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NONLINEAR SINGULAR HYPERBOLIC INITIAL BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we aim to prove the well posedness of a one point initial boundary value problem for a nonlinear hyperbolic singular integro-differential partial differential equation. Our proofs are mainly based on a fixed point theorem and some a priori bounds. The solvability of the problem is proved when the given data are related to a weighted Sobolev space. An additional result may allow us to study the regularity of the solution.

1. Introduction

A big number of physical phenomena can be modeled and formulated by some nice partial differential equations. It happens sometimes that such phenomena cannot be modeled by classical partial differential equations while describing the system as a function at a given time; it fails to take into account the effect of past history, such as in thermo-elasticity and heat diffusion. In such situations, a memory term on the form of an integral should be included the equation. This integral term can be considered as a damping term in the equation. A range of elliptic integro-differential equations were studied in the works of numerous researchers, including for example Bokalo and Dmytriv [7], Barles and Imbert [4], Caffarelli and Silvestre [8], Chipot and Guesmia [10], Balachandran and Park [3], Bakalo and Dmytriv [7]. Integro-differential equations of hyperbolic and parabolic type are studied by many authors see for example Adolfsson [1], Baker [2], Sloan and Thomee [18], Berrimi and Messaoudi [5], Cannarsa and Sforza [9] and Weiking [19]. In the present work, we consider a hyperbolic integro-differential equation with an unknown function that enters both into the differential part and into the integral part of the equation. It can occur in visco-elasticity see Renardi and Hrusa [15, 16] and other areas as fluid dynamics, biological models, and chemical kinetics. Well-posedness of problem (1.1)–(1.3) is studied, in fact, we prove the existence and uniqueness of solutions for a posed nonlinear hyperbolic integro-differential equation with a Neumann and Dirichlet conditions. The used tool is a variant of fixed point theorems.

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In the rectangle $Q_T = (0,1)\times(0,T)$, where $0 < T < \infty$, we consider the nonlinear singular second order hyperbolic integro-differential equation

$$\begin{aligned}\mathcal{L}v &= \frac{\partial^2 v}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial v}{\partial x}) - \frac{d}{dt}\max\Big(\int_0^x \xi v(\xi,t)d\xi, 0\Big)\\ &= f(x,t),\end{aligned} \tag{1.1}$$

With (1.1), we associate the following initial conditions

$$v(x,0) = Z_1(x), \quad v_t(x,0) = Z_2(x), \quad x \in (0,1), \tag{1.2}$$

and the boundary conditions

$$v_x(1,t) = 0, \quad v(1,t) = 0, \quad t \in (0,T), \tag{1.3}$$

where $Z_1(x)$, $Z_2(x)$ and $f(x,t)$ are given functions which will be specified later on.

In Section 2, we shall give some function spaces and tools which will be used repeatedly below. Section 3 is devoted to the proof of uniqueness of solution of the given problem in the classical Sobolev space $W^{1,1}_{\gamma,2}(Q_T)$. In section 4, we established the existence of solution and the proof was mainly based on the Schauder fixed point theorem. In the last section a priori bound is obtained and can help to establish some regularity results for the solution of problem (1.1)–(1.3)

2. Preliminaries

For the investigation of problem (1.1)–(1.3), we need the following functions spaces: Let $L^2_\gamma(Q_T)$, $L^2_\sigma(Q_T)$, $L^2_\rho(Q_T)$ be the weighted Hilbert spaces of square integrable functions on Q_T with $\gamma = x+x^2$, $\sigma = x^2$, and $\rho = x$. The inner products in $L^2_\gamma(Q_T)$, $L^2_\sigma(Q_T)$, $L^2_\rho(Q_T)$ are respectively denoted by $(\cdot,\cdot)_{L^2_\gamma(Q_T)}$, $(\cdot,\cdot)_{L^2_\sigma(Q_T)}$, $(\cdot,\cdot)_{L^2_\rho(Q_T)}$ such that

$$(u,v)_{L^2_A(Q_T)} = \int_0^1 A(x)uvdx, \quad A(x) = \gamma, \sigma, \rho. \tag{2.1}$$

