Q act in the scale $\{H_q\}_{q\in\mathbb{R}}$ onto the vector-functions $v=\sum_{k\in\mathbb{Z}^p} v_k e^{ik\cdot x}$, whose components v_k belong to $\mathbb{C}^n$, as follows:

$$Pv = \sum_{k\in\mathbb{Z}^p} P_k v_k e^{ik\cdot x} \equiv \sum_{k:\ \det M_k=0} P_k v_k e^{ik\cdot x} + \sum_{k:\ \det M_k\ne0} v_k e^{ik\cdot x}, \tag{3.3}$$

$$Qv = \sum_{k\in\mathbb{Z}^p} Q_k v_k e^{ik\cdot x} \equiv \sum_{k:\ \det M_k=0} Q_k v_k e^{ik\cdot x} + \sum_{k:\ \det M_k\ne0} v_k e^{ik\cdot x}, \tag{3.4}$$

where $P_k = M_k^- M_k$ and $Q_k = M_k M_k^-$ are projectors in the $\mathbb{C}^n$ space.

Now we shall state the theorem of existence of generalized solution of problem (2.1), (2.2), which is true for an arbitrary vector of coefficients $\vec{a}$.

Theorem 3.1. *The generalized solution of problem* (2.1), (2.2) *exists if and only if the orthogonality condition* $(I-Q)\varphi = (I-Q)\operatorname{col}(\varphi_1,\dots,\varphi_n) = 0$ *is satisfied. One can be represented by the following formula:*

$$u = \sum_{k\in\mathbb{Z}^p} E_k(t) M_k^- \hat{\varphi}_k e^{ik\cdot x} + \sum_{k\in\mathbb{Z}^p} E_k(t)((I-P)U)_k e^{ik\cdot x}, \tag{3.5}$$

where $U = \operatorname{col}(U_1,\dots,U_n)$ *is an arbitrary vector whose components belong to the space* H', *and* $\hat{\varphi}_k$ *and* $((I-P)U)_k$ *are the Fourier coefficients of the vector-functions* φ *and* $(I-P)U$.

Proof. Since the solution of problem (2.1), (2.2) has the form $u = \sum_{k\in\mathbb{Z}^p} u_k e^{ik\cdot x}$, the function $u_k = u_k(t)$ is a solution of the problem

$$L(d/dt, ik)u_k = 0, \quad \mathcal{M}(r_j; u_k) = \varphi_{jk}, \quad j = 1,\dots,n. \tag{3.6}$$

If $C_k \in \mathbb{C}^n$ are arbitrary vectors, then, from the general solution $u_k = E_k(t)C_k$ of the equation $L(d/dt, ik)u_k = 0$ in case $M_k C_k = \hat{\varphi}_k$, we obtain the solution of problem (3.6).

It is known [14, p. 436], that the general solution of the latter system of linear algebraic equations is given by the formula $C_k = M_k^- \hat{\varphi}_k + (I-P_k)U_k$. In particular, in the case $\det M_k \neq 0$, it is the unique solution $C_k = M_k^{-1}\hat{\varphi}_k$. Substituting the calculated C_k value into the formula for u_k, and u_k into the formula for u, we obtain the solution (3.5). $\square$

Let us introduce in the space H' the projector $\Pi(\mathcal{Z})$, where $\mathcal{Z}$ is an arbitrary subset of $\mathbb{Z}^p$, acting onto the element $\varphi = \sum_{k\in\mathbb{Z}^p} \varphi_k e^{ik\cdot x}$ by the formula

$$\Pi(\mathcal{Z})\varphi = \sum_{k\in\mathcal{Z}} \varphi_k e^{ik\cdot x}.$$

If $\mathcal{Z}$ is a finite set, then the element $\Pi(\mathcal{Z})\varphi$ is a polynomial for each function φ from the space H', i.e. $H' \xrightarrow{\Pi(\mathcal{Z})} H$.

