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CHARACTERIZATION OF SOLUTIONS TO EQUATIONS INVOLVING THE $p(x)$-LAPLACE OPERATOR

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Abstract. In this article we study two problems, a nonlinear eigenvalue problem involving the $p(x)$-Laplacian and a subcritical boundary value problem for the same operator. We work on the variable exponent Sobolev spaces and use one of the variants of the Mountain-Pass Lemma.

1. Introduction

In the previous few decades, variable exponent Sobolev spaces attracted a lot of interest in the study of the partial differential equations. Problems involving the $p(x)$-Laplace operator such as

$$\Delta_{p(x)} := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

where p is a continuous nonconstant function, were intensely studied. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p := \operatorname{div}(|\nabla u|^{p-2})\nabla u)$, where $p > 1$ is a real constant. Due to the fact that the $p(x)$-Laplacian is nonhomogeneous, it possess more complicated nonlinearities than the p-Laplace operator. For more details we refer to [1, 2, 5, 6, 10, 11, 12, 14, 18, 20].

The variable exponent Sobolev spaces are used to model various phenomenona which are the image restoration and the modeling of the electrorheoleogical and thermorheological fluids. The first major discovery on the electrorheological fluids (or smart fluids) was in 1949, known as the Winslow effect, and it describes the behavior of certain fluids that become solids or quasi-solids when they are subjected to an electric field. Electrorheological fluids have been used in robotics and space technology. The experimental research has been mainly in the United States, for instance in NASA laboratories.

In this article we establish two results. The first one proves an alternative for a nonlinear eigenvalue problem involving the $p(x)$-Laplacian. Several ideas developed in the study of the spectrum of such general operators in divergence form are developed by Mihăilescu, Rădulescu, Repovš in [16], Molica Bisci, Repovš in [17], and Stăncuţ, Stîrcu in [30]. In the second part we study an existence result of a subcritical boundary value problem for the same operator. To prove our first result

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we use a mountain pass lemma on the product space $W_0^{1,p(\cdot)}(\Omega)\times\mathbb{R}$, considering a special hyperplane which is intended to separating surface instead of a sphere [18, 26]. The result obtained in the second problem is based on a special version of the mountain pass lemma of Ambrosetti-Rabinowitz [19]. For more details about the Mountain-Pass Lemma we refer to [5, 23, 24].

This article is organized as follows. In the next section we make a brief introduction of the variable exponent Sobolev spaces that is natural to look for weak solutions of this kind of problems. In section 3 we establish the main results concerning our first problem and we prove them. Finally, section 4 is dedicated to the study of the second problem of this paper which implies the $p(x)$-Laplace operator.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$. We define

$$C_+(\Omega)=\big\{p\in C(\overline{\Omega}):\min_{x\in\Omega}p(x)>1\big\}$$

and for any continuous function $p:\overline{\Omega}\to(1,\infty)$, denote

$$p^-=\inf_{x\in\Omega}p(x)\quad\text{and}\quad p^+=\sup_{x\in\Omega}p(x).$$

For any $p\in C_+(\overline{\Omega})$ we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega)=\big\{u:\Omega\to\mathbb{R}\text{ a measurable function}:\int_\Omega|u|^{p(x)}\,dx<\infty\big\}.$$

Equipped with the Luxemburg norm

$$|u|_{p(x)}=\inf\big\{\mu>0:\int_\Omega\big|\frac{u(x)}{\mu}\big|^{p(x)}\,dx\le1\big\},$$

$L^{p(x)}(\Omega)$ becomes a Banach space.

If $p(x)=p\equiv$constant for every $x\in\Omega$, then the $L^{p(x)}(\Omega)$ space is reduced to the classic Lebesgue space $L^p(\Omega)$ and the Luxemburg norm becomes the standard norm in $L^p(\Omega)$, $\|u\|_{L^p}=\big(\int_\Omega|u(x)|^p\,dx\big)^{1/p}$.

For $1<p^-\le p^+<\infty$, $L^{p(x)}(\Omega)$ is a reflexive uniformly convex Banach space, and for any measurable bounded exponent p, the $L^{p(x)}(\Omega)$ space is separable.

If p_1 and p_2 are two variable exponents such that $p_1(x)\le p_2(x)$ almost everywhere in Ω, with $|\Omega|<\infty$, then there exists a continuous embedding

$$L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$$

whose norm does not exceed $|\Omega|+1$.

We define the conjugate variable exponent $p':\overline{\Omega}\to(1,\infty)$, satisfying $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$, for every $x\in\overline{\Omega}$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of the $L^{p(x)}(\Omega)$.

If $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$ then the Hölder type inequality holds:

$$\Big|\int_\Omega uv\,dx\Big|\le\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)}\le2|u|_{p(x)}|v|_{p'(x)}.\tag{2.1}$$

The modular of the $L^{p(x)}(\Omega)$ space, defined by the mapping $\rho_{p(x)}:L^{p(x)}(\Omega)\to\mathbb{R}$,

$$\rho_{p(x)}(u)=\int_\Omega|u(x)|^{p(x)}\,dx,$$

has an important role in manipulating the generalized Lebesgue spaces. If $p(x) = p \equiv$constant for every $x \in \Omega$, then the modular $\rho_{p(x)}(u)$ becomes $\|u\|_{L^p}^p$.

If $p(x) \not\equiv$constant in Ω and $u, u_n \in L^{p(x)}(\Omega)$ then the following relations hold:

$$|u|_{p(x)} < 1 \;\Rightarrow\; |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$|u|_{p(x)} > 1 \;\Rightarrow\; |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} = 1 \;\Rightarrow\; \rho_{p(x)}(u) = 1, \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \;\Leftrightarrow\; \rho_{p(x)}(u_n - u) \to 0. \tag{2.5}$$

For more details about these variable exponent Lebesgue spaces see [8, 15, 22].