Let $L^2(0,T;\ W^{1,1}_{\gamma,2}(0,1))$ be the space of (classes) of function u measurable on $[0,T]$ for the Lebesgue measure and having their values in $W^{1,1}_{\gamma,2}(0,1)$,and such that

$$\|u\|_{L^2(0,T;W^{1,1}_{\gamma,2}(0,1))} = \Big(\int_0^T \|u(\cdot,t)\|^2_{W^{1,1}_{\gamma,2}(0,1)}dt\Big)^{1/2} < \infty$$

The space $L^2(0,T;\ W^{1,1}_{\gamma,2}(0,1))$ is a Hilbert space having the inner product

$$(u,v)_{L^2(0,T;\ W^{1,1}_{\gamma,2}(0,1))} = \int_0^T (u,v)_{W^{1,1}_{\gamma,2}(0,1)}dt, \tag{2.2}$$

where

$$(u,v)_{W^{1,1}_{\gamma,2}(0,1)} = \int_0^1 (x+x^2)(uv + u_xv_x + u_tv_t)dx. \tag{2.3}$$

The space $W^{1,1}_{\gamma,2}(Q_T)$, is the set of functions $u \in L^2_\gamma(Q_T)$ such that $u_x, u_t \in L^2_\gamma(Q_T)$. In general the elements of $W^{m,n}_{\gamma,2}(Q_T)$, with m,n nonnegative integers are functions having $x-$derivatives up to m^{th} order in $L^2_\gamma(Q_T)$, and $t-$derivative up to n^{th} order in $L^2_\gamma(Q_T)$. We also use weighted spaces in $(0,1)$ such as: $L^2_\gamma(0,1)$, $L^2_\sigma(0,1)$, $L^2_\rho(0,1)$, $W^{1,1}_{\gamma,2}(0,1)$ and $W^{1}_{\gamma,2}(0,1)$.

The following inequalities are needed:

(1) Cauchy ε-inequality which holds for all $\varepsilon > 0$ and for arbitrary A and B.

$$AB \le \frac{\varepsilon}{2}A^2 + \frac{1}{2\varepsilon}B^2, \tag{2.4}$$

(2) A Poincaré type inequality (see [14]).

$$\begin{gathered} \|\Lambda_x(\xi u)\|^2_{L^2(0,1)} \le \frac{1}{2}\|u\|^2_{L^2_\rho(Q_T)}, \\ \|\Lambda^2_x(\xi u)\|^2_{L^2(0,1)} \le \frac{1}{2}\|\Lambda_x(\xi u)\|^2_{L^2(Q_T)}, \end{gathered} \tag{2.5}$$

where

$$\Lambda_x(\xi u) = \int_0^x \xi u(\xi,t)d\xi, \quad \Lambda^2_x(\xi u) = \int_0^x \int_0^\xi \eta u(\eta,t)d\eta d\xi.$$

(3) Gronwall's inequality (see [14, Lemma 4.1]).

3. Uniqueness of solution

Theorem 3.1. *Let $Z_1(x) \in W^1_{\gamma,2}((0,1))$, $Z_2(x) \in L^2_\gamma((0,1)$ and $f(x,t) \in L^2_\gamma(Q_T)$. Then the posed problem (1.1)–(1.3) has at most one solution in $W^{1,1}_{\gamma,2}(Q_T)$, if it exists.*

Proof. Let v_1 and v_2 be two solutions of problem (1.1)–(1.3) and let $\eta(x,t) = V_1(x,t) - V_2(x,t)$, where

$$V_j(x,t) = \int_0^t v_j(x,\tau)d\tau, \quad j = 1,2, \tag{3.1}$$

then the function $\eta(x,t)$ satisfies

$$\mathcal{L}\eta = \frac{\partial^2\eta}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial\eta}{\partial x}\big) = \gamma_1(x,t) - \gamma_2(x,t), \tag{3.2}$$