Let $\mathcal{K}_0 \equiv \mathcal{K}_0(T) = \{k \in \mathbb{Z}^p\colon \det M_k = 0\}$, and $\bar{\mathcal{K}}_0 = \mathbb{Z}^p \setminus \mathcal{K}_0$ be the complement to the set $\mathcal{K}_0$. Then, the formula (3.5) could be written as

$$\begin{aligned} u &\equiv \Pi(\mathcal{K}_0)u + \Pi(\bar{\mathcal{K}}_0)u \\ &= \sum_{k\in\mathcal{K}_0} E_k(t)(M_k^- \hat{\varphi}_k + (I-P_k)U_k)e^{ik\cdot x} + \sum_{k\in\bar{\mathcal{K}}_0} E_k(t)M_k^{-1}\hat{\varphi}_k e^{ik\cdot x}, \end{aligned} \tag{3.7}$$

which implies the following obvious consequences.

Corollary 3.2. *The null space of problem* (2.1), (2.2) *in the space* $C^n([0,T]; H')$ *consists of elements*

$$u = \sum_{k\in\mathcal{K}_0} E_k(t)(I-P_k)U_k e^{ik\cdot x}, \tag{3.8}$$

where U_k *are arbitrary vectors from* $\mathbb{C}^n$.

Corollary 3.3. *Problem* (2.1), (2.2) *is a Fredholm problem if and only if the set* $\mathcal{K}_0$ *is finite. In this case, the null space of the problem has a finite dimension, that is equal to* $\sum_{k\in\mathcal{K}_0} \operatorname{rank}(I-P_k)$, *where* $\operatorname{rank}(I-P_k)$ *is the rank of the matrix* $I-P_k$.

Corollary 3.4. *If $\mathcal{K}_0 = \emptyset$, i.e. $\det M_k \neq 0$ for all $k \in \mathbb{Z}^p$, then the generalized solution of problem (2.1), (2.2) is unique and has the form*

$$u = \sum_{k\in\mathbb{Z}^p} E_k(t) M_k^{-1} \hat{\varphi}_k e^{ik\cdot x}. \tag{3.9}$$

The inverse statement is also true: the uniqueness implies that the set $\mathcal{K}_0$ is empty.

4. Structure of characteristic matrices. Existence of solutions

For estimating the functions $u_k = u_k(t)$, let us investigate the structure of M_k matrices, determine the existence of inverse matrices M_k^{-1} and find their structure.

If $r_1 \geq n$, then integrating by parts n times allows us to write the formula

$$\mathcal{M}(r_j; E_k) = \mathcal{M}(r_j - n; E_k)\Lambda_k^{-n} - T^{n-1}\mathbf{1}_k \sum_{\alpha=0}^{n-1} \overset{r_j-\alpha}{T} \Lambda_k^{n-\alpha-1}, \quad j = 1,\dots,n.$$

In matrix form, denoted $\vec{\lambda}(k) = (\lambda_1,\dots,\lambda_n)$ with $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$, $\alpha \leq l$, $\beta \leq l$ $\big(l = l(k) \leq n\big)$, we obtain

$$M_k = M_{kn} - T^{n-1} W^\top \big[\overset{r_1}{T},\dots,\overset{r_n}{T}\big] JW\big(\vec{\lambda}(k)\big), \tag{4.1}$$

where $W\big[f_1,\dots,f_n\big]$ is the Wronski matrix $\big(f_j^{(i-1)}\big)_{i,j=1,\dots,n}$ of the system of functions f_j, $W(\vec{\lambda})$ is the Vandermonde matrix $\operatorname{col}\big(\mathbf{1}_k, \mathbf{1}_k\Lambda_k,\dots,\mathbf{1}_k\Lambda_k^{n-1}\big)$ with generators $\lambda_1,\dots,\lambda_l$ of multiplicities $n_1,\dots,n_l$ (if $l = n$, then $W(\vec{\lambda}) = (\lambda_j^{i-1})_{i,j=1,\dots,n}$), the antidiagonal matrix $J = (\delta_{i,n+1-j})_{i,j=1,\dots,n}$ is formed of Kronecker delta δ_{ij} symbols, $W^\top$ is a transpose matrix to W, and the matrix