Finally, we define the *variable exponent Sobolev space* $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\big\}$$

equipped with the equivalent norms

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

$$\|u\| = \inf\big\{\mu > 0 : \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\mu}\Big|^{p(x)} + \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\Big)\,dx \le 1\Big\}.$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$ or

$$W_0^{1,p(x)}(\Omega) = \big\{u : u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega)\big\}.$$

If $p^- > 1$, the function spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive, uniformly convex Banach spaces. Furthermore, for any measurable bounded exponent p, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable.

For the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(x)}(\Omega)$ we consider $p \in C_+(\overline{\Omega})$ to be logarithmic Hölder continuous, so there exists $M > 0$ such that

$$|p(x) - p(y)| \le \frac{-M}{\log(|x-y|)}, \quad \text{for every } x, y \in \Omega \text{ with } |x-y| \le \frac{1}{2}.$$

Moreover, if Ω is bounded and p is global logarithmic Hölder continuous, which means, there exist $C_1, C_2 > 0$ two constants and p_∞ a real constant such that

$$|p(x) - p(y)| \le \frac{C_1}{\log\big(e + \frac{1}{|x-y|}\big)}, \quad \text{for every } x, y \in \Omega,$$

$$|p(x) - p_\infty| \le \frac{C_2}{\log(e + |x|)}, \quad \text{for every } x \in \Omega$$

then, on the space $W^{1,p(x)}(\Omega)$ we have the Poincaré type inequality, so, there exists a constant $C > 0$ such that

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{2.6}$$

If $\Omega \subset \mathbb{R}^N$ is a bounded domain and p is global log-Hölder continuous on $W_0^{1,p(x)}(\Omega)$, we can work with the norm $|\nabla u|_{p(x)}$ equivalent with $\|u\|_{p(x)}$.

As well, we remark that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for every $x \in \overline{\Omega}$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$, the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact and continuous.

We define the *modular* of the space $W^{1,p(x)}(\Omega)$ as the mapping $\varrho : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varrho_{p(x)}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx.$$

Then, if u, $(u_n) \in W^{1,p(x)}(\Omega)$, the following relations hold:

$$\|u\|_{p(x)} < 1 \ \Rightarrow\ \|u\|_{p(x)}^{p^+} \le \varrho_{p(x)}(u) \le \|u\|_{p(x)}^{p^-}, \tag{2.7}$$

$$\|u\|_{p(x)} > 1 \ \Rightarrow\ \|u\|_{p(x)}^{p^-} \le \varrho_{p(x)}(u) \le \|u\|_{p(x)}^{p^+}, \tag{2.8}$$

$$\|u_n - u\|_{p(x)} \to 0 \ \Leftrightarrow\ \varrho_{p(x)}(u_n - u) \to 0. \tag{2.9}$$

For more properties about these spaces we refer [3, 9, 13, 25, 27, 28, 29]. We note that, for simplicity, throughout this paper we use $\|\cdot\|$ instead of $\|\cdot\|_{W_0^{1,p(x)}}$.

3. A nonlinear eigenvalue problem involving the $p(x)$-Laplacian

Throughout this paper we assume that p satisfies the following properties:

$$\begin{gathered} p \in C_+(\overline{\Omega}), \\ 1 < p^- \le p(x) \le p^+ < \infty, \\ p \text{ is global log-Hölder continuous.} \end{gathered} \tag{3.1}$$

Let Ω be a bounded domain in $\mathbb{R}^N$. In this section we are concerned in the study of the following nonlinear eigenvalue problem involving the $p(x)$-Laplacian

$$\begin{gathered} -\Delta_{p(x)} u = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ 0 < \lambda \le a, \end{gathered} \tag{3.2}$$

with constraints on eigenvalues, where a is a positive constant and the function f satisfies the following conditions

- (H1) f is a measurable function in $x \in \Omega$ and continuous in $u \in \mathbb{R}$, with $f(x,0) \ne 0$ on a subset of Ω (where $|\Omega| > 0$); then, f is a *Carathéodory* function;
- (H2) $|f(x,u)| \le c_1 + c_2|u|^{q(x)-1}$ for almost everywhere in Ω and all $u \in \mathbb{R}$, where c_1 and c_2 are two positive constants, $q \in C_+(\overline{\Omega})$ and $1 < p^- \le p(x) \le p^+ < q^- \le q(x) \le q^+ < p^*(x)$, where
$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \ge N. \end{cases}$$
- (H3) for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$, there exist $b_1 \ge 0$ and $b_2 \ge 0$ two constants, β a continuous function and ν a constant with $1 \le \beta(x) < p(x) < \nu$ such that
$$f(x,u)u - \nu \int_0^u f(x,t)dt \ge -b_1 - b_2|u|^{\beta(x)}.$$

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a *weak solution* of problem (3.2) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx - \lambda \int_\Omega f(x,u)\varphi\, dx = 0, \quad \text{for all } \varphi \in W_0^{1,p(x)}(\Omega).$$

Remark 3.1. If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous, $p^+ < N$ and $p(x) \le q(x) \le p^*(x)$ for every $x \in \Omega$, then there exists a continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see [28]). Thus, there exists a positive constant $C > 0$ such that

$$|u|_{q(x)} \le C\|u\|_{W_0^{1,p(x)}}, \quad \text{for any } u \in W_0^{1,p(x)}(\Omega). \tag{3.3}$$

For using them later, we denote:

$$a_1 = 2c_1|1|_{q'(x)} \quad \text{and} \quad a_2 = C\left(2c_1|1|_{q'(x)} + c_2(q^-)^{-1}\right). \tag{3.4}$$

We first state a version of the Mountain-Pass Theorem by Ambrosetti and Rabinowitz.