$$\eta_x(1,t) = 0, \quad \eta(1,t) = 0, \quad t \in (0,T), \tag{3.3}$$

$$\eta(x,0) = 0, \quad \eta_t(x,0) = 0, \quad x \in (0,1), \tag{3.4}$$

where

$$\gamma_j(x,t) = \max(\Lambda_x(\xi v_j), 0), \quad j = 1,2. \tag{3.5}$$

Consider the identity

$$\begin{aligned} &\Big(\frac{\partial\eta}{\partial t}, \frac{\partial^2\eta}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial\eta}{\partial x})\Big)_{L^2_\rho(Q_T)} + 2\Big(\frac{\partial\eta}{\partial t}, \frac{\partial^2\eta}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial\eta}{\partial x})\Big)_{L^2_\sigma(Q_T)} \\ &\quad + \Big(\frac{\partial\eta}{\partial x}, \frac{\partial^2\eta}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial\eta}{\partial x})\Big)_{L^2_\sigma(Q_T)} \\ &= \Big(\frac{\partial\eta}{\partial t}, \gamma_1 - \gamma_2\Big)_{L^2_\rho(Q_T)} + 2\Big(\frac{\partial\eta}{\partial t}, \gamma_1 - \gamma_2\Big)_{L^2_\sigma(Q_T)} + \Big(\frac{\partial\eta}{\partial x}, \gamma_1 - \gamma_2\Big)_{L^2_\sigma(Q_T)}. \end{aligned} \tag{3.6}$$

Integrating by parts, and taking into account conditions (3.3) and (3.4), we deduce the following expressions for the terms on the left-hand side of (3.6)

$$\Big(\frac{\partial\eta}{\partial t}, \frac{\partial^2\eta}{\partial t^2}\Big)_{L^2_\rho(Q_T)} = \frac{1}{2}\|\eta_t(x,T)\|^2_{L^2_\rho((0,1))}, \tag{3.7}$$

$$-\Big(\frac{\partial\eta}{\partial t}, \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial\eta}{\partial x}\big)\Big)_{L^2_\rho(Q_T)} = \frac{1}{2}\|\eta_x(x,T)\|^2_{L^2_\rho((0,1))}, \tag{3.8}$$

$$2\Big(\frac{\partial\eta}{\partial t},\frac{\partial^2\eta}{\partial t^2}\Big)_{L^2_\sigma(Q_T)}=\|\eta_t(x,T)\|^2_{L^2_\sigma((0,1))},\tag{3.9}$$

$$-2\Big(\frac{\partial\eta}{\partial t},\frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial\eta}{\partial x}\big)\Big)_{L^2_\sigma(Q_T)}=\|\eta_x(x,T)\|^2_{L^2_\sigma((0,1))}+2\big(\frac{\partial\eta}{\partial t},\frac{\partial\eta}{\partial x}\big)_{L^2_\rho(Q_T)},\tag{3.10}$$

$$\Big(\frac{\partial\eta}{\partial x},\frac{\partial^2\eta}{\partial t^2}\Big)_{L^2_\sigma(Q_T)}=\|\eta_t\|^2_{L^2_\rho(Q_T)}+\int_0^1 x^2\eta_x(x,T)\eta_t(x,T)dx,\tag{3.11}$$

$$-\Big(\frac{\partial\eta}{\partial x},\frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial\eta}{\partial x}\big)\Big)_{L^2_\sigma(Q_T)}=0.\tag{3.12}$$

Substituting formulas (3.7)–(3.12) into (3.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\|\eta_t(x,T)\|^2_{L^2_\rho((0,1))}+\frac{1}{2}\|\eta_x(x,T)\|^2_{L^2_\rho((0,1))}+\|\eta_t(x,T)\|^2_{L^2_\sigma((0,1))}\\
&+\|\eta_x(x,T)\|^2_{L^2_\sigma((0,1))}+\|\eta_t\|^2_{L^2_\rho(Q_T)}\\
&=\Big(\frac{\partial\eta}{\partial t},\gamma_1-\gamma_2\Big)_{L^2_\rho(Q_T)}+2\Big(\frac{\partial\eta}{\partial t},\gamma_1-\gamma_2\Big)_{L^2_\sigma(Q_T)}+\Big(\frac{\partial\eta}{\partial x},\gamma_1-\gamma_2\Big)_{L^2_\sigma(Q_T)}\\
&\quad-\int_0^1 x^2\eta_x(x,T)\eta_t(x,T)dx-2\big(\frac{\partial\eta}{\partial t},\frac{\partial\eta}{\partial x}\big)_{L^2_\rho(Q_T)}.
\end{aligned}\tag{3.13}$$