$$M_{kn} \equiv \operatorname{col}\big(\mathcal{M}(r_1 - n; E_k),\dots,\mathcal{M}(r_n - n; E_k)\big)\Lambda_k^{-n}$$

has following form

$$M_{kn} = T^{n-1} \operatorname{col}\big(\mathbf{1}_k\mathcal{M}\big(r_1 - n; e^{(T-t)\Lambda_1}\big),\dots,\mathbf{1}_k\mathcal{M}\big(r_n - n; e^{(T-t)\Lambda_n}\big)\big).$$

Let $S_1 = \big(\big\{{i-1 \atop j-1}\big\}\big)_{i,j=1,\dots,n}$ and $S_2 = \big((-1)^{i-j}\big[{i-1 \atop j-1}\big]\big)_{i,j=1,\dots,n}$ be the matrices of Stirling numbers of the first $\big\{{i \atop j}\big\}$ and the second $\big[{i \atop j}\big]$ kind respectively. Thus, $S_1S_2 = I_n$, and from the formula (6.13) in [4, p. 263] we have

$$W\big[\overset{r_1}{T},\dots,\overset{r_n}{T}\big] = \operatorname{diag}\big(T^{1-j}\big)_{j=1,\dots,n} S_2 W(\vec{r}) \operatorname{diag}\big(\overset{r_j}{T}\big)_{j=1,\dots,n}.$$

Based on the latter, and (4.1), we obtain the factorization

$$\operatorname{diag}\big(1/\overset{r_j}{T}\big)_{j=1,\dots,n} M_k = -W^\top(\vec{r}) S_2^\top (I_n - H) JW\big(T\vec{\lambda}(k)\big), \tag{4.2}$$

where we denote $W^{-\top} = (W^\top)^{-1} = (W^{-1})^\top$ and

$$H = S_1^\top W^{-\top}(\vec{r}) \operatorname{diag}\big(1/\overset{r_j}{T}\big)_{j=1,\dots,n} M_{kn} W^{-1}\big(T\vec{\lambda}(k)\big) J. \tag{4.3}$$

For the matrix $B = (b_{ij})_{i,j=1,\dots,n}$, denote by $\|B\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^n |b_{ij}|$ the matrix ∞-norm [5, p. 108, 109], then $\|B^\top\|_\infty \leq n\|B\|_\infty$.

Based on the formula $\left\{{i \atop j}\right\} = j\left\{{i-1 \atop j}\right\} + \left\{{i-1 \atop j-1}\right\}$ from [20, p. 259], we obtain for matrix S_1 the estimation $\|S_1\|_\infty \le n!$; and basing on formula (22.3) from [5, p. 417], we estimate the norm of the Vandermonde matrix:

$$\|W^{-1}(\vec{r})\|_\infty \le \max_{i=1,\dots,n} \prod_{j\neq i,\, j=1}^{n} \frac{1+r_j}{|r_i - r_j|}.$$

Thus, the multipliers of the matrix H in the case when $\tilde{k} \ge K$, and the condition (2.4) is true, have the following estimates:

$$\begin{aligned}
&\|S_1^\top\|_\infty \le n\cdot n!,\\
&\|W^{-\top}(\vec{r})\|_\infty \le n\|W^{-1}(\vec{r})\|_\infty < \prod_{j=1}^{n}(1+r_j) \equiv R_1 < \infty,\\
&\|W^{-1}\big(T\vec{\lambda}(k)\big)\|_\infty \le \|W^{-1}\big(-iT\vec{\mu}(k)\big)\|_\infty \cdot \|\operatorname{diag}(\tilde{k}^{1-j})_{j=1,\dots,n}\|_\infty\\
&= \|W^{-1}\big(-iT\vec{\mu}(k)\big)\|_\infty < \Big(\frac{1+m_1T}{mT}\Big)^{n-1}\\
&\le \Big(\frac{1+m_1T_0}{mT_0}\Big)^{n-1} \equiv R_2 < \infty.
\end{aligned}$$