Lemma 3.2 ([18])**.** *Let X be a Banach space and let $J \in C^1(X \times \mathbb{R}, \mathbb{R})$ be a functional satisfying the hypotheses:*

(i) there exist $\rho > 0$ and $\alpha > 0$ two constants such that $J(v, \rho) \ge \alpha$, for every $v \in X$;

(ii) there exists $r > \rho$ with $J(0, 0) = J(0, r) = 0$. Then we have a critical value of J, denoted by

$$c := \inf_{g\in\Gamma} \max_{0\le\tau\le1} J(g(\tau)),$$

where

$$\Gamma = \{g \in C([0, 1]), X \times \mathbb{R}); g(0) = (0, 0), g(1) = (0, r)\}$$

and

$$c \ge \inf_{v\in X} J(v, \rho) \ge \alpha > 0.$$

Now, we give our result concerning the nonlinear eigenvalue problem (3.2).

Theorem 3.3. *Suppose that relation* (3.1) *holds and the hypotheses* (H1)–(H3) *are satisfied by the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that, for some constants $0 < \rho < r$, $\sigma > 0$, the following relations hold:*

(1) $\gamma(0) = \gamma(r) = 0$;

(2) $\gamma(\rho) = \dfrac{a_1 + a_2}{\sigma + 1}$;

(3) $\lim_{|t|\to\infty} \gamma(t) = +\infty$;

(4) $\gamma'(t) < 0$ *if and only if* $t < 0$ *or* $\rho < t < r$.

Then, for every $a > 0$, the one the following alternatives holds:

(a) *for the problem* (3.2)*, $a > 0$ is an eigenvalue with the corresponding eigenfunction $u \in W_0^{1,p(x)}(\Omega)$ established by*

$$\alpha \le -\int_\Omega \int_0^{u(x)} f(x, t)\, dt\, dx + \frac{1}{a}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx \le a_1 + \alpha$$

or

(b) *one can state $z > 0$ a number which satisfies*

$$\rho < z < r \tag{3.5}$$

and determines by means of the following relations an eigensolution $(u, \lambda) \in W_0^{1,p(x)}(\Omega) \times (0, a]$ of the problem (3.2)*:*

$$\|u\| = |z|^{-\sigma/q^-}\left(-\gamma'(z)\right)^{1/q^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right)^{-1/q^-}, \tag{3.6}$$

$$\lambda^{-1} = z\big(-\gamma'(z)\big)\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)^{-1} + a^{-1}, \tag{3.7}$$

$$\begin{aligned} \alpha \le z^{\sigma+1}\|u\|^{q^-} \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + (\sigma+1)\gamma(z) \\ - \int_\Omega \int_0^{u(x)} f(x,t)\,dt\; dx + \frac{1}{a}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \le a_1 + \alpha. \end{aligned} \tag{3.8}$$

Proof. Our purpose is to establish problem (3.2) in terms of Lemma 3.2. Therefore, we set a C^1 functional $J : W_0^{1,p(x)}(\Omega) \times \mathbb{R} \to \mathbb{R}$ associated to our problem, defined by

$$\begin{aligned} J(v,t) = |t|^{\sigma+1}\|v\|^{q^-} \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx + (\sigma+1)\gamma(t) \\ - \int_\Omega \int_0^{v(x)} f(x,t)\,dt\; dx + \frac{1}{a}\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx. \end{aligned} \tag{3.9}$$

First, we remark that from (1) in the assumptions and (3.9), condition (ii) of Lemma 3.2 is satisfied.

We may assume, without loss of generality, that $|u|_{q(x)} < 1$ and $\|u\| < 1$, for all $u \in W_0^{1,p(x)}(\Omega)$.

Therefore, by (H2), (3.3) and (3.4) we deduce that for any $v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned} \int_\Omega \int_0^{v(x)} f(x,t)\,dt\; dx &\le c_1 \int_\Omega v(x)\,dx + c_2 \int_\Omega \frac{1}{q(x)}|v(x)|^{q(x)}\,dx \\ &\le 2c_1|1|_{q'(x)}|v|_{q(x)} + \frac{c_2}{q^-}\int_\Omega |v(x)|^{q(x)}\,dx \\ &\le 2c_1|1|_{q'(x)}|v|_{q(x)} + \frac{c_2}{q^-}|v|_{q(x)}^{q^-} \\ &\le 2c_1|1|_{q'(x)} + \left(2c_1|1|_{q'(x)} + c_2(q^-)^{-1}\right)|v|_{q(x)}^{q^-} \\ &\le 2c_1|1|_{q'(x)} + C\left(2c_1|1|_{q'(x)} + c_2(q^-)^{-1}\right)\|v\|^{q^-} \\ &= a_1 + a_2\|v\|^{q^-}. \end{aligned} \tag{3.10}$$

To apply the mountain pass theorem with a separating surface, established in Lemma 3.2, we need to prove that the functional J satisfies the condition $J(v,\rho) > \alpha > 0$, for every $v \in W_0^{1,p(x)}(\Omega)$ and ρ a fixed constant. So, by (3.9), (3.10) and assumption (2) of this theorem,

$$\begin{aligned} J(v,\rho) &\ge \frac{\rho^{\sigma+1}}{p^+}\|v\|^{q^-}\int_\Omega |\nabla v|^{p(x)}\,dx + (\sigma+1)\gamma(\rho) - a_1 - a_2\|v\|^{q^-} \\ &\quad + \frac{1}{ap^+}\int_\Omega |\nabla v|^{p(x)}\,dx \\ &\ge \|v\|^{q^-}\Big(\frac{\rho^{\sigma+1}}{p^+}\|v\|^{p^+} - a_2\Big) + (\sigma+1)\gamma(\rho) - a_1 \ge \alpha, \end{aligned}$$

for every $v \in W_0^{1,p(x)}(\Omega)$. Therefore, the hypothesis (i) in Lemma 3.2 is satisfied.