By using Cauchy $\varepsilon-$inequality and the Poincaré type inequality (2.5), we estimate the terms on the right-hand side of (3.13) as follows

$$\begin{aligned}
&-\int_0^1 x^2\eta_x(x,T)\eta_t(x,T)dx\\
&\le\frac{\varepsilon_1}{2}\|\eta_t(x,T)\|^2_{L^2_\sigma((0,1))}+\frac{1}{2\varepsilon_1}\|\eta_x(x,T)\|^2_{L^2_\sigma((0,1))},
\end{aligned}\tag{3.14}$$

$$-2\Big(\frac{\partial\eta}{\partial t},\frac{\partial\eta}{\partial x}\Big)_{L^2_\rho(Q_T)}\le\varepsilon_2\|\eta_x\|^2_{L^2_\rho(Q_T)}+\frac{1}{\varepsilon_2}\|\eta_t\|^2_{L^2_\rho(Q_T)},\tag{3.15}$$

$$\Big(\frac{\partial\eta}{\partial t},\gamma_1-\gamma_2\Big)_{L^2_\rho(Q_T)}\le\frac{\varepsilon_3}{2}\|\eta_t\|^2_{L^2_\rho(Q_T)}+\frac{1}{4\varepsilon_3}\|\eta_t\|^2_{L^2_\rho(Q_T)},\tag{3.16}$$

$$2\Big(\frac{\partial\eta}{\partial t},\gamma_1-\gamma_2\Big)_{L^2_\sigma(Q_T)}\le\varepsilon_4\|\eta_t\|^2_{L^2_\sigma(Q_T)}+\frac{1}{\varepsilon_4}\|\eta_t\|^2_{L^2_\sigma(Q_T)},\tag{3.17}$$

$$\Big(\frac{\partial\eta}{\partial x},\gamma_1-\gamma_2\Big)_{L^2_\sigma(Q_T)}\le\frac{\varepsilon_5}{2}\|\eta_t\|^2_{L^2_\sigma(Q_T)}+\frac{1}{2\varepsilon_5}\|\eta_x\|^2_{L^2_\sigma(Q_T)}.\tag{3.18}$$

Let $\varepsilon_1=1$, $\varepsilon_2=8$, $\varepsilon_3=1$, $\varepsilon_4=1$, $\varepsilon_5=1$, and combine (3.14)–(3.18) and (3.13), we obtain

$$\begin{aligned}
&\|\eta_t(x,T)\|^2_{L^2_\rho((0,1))}+\|\eta_x(x,T)\|^2_{L^2_\rho((0,1))}+\|\eta_t(x,T)\|^2_{L^2_\sigma((0,1))}\\
&+\|\eta_x(x,T)\|^2_{L^2_\sigma((0,1))}+\|\eta_t\|^2_{L^2_\rho(Q_T)}\\
&\le 64\Big(\|\eta_t\|^2_{L^2_\rho(Q_T)}+\|\eta_x\|^2_{L^2_\rho(Q_T)}+\|\eta_t\|^2_{L^2_\sigma(Q_T)}+\|\eta_x\|^2_{L^2_\sigma(Q_T)}\Big).
\end{aligned}\tag{3.19}$$

Application of Gronwall's lemma [14, Lemma 4.1] to (3.19), yields

$$\|\eta_t\|^2_{L^2_\rho(Q_T)}=0\Rightarrow\eta_t=v_1-v_2=0\quad\text{for all }(x,t)\in Q_T,$$

hence, we conclude the uniqueness of the solution. □

4. Existence of the solution

Theorem 4.1. *Let $Z_1(x) \in W^1_{\gamma,2}((0,1))$, $Z_2(x) \in L^2_\gamma((0,1)$ and $f(x,t) \in L^2_\gamma(Q_T)$ be given and satisfy*

$$\|Z_1\|^2_{W^1_{\gamma,2}((0,1))} + \|Z_2\|^2_{L^2_\gamma((0,1))} + \|f\|^2_{L^2_\gamma(Q_T)} \le C_2^2, \tag{4.1}$$

for $C_2 > 0$ small enough, and that

$$\frac{\partial Z_1(1)}{\partial x} = 0, \quad Z_1(1) = 0, \quad \frac{\partial Z_2(1)}{\partial x} = 0, \quad Z_2(1) = 0, \tag{4.2}$$

Then problem (1.1)–(1.3) has a unique solution $v \in W^{1,1}_{\gamma,2}(Q_T)$.