Taking into account $\|J\|_\infty = 1$, by formula (4.3), we establish the inequality

$$\|H\|_\infty \le n\cdot n! R_1R_2 \|\operatorname{diag}\big(1/\overset{r_j}{T}\big)_{j=1,\dots,n} W_{kn}\|_\infty\,.$$

Taking into account $\big|\mathcal{M}\big(r_j; e^{(T-t)\lambda_\alpha}\big)\big| \le R\,\overset{r_j+1}{T}$, we estimate the elements of $\mathcal{M}\big(r_j - n; e^{(T-t)\lambda_\alpha}\big)/\overset{r_j}{T}$, placed in the j-th row of the matrix $\operatorname{diag}\big(1/\overset{r_j}{T}\big)_{j=1,\dots,n} W_{kn}$:

$$\big|\mathcal{M}\big(r_j - n; e^{(T-t)\lambda_\alpha}\big)/\overset{r_j}{T}\big| \le \frac{\mathcal{M}\big(r_j-n-1;1\big) + R\,\overset{r_j-n}{T}}{\tilde{k}|\mu_\alpha|\overset{r_j}{T}} \le \frac{2R\tilde{k}^{-1}r_j!}{m_0T_0^n(r_j-n)!}.$$

From this, we obtain

$$\|\operatorname{diag}\big(1/\overset{r_j}{T}\big)_{j=1,\dots,n} M_{kn}\|_\infty \le \frac{2nR\tilde{k}^{-1}r_n!}{m_0T_0^n(r_n-n)!} \equiv 2n\tilde{k}^{-1}RR_3 < \infty.$$

Therefore, if $k \in \mathcal{K}_1$, where $\bar{\mathcal{K}}_1$ is a finite set, and

$$\mathcal{K}_1 \equiv \big\{k \in \mathbb{Z}^p \colon \tilde{k} \ge K_1 = \max\big(K, 4n^2n!RR_1R_2R_3\big)\big\}, \tag{4.4}$$

then it holds the inequality

$$\|H\|_\infty \le 2n^2\tilde{k}^{-1}n!RR_1R_2R_3 \le 1/2,$$

thus for such vectors k there exist the matrices $(I_n - H)^{-1}$ and M_k^{-1}, in particular, $\|(I_n-H)^{-1}\|_\infty \le 2$ and

$$\begin{aligned}
&\|\big(\operatorname{diag}\big(1/\overset{r_j}{T}\big)_{j=1,\dots,n} M_k\big)^{-1}\|_\infty\\
&= \|W\big(T\vec{\lambda}(k)\big)^{-1} J (I_n-H)^{-1} S_1^\top W^{-\top}(\vec{r})\|_\infty\\
&\le \|W\big(T\vec{\lambda}(k)\big)^{-1}\|_\infty \|(I_n-H)^{-1}\|_\infty \|S_1^\top\|_\infty \|W^{-\top}(\vec{r})\|_\infty\\
&\le 2n^2n!R_1R_2.
\end{aligned} \tag{4.5}$$

Theorem 4.1. *Let $\{\varphi_1,\dots,\varphi_n\}\subset H_{q+n}$, the fixed vector $\vec{a}$ of coefficients of differential equation (2.1) satisfy conditions (2.4), and $r_1\ge n$. Denote $\mathcal{T}_0$ be a set of numbers $T\in[T_0,T_1]$, for which $\det M_k=0$ at least for one $k\in\bar{\mathcal{K}}_1$, where $\mathcal{K}_1$ is defined by (4.4). Then, $\mathcal{T}_0$ is a finite set, and for each $T\in[T_0,T_1]\setminus\mathcal{T}_0$ in the space H^n_q there exist a unique solution (3.9) of problem (2.1), (2.2), and for each $T\in\mathcal{T}_0$ if $(I-Q)\varphi=0$, then exist a unique, accurate within the polynomial $\Pi(\bar{\mathcal{K}}_1)u$, solution (3.7), where*

$$\begin{aligned}\Pi(\bar{\mathcal{K}}_1)u=&\sum_{k\in\mathcal{K}_0(T)}E_k(t)(M_k^-\hat{\varphi}_k+(I-P_k)U_k)e^{ik\cdot x}\\&+\sum_{k\in\bar{\mathcal{K}}_1\setminus\mathcal{K}_0(T)}E_k(t)M_k^{-1}\hat{\varphi}_ke^{ik\cdot x}.\end{aligned}\tag{4.6}$$