Now we verify if the functional J satisfies the Palais-Smale condition. Let be (v_n, t_n) in $W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ a sequence such that $J(v_n,t_n)$ is bounded and

$$J'(v_n,t_n) = (J_v(v_n,t_n), J_t(v_n,t_n)) \to 0 \quad \text{in } W^{-1,p'(x)}(\Omega) \times \mathbb{R},$$

where $p'(x) = \frac{p(x)}{p(x)-1}$. Hence,

$$|J(v_n, t_n)| \le M, \tag{3.11}$$

$$\begin{aligned} -J_v(v_n, t_n) &= |t_n|^{\sigma+1} \|v_n\|^{q^-} \Delta_{p(x)} v_n + f(\cdot, v_n) + a^{-1} \Delta_{p(x)} v_n \\ &\to 0 \quad \text{in } W^{-1,p'(x)}(\Omega), \end{aligned} \tag{3.12}$$

$$J_t(v_n, t_n) = |t_n|^{\sigma} (\operatorname{sgn} t_n) \|v_n\|^{q^-} \int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)}\, dx + \gamma'(t_n) \to 0 \quad \text{in } \mathbb{R}. \tag{3.13}$$

By (3.9), (3.10) and (3.11) we deduce that

$$\|v_n\|^{q^-} \Big(|t_n|^{\sigma+1} \int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)}\, dx - a_2 \Big) + (\sigma+1)\gamma(t_n) - a_1 \le M.$$

Using hypothesis (3) in this theorem, we can prove the sequence (t_n) is bounded in $\mathbb{R}$. We may suppose that (v_n) is bounded away from zero. Forwards, we consider two cases.

Case 1: We suppose that along a subsequence we have $t_n \to 0$. Thus, by (4), we obtain that $\gamma'(t_n) \to \gamma(0) = 0$. Therefore, by (3.13),

$$|t_n|^{\sigma} \|v_n\|^{q^-} \int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)}\, dx \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

From (3.9), (3.11) and (3.14) we infer that

$$\int_\Omega \int_0^{v_n(x)} f(x,\tau) d\tau\, dx - \frac{1}{a} \int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)}\, dx \quad \text{is bounded in } \mathbb{R}. \tag{3.15}$$

By [7, Proposition 12.3.2] there exists a function $g \in L^{p'(x)}(\Omega)$ such that $\|\Delta_{p(x)} v_n\|_{W^{-1,p'(x)}} \approx \|g\|_{L^{p'(x)}}$. We know that $t_n \to 0$ and v_n is bounded away from zero, then from (3.14) it follows that

$$\begin{aligned} &|t_n|^{\sigma+1} \|v_n\|^{q^-} \|\Delta_{p(x)} v_n\|_{W^{-1,p'(x)}} \\ &\approx |t_n| |t_n|^{\sigma} \|v_n\|^{q^-} \int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)}\, dx \Big(\int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)}\, dx \Big)^{-1} \|g\|_{L^{p'(x)}} \to 0 \end{aligned}$$

as $n \to \infty$. Then, by (3.12) we obtain

$$f(\cdot, v_n) + a^{-1} \Delta_{p(x)} v_n \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Taking into account (3.15) and (3.16) we have that for any constant $M > 0$, considering $\nu > 2$ in (H3),

$$\begin{aligned} &M + \nu^{-1} \|v_n\| \\ &\ge \frac{1}{a} \int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)}\, dx - \int_\Omega \int_0^{v_n(x)} f(x,\tau) d\tau\, dx \\ &\quad + \frac{1}{\nu} \Big(\int_\Omega f(x, v_n) v_n\, dx + \frac{1}{a} \int_\Omega \left(\Delta_{p(x)} v_n \right) v_n\, dx \Big) \\ &\ge \frac{1}{ap^+} \|v_n\|^{p^+} - \int_\Omega \int_0^{v_n(x)} f(x,\tau) d\tau\, dx + \frac{1}{\nu} \int_\Omega f(x, v_n) v_n\, dx - \frac{1}{a\nu} \|v_n\|^{p^-} \\ &= \frac{1}{a} \Big(\frac{1}{p^+} \|v_n\|^{p^+ - p^-} - \frac{1}{\nu} \Big) \|v_n\|^{p^-} + \frac{1}{\nu} \int_\Omega \Big(f(x, v_n) v_n - \nu \int_0^{v_n(x)} f(x,\tau) d\tau \Big)\, dx, \end{aligned}$$

for n large enough. By (H3) and (3.3) we can provide that there exist two constants $e_1 \geq 0$ and $e_2 \geq 0$ such that

$$\begin{aligned} M + \nu^{-1}\|v_n\| &\geq \frac{1}{a}\Big(\frac{1}{p^+}\|v_n\|^{p^+-p^-} - \frac{1}{\nu}\Big)\|v_n\|^{p^-} - \frac{1}{\nu}\int_\Omega \big(b_1 + b_2|v_n|^{\beta(x)}\big)\,dx \\ &\geq \frac{1}{a}\Big(\frac{1}{p^+}\|v_n\|^{p^+-p^-} - \frac{1}{\nu}\Big)\|v_n\|^{p^-} - e_1 - e_2\|v_n\|^{\beta^-}. \end{aligned}$$

Since $1 \leq \beta^- \leq \beta(x) \leq \beta^+ < p^- \leq p(x) \leq p^+ < \nu$, the last inequality ensures that (v_n) is bounded in $W_0^{1,p(x)}(\Omega)$.

We remark that hypothesis (H2) ensures that the restriction of Nemytskii's operator to $W_0^{1,p(x)}(\Omega)$,

$$v \in W_0^{1,p(x)}(\Omega) \mapsto f(\cdot, v(\cdot)) \in W^{-1,p'(x)}(\Omega),$$

is a compact mapping, namely, it maps any bounded set onto a relatively compact set (see [10]). So, passing eventually to a subsequence,

$$f(\cdot, v_n(\cdot)) \text{ converges in } W^{-1,p'(x)}(\Omega). \tag{3.17}$$

Therefore, relations (3.16) and (3.17) ensure that there exists a convergent subsequence of (v_n) in $W_0^{1,p(x)}(\Omega)$.