Proof. Consider the class of functions

$$K(A) = \{v \in L^2_\gamma(Q_T) : \|v\|_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))} \le A,\ \|v_t\|_{L^2_\gamma(Q_T)} \le A\}, \tag{4.3}$$

satisfying conditions (1.2) and (1.3), where A is a positive constant which will be specified later on. For any $u \in K(A)$, problem

$$\frac{\partial^2 v}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial v}{\partial x}\big) = \mathcal{I}u + f(x,t), \tag{4.4}$$

$$v(x,0) = Z_1(x), \quad v_t(x,0) = Z_2(x), \quad x \in (0,1) \tag{4.5}$$

$$v_x(1,t) = 0, \quad v(1,t) = 0, \quad t \in (0,T), \tag{4.6}$$

where

$$\mathcal{I}u = \frac{d}{dt}\max(\Lambda_x(\xi u), 0), \tag{4.7}$$

has a unique solution in $W^{1,1}_{\gamma,2}(Q_T)$.

Define a mapping $\mathcal{M}$ such that $v = \mathcal{M}u$. If we show that the mapping $\mathcal{M}$ admits a fixed point v in the closed bounded convex subset $K(A)$, then v will be our solution. Let us first show that the mapping $\mathcal{M}$ maps the set $K(A)$ to $K(A)$. Let $v = W + S$, such that W solves

$$\frac{\partial^2 W}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial W}{\partial x}\big) = \mathcal{I}u, \quad (x,t) \in Q_T, \tag{4.8}$$

$$W(x,0) = 0, \quad W_t(x,0) = 0, \quad x \in (0,1), \tag{4.9}$$

$$W_x(1,t) = 0, \quad W(1,t) = 0, \quad t \in (0,T), \tag{4.10}$$

and S solves

$$\frac{\partial^2 S}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial S}{\partial x}\big) = f(x,t), \quad (x,t) \in Q_T, \tag{4.11}$$

$$S(x,0) = Z_1(x), \quad S_t(x,0) = Z_2(x), \quad x \in (0,1) \tag{4.12}$$

$$S_x(1,t) = 0, \quad S(1,t) = 0, \quad t \in (0,T) \tag{4.13}$$

We now consider the following inner products in $L^2(Q_y)$, $y \in (0,T)$

$$(\mathcal{L}W, M_1W)_{L^2(Q_y)} = (\mathcal{I}u, M_1W)_{L^2(Q_y)}, \tag{4.14}$$

$$(\mathcal{L}S, M_2S)_{L^2(Q_y)} = (f, M_2S)_{L^2(Q_y)}, \tag{4.15}$$

where

$$M_1W = 2x^2W_t + x^2W_x + 2xW_t, \quad M_2S = 2x^2S_t + x^2S_x + 2xS_t.$$

By integrating by parts in (4.14) and evoking conditions (4.9) and (4.10), we have

$$
\begin{aligned}
&\|W_t(x,y)\|^2_{L^2_\sigma((0,1))}+\|W_x(x,y)\|^2_{L^2_\sigma((0,1))}+\|W_t(x,y)\|^2_{L^2_\rho((0,1))}\\
&+\|W_x(x,y)\|^2_{L^2_\rho((0,1))}+\|W_t\|^2_{L^2_\rho(Q_y)}\\
&=-\int_0^1 x^2W_x(x,y)W_t(x,y)dx-2(W_t,W_x)_{L^2_\rho(Q_y)}+2(W_t,\mathcal{I}u)_{L^2_\sigma(Q_y)}\\
&\quad+(W_x,\mathcal{I}u)_{L^2_\sigma(Q_y)}+2(W_t,\mathcal{I}u)_{L^2_\rho(Q_y)}.
\end{aligned}
\tag{4.16}
$$