For these solutions, the following estimate holds:

$$\|\Pi(\mathcal{K}_1)u;H^n_q\|^2\le n\frac{R_4^2}{T^2}\sum_{\alpha=1}^n\frac{(r_\alpha!)^2}{T^{2(r_\alpha-n)}}\|\Pi(\mathcal{K}_1)\varphi_\alpha;H_{q+n}\|^2,\tag{4.7}$$

where $R_4=2n^2n!\frac{m_1^nRR_1R_2}{\left(\sum_{j=0}^n m_1^{2j}\right)^{-1/2}}$, $R_1=\prod_{j=1}^n(1+r_j)$ $R_2=\left(m_1/m+1/mT_0\right)^{n-1}$.

Proof. The finiteness of the set $\mathcal{T}_0$ follows from the finiteness of the set of zeros of the entire function $\det M_k\equiv\det M_k(T)$ on the finite interval $[T_0,T_1]$, where $k\in\bar{\mathcal{K}}_1$, and from the finiteness of the set $\bar{\mathcal{K}}_1$.

If $T\in[T_0,T_1]\setminus\mathcal{T}_0$, then $\mathcal{K}_0=\emptyset$ and, by corollary 3.4, there exist a unique solution of the problem (2.1), (2.2) of the form (3.9) in the space $C^n([0,T];H')$, and

$$\max_{t\in[0,T]}|u_k^{(\alpha)}(t)|^2\le\max_{t\in[0,T]}\|E_k^{(\alpha)}(t)\|^2\|\left(\operatorname{diag}\left(1/\overset{r_j}{T}\right)_{j=1,\dots,n}M_k\right)^{-1}\|^2\sum_{\beta=1}^n\left|\varphi_{\beta k}/\overset{r_\beta}{T}\right|^2,$$

where, for the vectors $k\in\mathcal{K}_1$ we have

$$\begin{aligned}\max_{t\in[0,T]}\|E_k^{(\alpha)}(t)\|^2&=T^{2(n-1)}\max_{t\in[0,T]}\sum_{j=1}^n|\lambda_j^2|^{n+\alpha}\left|e^{2(T-t)\lambda_j}\right|\\&\le nT^{2(n-1)}R^2(\tilde{k}m_1)^{2(n+\alpha)}.\end{aligned}$$

Based on these estimates, and the estimate (4.5), we conclude the inequalities

$$\tilde{k}^{2(q-j)}\max_{t\in[0,T]}|u_k^{(j)}(t)|^2\le n\left(2n^2n!\,m_1^{n+j}RR_1R_2\right)^2\sum_{\alpha=1}^n\left|T^{n-1}\tilde{k}^{q+n}\varphi_{\alpha k}/\overset{r_\alpha}{T}\right|^2$$

for $j=0,1,\dots,n$, which imply formula (4.7).

In case when $T\in\mathcal{T}_0$, then $\mathcal{K}_0$ is a finite non-empty set, and problem (2.1), (2.2) has a non-trivial finite-dimensional null space described by formula (4.6), where we assume $\varphi=0$. Formula (4.7) follows of formula (3.7). □

5. Examples

Let $n=2$ and problem (2.1), (2.2) be as follows:

$$\partial_tu=\Delta u-u,\quad\mathcal{M}(r_1;u)=\varphi_1,\quad\mathcal{M}(r_2;u)=\varphi_2,\tag{5.1}$$

where $\Delta=\sum_{j=1}^p\partial^2_{x_j}$ is the Laplace operator.