Case 2: We suppose that (t_n) is bounded away from zero. By (3.13) we deduce that (v_n) is bounded in $W_0^{1,p(x)}(\Omega)$ and consequently, (3.17) comes true. From (3.12) we obtain that

$$\Delta_{p(x)} v_n \left(a|t_n|^{\sigma+1}\|v_n\|^{q^-} + 1\right) \text{ converges in } W^{-1,p'(x)}(\Omega).$$

Then, $(\Delta_{p(x)} v_n)$ is convergent in $W^{-1,p'(x)}(\Omega)$. Hence, we finally obtain that, up to a subsequence, (v_n) is convergent in $W_0^{1,p(x)}(\Omega)$. This ends the proof that the functional J satisfies the Palais-Smale condition.

Taking into account that the hypotheses of Lemma 3.2 are satisfied, there exists (u, z) in $W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ which satisfies

$$-\Delta_{p(x)} u = \frac{1}{|z|^{\sigma+1}\|u\|^{q^-} + a^{-1}} f(\cdot, u), \tag{3.18}$$

$$|z|^\sigma (\operatorname{sgn} z)\|u\|^{q^-} \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \gamma'(z) = 0, \tag{3.19}$$

$$\begin{aligned} &|z|^{\sigma+1}\|u\|^{q^-} \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + (\sigma+1)\gamma(z) \\ &- \int_\Omega \int_0^{u(x)} f(x, z)\,dt\,dz + \frac{1}{a}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \geq \alpha. \end{aligned} \tag{3.20}$$

Relation (3.19) leads us to

$$z\gamma'(z) \leq 0. \tag{3.21}$$

We consider two possibilities:

(i) If $z = 0$ the statement (a) of Theorem 3.3 follows from (3.18) and (3.20). From the definition of c and Γ in Lemma 3.2, taking into account the path $g \in \Gamma$ given by $g(t) = (0, tr)$, for $0 \leq t \leq 1$, we obtain the second inequality of (a) in the Theorem 3.3.

(ii) In the case that $z \neq 0$, we argue by contradiction. If $z < 0$ then, following the assumption (γ_4), we obtain that $\gamma'(z) < 0$, which is a contradiction with (3.21). Hence, we only consider that $z > 0$. Again, by (4) in Theorem 3.3 we obtain that

$$\rho \leq t \leq r. \tag{3.22}$$

If $t = \rho$ or $t = r$, by (3.19) and (γ_4), we have $u = 0$. Thus, we obtain a contradiction between (3.18) and hypothesis (H1). So, we showed that (3.22) is reduced to (3.5). Because $z > 0$, (3.19) yields to (3.6).

Relation (3.12) proves that $(u, \lambda) \in W_0^{1,p(x)}(\Omega) \times \mathbb{R}$ is an eigensolution of problem (3.2), where

$$\lambda = \frac{1}{|z|^{\sigma+1}\|u\|^{q^-} + a^{-1}}. \tag{3.23}$$

Replacing $\|u\|$ as determined by (3.6) in (3.23) we obtain (3.7). From Lemma 3.2, making use of the path $g(t) = (0, tr)$, $0 \leq t \leq 1$, the inequality (3.8) follows. $\square$

Corollary 3.4. *Suppose that the hypotheses* (H1)–(H3) *are satisfied by a function* $f : \Omega \times \mathbb{R} \to \mathbb{R}$*, with the assumption that* (3.1) *holds. Let be a number* $a > 0$ *which is not an eigenvalue of problem 3.2. So, there exists a sequence* $(u_n, \lambda_n) \in W_0^{1,p(x)}(\Omega) \times (0, a)$ *of eigensolutions of* (3.2) *which satisfies*

$$u_n \to 0 \text{ in } W_0^{1,p(x)}(\Omega) \quad \text{and } \lambda_n^{-1}\|u_n\|^{p^-} \to 0 \text{ as } n \to \infty.$$

Proof. Let be $\varepsilon > 0$. For any such ε, we can establish $\gamma_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ which satisfies the hypotheses (1)–(4) of Theorem 3.3 with $\rho = \rho_\varepsilon < r = r_\varepsilon$, depending on ε, and $\sigma > 0$, $\alpha > 0$ independent of ε such that

$$|\gamma_\varepsilon'(t)| \leq \varepsilon^{q^-} t^{-1} \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx, \quad \forall t \geq \Big(\frac{p^+ a_2}{\|u\|^{p^+}}\Big)^{1/(\sigma+1)}. \tag{3.24}$$

By Theorem 3.3, there exists the number $z = z_\varepsilon \in (\rho_\varepsilon, r_\varepsilon)$ that describes an eigensolution $(u_\varepsilon, \lambda_\varepsilon)$ of problem (3.2) by relations (3.6) and (3.7) with $u = u_\varepsilon$ and $\lambda = \lambda_\varepsilon$. Obviously, we can suppose that

$$z_\varepsilon \to +\infty \quad \text{as } \varepsilon \to 0. \tag{3.25}$$

Therefore, by (3.6), (3.24) and (3.25) we obtain

$$\begin{aligned} \|u_\varepsilon\| &= z_\varepsilon^{-\sigma/q^-}\big(-\gamma'(z_\varepsilon)\big)^{1/q^-}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\Big)^{-1/q^-} \\ &\leq \varepsilon z_\varepsilon^{-(\sigma+1)/q^-} \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{3.26}$$

We have the equality

$$-\frac{1}{\lambda_\varepsilon}\Delta_{p(x)} u_\varepsilon = f(x, u_\varepsilon).$$