Then $\mu_1(k) = -\mu_2(k) = 1$ and the assumption (2.4) is true for such values $K = R = m_0 = m_1 = 1$ and $m = 2$. Provided $r_2 > r_1 \geq 2$, theorem (4.1) holds.

Since $R_1 = (r_1+1)(r_2+1)$, $R_2 = 1/2 + 1/2T_0$, $R_3 = (r_2-1)r_2/T_0^2$, we conclude that

$$K_1 = \max\Big(1, 16(r_1+1)(r_2^2-1)r_2\frac{1+T_0}{T_0^3}\Big) \quad \text{and} \quad R_4 = 8\sqrt{3}(r_1+1)(r_2+1)\frac{1+T_0}{T_0}$$

in formulas (4.4) and (4.7) respectively.

The example of another (ill-posed) problem is problem (5.1), in which the vector of moment orders does not meet the condition $r_1 \geq 2$ and is defined by the formula $(r_1, r_2) = (0, 1)$.

The characteristic matrix M_k is determined by the formula

$$M_k = 2iT\begin{pmatrix} i\tilde{k}e^{-i\theta_k}\sin\theta_k & i\tilde{k}e^{i\theta_k}\sin\theta_k \\ e^{-i\theta_k}\sin\theta_k - \theta_k & \theta_k - e^{i\theta_k}\sin\theta_k \end{pmatrix},$$

moreover $\det M_k = i8\tilde{k}T^2(\sin\theta_k - \theta_k\cos\theta_k)\sin\theta_k$, where $\theta_k = \tilde{k}T/2$.

Let $\varphi_1 = 0$, $\varphi_2(x) = \sum_{k\in\mathbb{Z}^p}\varphi_{2k}e^{ik\cdot x}$ and the following condition holds:

$$\sin\theta_k\big(\sin\theta_k - \theta_k\cos\theta_k\big) \neq 0, \quad k \in \mathbb{Z}^p,$$

then this problem has a unique generalized solution

$$u = \sum_{k\in\mathbb{Z}^p}\frac{\sin\tilde{k}(t-T/2)}{\sin\tilde{k}T/2 - (\tilde{k}T/2)\cos\tilde{k}T/2}\frac{\tilde{k}^2}{2}\varphi_{2k}\, e^{ix\cdot k}.$$

The subsequences of denominators $\sin\tilde{k}T/2 - (\tilde{k}T/2)\cos\tilde{k}T/2$ of the solution tend to zero *ultrafast* for certain values of T, so for such values of T the solution does not belong to any space from the scale $\{H_q^n(Q_p)\}_{q\in\mathbb{R}}$, or another given scale.

Conclusions. We establish the conditions of unique solvability of the problem with nonlocal integral conditions in the form of moments for hyperbolic type partial differential equations in the space of generalized periodic functions and in the scale of Sobolev spaces of periodic functions with respect to spatial variables. This problem is ill-posed in the sense of Hadamard in case of small values of the moment orders, and, in the case when the moment orders are greater than the order of differential equation, the problem is well-posed with loss of derivatives. A similar result was previously obtained by the authors for the problem with integral boundary conditions for the system of Lame equations in the spaces of almost periodic function with respect to spatial variables.

We also investigate the pattern of dependence of the solution norm on the problem parameters. We also consider the issue of Fredholmity of the problem, and also determine the form of elements of its null space.

References

[1] Avalishvili, G.; Avalishvili, M.; Gordeziani, D.; *On integral nonlocal boundary value problems for some partial differential equations*, Bull. Georg. Nation. Acad. Sec., 5 (2011), no 11, 31–37.

[2] Fardigola, L. V.; *Test for propriety in a layer of a boundary problem with integral condition*, Ukr. Math. Journ., 42 (1990), no 11, 1388–1394.

[3] Fardigola, L. V.; *An integral boundary-value problem in a layer for a system of linear partial differential equations*, Sbornik Math, 53 (1995), no 6, 1671–1692.