When $\varepsilon \to 0$, taking into account that $u_\varepsilon \to 0$ in $W_0^{1,p(x)}(\Omega)$ and the hypothesis (H1), it results that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, by (3.7) we obtain

$$\lambda_\varepsilon^{-1} - a^{-1} = z_\varepsilon\big(-\gamma'(z_\varepsilon)\big)\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\Big)^{-1} \leq \varepsilon^{q^-}. \tag{3.27}$$

By (3.26) and (3.27) we remark that

$$\|u_\varepsilon\|^{p^-}(\lambda_\varepsilon^{-1} - a^{-1}) \leq \varepsilon^{p^-} z_\varepsilon^{\frac{-p^-(\sigma+1)}{q^-}} \varepsilon^{q^-} = \varepsilon^{q^-+p^+} z_\varepsilon^{\frac{-p^-(\sigma+1)}{q^-}}.$$

which means, by (3.26), that $\lambda_\varepsilon^{-1}\|u_\varepsilon\|^{p^-} \to 0$ as $\varepsilon \to 0$, and this completes the proof. $\square$

Corollary 3.5. *Considering the hypotheses of Corollary 3.4, for any C^1 function $\gamma : \mathbb{R} \to \mathbb{R}$ which satisfies relations (1)–(4) in theorem 3.3, with fixed constants ρ, r, σ, α, there exists a one-to-one mapping from $[1, +\infty)$ into the set of eigensolutions (u, λ) of problem (3.2). Especially, there exist uncountable many solutions (u, λ) of (3.2).*

Proof. We first remark that if $\gamma \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the hypotheses $(\gamma_1) - (\gamma_4)$, where ρ, r, σ, α are given numbers, then this is true for each function $\eta\gamma$, where $\eta \geq 1$ is an arbitrary number.

Assume that there exists some $a > 0$ which is not an eigenvalue of (3.2). If we apply the Theorem 3.3 with $\eta\gamma$, for $\eta \geq 1$, replacing γ, we can find an eigensolution $(u_\eta, \lambda_\eta) \in W_0^{1,p(x)}(\Omega) \times (0, a)$ and a number $z_\eta \in (0, r)$ such that

$$\|u_\eta\| = z_\eta^{-\sigma/q^-}\big(-\gamma'(z_\eta)\big)^{1/q^-}\eta^{1/q^-}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_\eta|^{p(x)}\,dx\Big)^{-1/q^-}. \tag{3.28}$$

From (3.23), we have

$$\lambda_\eta^{-1} = |z_\eta|^{\sigma+1}\|u_\eta\|^{q^-} + a^{-1}. \tag{3.29}$$

Let us consider $\eta_1, \eta_2 \geq 1$ where $\eta_1 \neq \eta_2$. Hence, (3.29) proves that $z_{\eta_1} = z_{\eta_2}$. Therefore, by (3.28), it follows that $\eta_1 = \eta_2$. So, we obtain a contradiction which completes the proof. $\square$

4. A subcritical boundary value problem with variable exponent

We consider now the other problem related to the $p(x)$-Laplace operator:

$$\begin{gathered}
-\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u + |u|^{q(x)-2}u \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,\\
u \not\equiv 0 \quad \text{in } \Omega,
\end{gathered} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N (N > 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, p, q are continuous functions on $\overline{\Omega}$ which satisfy

$$1 < p(x) < q(x) < p^*(x),$$

where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$, for all $x \in \overline{\Omega}$.

Definition 4.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (4.1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\,dx = \lambda\int_\Omega |u|^{p(x)-2}u\varphi\,dx + \int_\Omega |u|^{q(x)-2}u\varphi\,dx,$$

for every $\varphi \in W_0^{1,p(x)}(\Omega)$.

Finally, we give our existence result.

Theorem 4.2. *If $\lambda < \lambda_{P^*}$, where*

$$\lambda_{P^*} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx},$$
$$1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < p^*(x),$$

with p satisfying hypothesis (3.1)*, then there exists a weak solution for the problem* (4.1)*.*

The main tool that we use in the proof of the second result is the Mountain-Pass Theorem in the following variant.

Theorem 4.3 ([19])**.** *Let X be a real Banach space and $F \in C^1(X,\mathbb{R})$ be a functional which satisfies the Palais-Smale condition. If F satisfies the following geometric conditions*

(1) *there exist two constants $R, c_0 > 0$ such that $F(u) \geq c_0$, for every $u \in X$ with $\|u\| = R$,*
(2) *$F(0) < c_0$ and there exists $v \in X$ with $\|v\| > R$ such that $F(v) < c_0$, then there exists at least a critical point for the functional F.*

Proof of Theorem 4.2. We set

$$a(u,x) = \begin{cases} u^{q(x)-1} & \text{if } u \geq 0, \\ 0 & \text{if } u < 0, \end{cases}$$

and define $A(u,x) = \int_0^{u(x)} a(t,x)dt$. Denote the functional

$$E(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx - \int_\Omega A(u,x)\, dx.$$

Note that

$$A(u,x) = \int_0^{u(x)} a(t,x)dt \leq \frac{1}{q(x)} |u|^{q(x)} = C|u|^{q(x)}$$

and by the fact that $1 < p(x) < q(x) < p^*(x)$, it yields that $W_0^{1,p(x)}(\Omega) \subset L^{q(x)}(\Omega)$, which implies that E is well defined on $W_0^{1,p(x)}(\Omega)$. From [10] we have that E is a C^1 functional and for every $\varphi \in W_0^{1,p(x)}(\Omega)$,

$$E'(u)(\varphi) = \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi - \lambda |u|^{p(x)-2} u\varphi \Big)\, dx - \int_\Omega a(u)\varphi\, dx.$$

By the Definition 4.1 we observe that the critical points of E are weak solutions of the problem (4.1). ☐

Before proceed to the proof of the Theorem 4.2 we will point out some results obtained by Fan, Zang, Zhao from the study of the following eigenvalue problem

$$\begin{gathered} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2} u \quad \text{in } \Omega, \\ u \not\equiv 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.2}$$

We introduce the following Rayleigh quotient for the above problem:

$$\lambda_{P^*} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx}$$

and the set

$$\Lambda = \Lambda_{p(x)} = \{\lambda \in \mathbb{R}\lambda \text{ is an eigenvalue of (4.2)}\}.$$

So, we can deduce in [11],

$$\lambda_{P^*} = \inf \Lambda = \inf \Big\{ \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx} : u \in W_0^{1,p(x)}(\Omega) \setminus \{0\} \Big\}.$$

Remark 4.4. From the assumption that $\lambda < \lambda_{P^*}$ we obtain a constant $C_\lambda > 0$ such that, for every $u \in W_0^{1,p(x)}(\Omega)$ one has

$$C_\lambda \int_\Omega |\nabla u|^{p(x)}\,dx \le \int_\Omega \Big(|\nabla u|^{p(x)} - \lambda |u|^{p(x)}\Big)\,dx. \tag{4.3}$$

The following results play a crucial role in obtaining the existence of a nontrivial weak solution for the problem (4.1).

Considering the fact that the space $X = W_0^{1,p(x)}(\Omega)$ equipped with the norm $|\nabla u|_{p(x)} = \|u\|$ is a separable and reflexive Banach space, from [10] we have the following proposition.

Proposition 4.5. (i) $-\Delta_{p(x)} : X \to X^*$ *is a strictly monotone operator;*

(ii) $-\Delta_{p(x)}$ *is a mapping of type* (S_+)*, i.e. if* $u_n \rightharpoonup u$ *in* X *and*

$$\limsup_{n\to\infty} \big\langle (-\Delta_{p(x)} u_n) - (-\Delta_{p(x)} u), u_n - u \big\rangle \le 0,$$

then $u_n \to u$ *in* X*;*

(iii) $-\Delta_{p(x)} : X \to X^*$ *is a homeomorphism.*

Lemma 4.6. *Assume that hypotheses of the Theorem 4.2 hold, then* E *admits a Palais-Smale sequence.*

Proof. Let (u_n) be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\sup_n |E(u_n)| < +\infty, \tag{4.4}$$

$$\|E'(u_n)\|_{W^{-1,p'(x)}} \to 0, \quad \text{as } n \to \infty. \tag{4.5}$$

First of all we show that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Observe that (4.5) implies that, for every $v \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned} &\int_\Omega \Big(|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla v - \lambda |u_n|^{p(x)-2} u_n v\Big)\,dx \\ &= \int_\Omega a(u_n,x) v\,dx + o(1)\|v\|, \end{aligned} \tag{4.6}$$

as $n \to \infty$. Taking $v = u_n$ we obtain

$$\int_\Omega \big(|\nabla u_n|^{p(x)} - \lambda |u_n|^{p(x)}\big)\,dx = \int_\Omega a(u_n,x) u_n\,dx + o(1)\|u_n\|. \tag{4.7}$$

Note that (4.4) means we can find $M > 0$ such that, for any $n \ge 1$,

$$\Big| \int_\Omega \frac{1}{p(x)} \Big(|\nabla u_n|^{p(x)} - \lambda |u_n|^{p(x)}\Big)\,dx - \int_\Omega A(u_n,x)\,dx \Big| \le M. \tag{4.8}$$

A direct computation shows that

$$\int_\Omega a(u_n,x) u_n\,dx = \int_\Omega q(x) A(u_n,x)\,dx. \tag{4.9}$$

By (4.7), (4.8), (4.9) and $1 < p^+ < q^- \le q^+$ we find that

$$\int_\Omega A(u_n,x)\,dx = O(1) + o(1)\|u_n\|. \tag{4.10}$$

Hence, using (4.7) and (4.10) we obtain

$$\int_\Omega \Big(|\nabla u_n|^{p(x)} - \lambda |u_n|^{p(x)}\Big)\,dx \le O(1) + o(1)\|u_n\|.$$

Therefore,

$$\int_\Omega |\nabla u_n|^{p(x)}\,dx = O(1) + o(1)\|u_n\|,$$

thus (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$.

Now, we point out that (u_n) is relatively compact. We can write (4.6) as

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla v\,dx = \int_\Omega \theta(u_n,x)v\,dx + o(1)\|v\|, \tag{4.11}$$

for every $v\in W_0^{1,p(x)}(\Omega)$, where $\theta(u,x)=a(u,x)+\lambda|u|^{p(x)-2}u$, and $\lambda<\lambda_{P^*}$.

It is clear that θ is continuous, because $q(x)<\frac{Np(x)}{N-p(x)}$, for every $x\in\overline{\Omega}$, and there exists $C>0$ such that

$$|\theta(u,x)|\le C\left(1+|u|^{(Np(x)-N+p(x))/(N-p(x))}\right), \tag{4.12}$$

for every $x\in\overline{\Omega}$ and $u\in\mathbb{R}$. Furthermore

$$\theta(u,x)=o\left(|u|^{Np(x)/(N-p(x))}\right),\quad \text{as } |u|\to\infty,\ \text{uniformly for } x\in\Omega. \tag{4.13}$$

We define $-\Delta_{p(x)}:W_0^{1,p(x)}(\Omega)\to W^{-1,p'(x)}(\Omega)$, by

$$-\Delta_{p(x)} = -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right).$$

Using Proposition 4.5 the operator $\left(-\Delta_{p(x)}\right)$ is invertible, continuous and the operator

$$\left(-\Delta_{p(x)}\right)^{-1}:W^{-1,p'(x)}(\Omega)\to W_0^{1,p(x)}(\Omega)$$

is continuous. Therefore it is sufficient to prove that $\theta(u_n,x)$ is relatively compact in $W^{-1,p'(x)}(\Omega)$. Using the Sobolev embeddings for variable exponent spaces, this will be obtained by proving that a subsequence of $\theta(u_n,x)$ is convergent in

$$\left(L^{Np(x)/(N-p(x))}(\Omega)\right)^* = L^{Np(x)/(Np(x)-N+p(x))}(\Omega).$$