[4] Graham, Ronald L.; Knuth, Donald E.; Patashnik, Oren; *Concrete mathematics: a foundation for computer science* 2nd ed. Reading, Mass.: Addison-Wesley, 1994. - xiii, 657 p.

[5] Higham, Nicholas J.; *Accuracy and Stability of Numerical Algorithms.* Second edition, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2002. xxx+680 pp.

[6] Il'kiv, V. S.; *Problem with integral conditions for the system of partial differential equations with variable coefficients*, Visnyk of Lviv State Polytechnic University. Series of Appl. Math., 364 (1999), 318–323.

[7] Il'kiv, V. S.; Maherovska, T. V.; *The problem with integral conditions for second order partial differential equation*, Journal of Lviv Polytechnic National University. Physical and mathematical sciences, 625 (2008), 12–19.

[8] Il'kiv, V. S.; Nytrebych, Z. M.; Pukach, P. Ya.; *A problem with integral boundary conditions for a Lamé equations system in almost periodic function spaces*, Bucovinian Math. Journ., 3 (2015), no 2, 27-41.

[9] Il'kiv, V. S.; Nytrebych, Z. M.; Pukach, P. Ya.; *Boundary-value problem with integral conditions for a system of Lamé equations in the space of almost periodic function*, Electron. J. Differential. Equations, 2016 (2016), no 304, 1-12.

[10] Il'kiv, V. S.; Ptashnyk, B. I.; *Problems for partial differential equations with nonlocal conditions. Metric approach to the problem of small denominators*, Ukr. Math. Journ., 58 (2006), no 12, 1847-1875.

[11] Kalenyuk, P. I.; Il'kiv, V. S.; Nytrebych, Z. M.; Kohut, I. V.; *Unique solvability of the problem with integral conditions for partial differential equation of the second order in time*, Journal of Lviv Polytechnic National University. Physical and mathematical sciences, 768 (2013), 5–11.

[12] Kuz, A. M.; Ptashnyk, B. I.; *A problem with integral conditions with respect to time for Gårding hyperbolic equations*, Ukr. Math. Journ, 65 (2013), no 2, 277-293.

[13] Kuz, A. M.; Ptashnyk, B. I.; *A problem with integral conditions with respect to time for a system of equations of the dynamic elasticity theory*, Journal of Mathematical Sciences, 208 (2015), no 3, 310-326.

[14] P. Lancaster, M. Tismenetsky; *The Theory of Matrices*, 2nd ed., Academic Press, Orlando, FL, 1985.

[15] Medvid', O. M.; Symotyuk, M. M.; *Problem with integral conditions for linear systems of partial differential equations*, Matematychni Methody ta Fizyko-mekhanichni Polya, 50 (2007), no 1., P. 32–39.

[16] Ptashnyk, B. I.; Il'kiv, V. S.; Kmit', I. Ya.; Polishchuk, V .M.; *Nonlocal boundary value problems for partial differential equations.*, Kyiv.: Naukova Dumka, 2002. – 416 p.

[17] Pulkina, L. S.; *A nonlocal problem with integral conditions for a hyperbolic equation*, Diff. Eq., 40 (2004), no 7, 947–953.

[18] Symotyuk, M. M.; Savka, I. Ya.; *Initial-nonlocal boundary value problems for factorized partial differential equations*, Journal of Lviv Polytechnic National University. Physical and mathematical sciences, 768 (2013), 19-25.

Volodymyr S. Il'kiv
Department of Mathematics, Lviv Polytechnyc National University, 12 Bandera str., Lviv 79013, Ukraine
E-mail address: ilkivv@i.ua

Zinovii M. Nytrebych
Department of Mathematics, Lviv Polytechnyc National University, 12 Bandera str., Lviv 79013, Ukraine
E-mail address: znytrebych@gmail.com

Petro Ya. Pukach
Department of Mathematics, Lviv Polytechnyc National University, 12 Bandera str., Lviv 79013, Ukraine
E-mail address: ppukach@gmail.com