Knowing that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)\subset L^{Np(x)/(N-p(x))}(\Omega)$, we can suppose, up to a subsequence eventually, that

$$u_n\to u\in L^{p^*(x)}(\Omega) \text{ a.e. in } \Omega.$$

Using [4, Egorov Theorem], for each δ, there exists $B\subset\Omega$, with $|B|<\delta$, such that $u_n\to u$, uniformly in $\Omega\setminus B$. So, it is sufficient to prove that

$$\int_B |\theta(u_n,x)-\theta(u,x)|^{Np(x)/(Np(x)-N+p(x))}\,dx<\xi,$$

for any fixed $\xi>0$. But by (4.12),

$$\int_B |\theta(u,x)|^{Np(x)/(Np(x)-N+p(x))}\,dx \le C\int_B\left(1+|u|^{Np(x)/(N-p(x))}\right)dx$$

which for a sufficiently small $\delta>0$, can be made small enough.

By (4.13), we obtain

$$\begin{aligned}&\int_B |\theta(u_n,x)-\theta(u,x)|^{Np(x)/(Np(x)-N+p(x))}\,dx\\ &\le \varepsilon\int_B |u_n-u|^{Np(x)/(N-p(x))}\,dx + C_\varepsilon|B|,\end{aligned}$$

which can be made arbitrarily small, by Sobolev embeddings for spaces with variable exponent and by the boundedness of (u_n) in $W_0^{1,p(x)}(\Omega)$. Therefore, E admits a Palais-Smale sequence. $\square$

Lemma 4.7. *Under the conditions of Theorem 4.2 for the energy functional $E : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$, there exist two constants $R, c_0 > 0$ such that $E(u) \geq c_0$, for every $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = R$.*

Proof. For every $u \in \mathbb{R}$, we can write $|a(u,x)| \leq |u|^{q(x)-1}$. Hence, for every $u \in \mathbb{R}$,

$$|A(u,x)| \leq \frac{1}{q(x)} |u|^{q(x)}. \tag{4.14}$$

Now, (4.3) and (4.14) yield

$$\begin{aligned}
E(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx - \lambda \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\,dx - \int_\Omega A(u,x)\,dx \\
&= \int_\Omega \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} - \lambda |u|^{p(x)} \Big)\,dx - \int_\Omega A(u,x)\,dx \\
&\geq \frac{C_\lambda}{p^+} \int_\Omega |\nabla u|^{p(x)}\,dx - \frac{1}{q^-} \int_\Omega |u|^{q(x)}\,dx \\
&= C_1 \int_\Omega |\nabla u|^{p(x)}\,dx - C_2 \int_\Omega |u|^{q(x)}\,dx,
\end{aligned}$$

where C_1 and C_2 are positive constants.

By the hypothesis $1 < p^- \leq p^+ < q^- \leq q^+ < p^*(x)$ the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, hence there exists a constant $\tilde{C} > 0$ such that

$$\|u\|_{L^{q(x)}} \leq \tilde{C} \|u\|_{W_0^{1,p(x)}}.$$

Therefore, we may find a constant $C_3 > 0$, such that

$$E(u) \geq C_1 \int_\Omega |\nabla u|^{p(x)}\,dx - C_3 (\|u\|^{q^+} + \|u\|^{q^-}).$$

Set $\|u\| = |\nabla u|_{p(x)} < 1$. Hence $|\nabla u|^{p^+} \leq \rho_{p(x)}(\nabla u)$, which leads to

$$E(u) \geq C_1 |\nabla u|_{p(x)}^{p^+} - C_3 (\|u\|^{q^+} + \|u\|^{q^-})$$

and from the hypothesis $1 < p^- \leq p^+ < q^- \leq q^+$ we have

$$E(u) \geq C_1 |\nabla u|_{p(x)}^{p^+} - C_3 (\|u\|^{q^+} + \|u\|^{q^-}).$$

For $R > 0$ small enough, taking $|\nabla u|_{p(x)} = \|u\| = R$, we deduce that $E(u) \geq c_0 > 0$. $\square$

Lemma 4.8. *Assuming that the hypotheses of Theorem 4.2 hold, for the energy functional $E : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$, there exist two constants $R, c_0 > 0$ such that $E(0) < c_0$ and there exists $v \in W_0^{1,p(x)}(\Omega)$ with $\|v\| > R$ such that $E(v) < c_0$.*

Proof. We choose $u_0 \in W_0^{1,p(x)}(\Omega)$, $u_0 > 0$ in Ω and $t > 0$. From a straightforward computation we obtain

$$E(tu_0) = \int_\Omega \frac{t^{p(x)}}{p(x)} \Big(|\nabla u_0|^{p(x)} - \lambda |u_0|^{p(x)} \Big)\,dx - \int_\Omega \frac{t^{q(x)}}{q(x)} |u_0|^{q(x)}\,dx$$

$$\leq \frac{t^{p^+}}{p^-}\int_\Omega \Big(|\nabla u_0|^{p(x)} - \lambda |u_0|^{p(x)}\Big)\,dx - \frac{t^{q^-}}{q^+}\int_\Omega |u_0|^{q(x)}\,dx.$$

Since $p^+ < q^- \leq q^+$, for t large enough we obtain $E(tu_0) < 0 < c_0$. Then we can consider $v = tu_0$ with $\|v\| = t\|u_0\| > R$ such that $E(v) < c_0$, for $t > 0$ chosen sufficiently large. □

Proof of Theorem 4.2 completed. Since the Palais-Smale condition and the mountain pass geometry are assured by Lemmas 4.6, 4.7 and 4.8, we only have to apply Theorem 4.3 and the existence of a nontrivial weak solution is assured. □

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