Thanks to Cauchy ε-inequality, (4.16) reduces to

$$
\begin{aligned}
&\|W_t(x,y)\|^2_{L^2_\sigma((0,1))}+\|W_x(x,y)\|^2_{L^2_\sigma((0,1))}+\frac{1}{2}\|W_t(x,y)\|^2_{L^2_\rho((0,1))}\\
&+\frac{1}{2}\|W_x(x,y)\|^2_{L^2_\rho((0,1))}\\
&\le\|W_t\|^2_{L^2_\rho(Q_y)}+\|W_x\|^2_{L^2_\rho(Q_y)}+\|W_t\|^2_{L^2_\sigma(Q_y)}+\frac{1}{2}\|W_x\|^2_{L^2_\sigma(Q_y)}\\
&\quad+\|\mathcal{I}u\|^2_{L^2_\rho(Q_y)}+\frac{3}{2}\|\mathcal{I}u\|^2_{L^2_\sigma(Q_y)}.
\end{aligned}
\tag{4.17}
$$

Consider the following two elementary inequalities

$$
\|W(x,y)\|^2_{L^2_\sigma((0,1))}\le\|W\|^2_{L^2_\sigma(Q_y)}+\|W_t\|^2_{L^2_\sigma(Q_y)},\tag{4.18}
$$

$$
\|W(x,y)\|^2_{L^2_\rho((0,1))}\le\|W\|^2_{L^2_\rho(Q_y)}+\|W_t\|^2_{L^2_\rho(Q_y)}.\tag{4.19}
$$

Inequalities (4.17)–(4.19), yield

$$
\begin{aligned}
&\|W(x,y)\|^2_{L^2_\gamma((0,1))}+\|W_t(x,y)\|^2_{L^2_\gamma((0,1))}+\|W_x(x,y)\|^2_{L^2_\gamma((0,1))}\\
&\le 3\Big(\|W\|^2_{L^2_\gamma(Q_y)}+\|W_t\|^2_{L^2_\gamma(Q_y)}+\|W_x\|^2_{L^2_\gamma(Q_y)}+\|\mathcal{I}u\|^2_{L^2_\gamma(Q_y)}\Big).
\end{aligned}
\tag{4.20}
$$

Application of Gronwall's lemma to (4.20) gives

$$
\begin{aligned}
&\|W(x,y)\|^2_{L^2_\gamma((0,1))}+\|W_t(x,y)\|^2_{L^2_\gamma((0,1))}+\|W_x(x,y)\|^2_{L^2_\gamma((0,1))}\\
&\le 3e^{3T}\|\mathcal{I}u\|^2_{L^2_\gamma(Q_T)}.
\end{aligned}
\tag{4.21}
$$

Integrating of both sides of (4.21) over $(0,T)$ gives

$$
\|W(x,t)\|^2_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))}\quad\le 3Te^{3T}\|\mathcal{I}u\|^2_{L^2_\gamma(Q_T)}.\tag{4.22}
$$

We now consider (4.15) and use conditions (4.12) and (4.13), to obtain

$$
\begin{aligned}
&\|S_t(x,y)\|^2_{L^2_\sigma((0,1))}+\|S_x(x,y)\|^2_{L^2_\sigma((0,1))}+\|S_t(x,y)\|^2_{L^2_\rho((0,1))}\\
&+\|S_t(x,y)\|^2_{L^2_\rho((0,1))}+\|S_t\|^2_{L^2_\rho(Q_y)}\\
&=-\int_0^1 x^2S_x(x,y)S_t(x,y)dx+2(S_t,S_x)_{L^2_\rho(Q_y)}+2(S_t,f)_{L^2_\sigma(Q_y)}\\
&\quad+2(S_x,f)_{L^2_\sigma(Q_y)}+2(S_t,f)_{L^2_\rho(Q_y)}+\|\frac{\partial Z_1}{\partial x}\|^2_{L^2_\gamma((0,1))}\\
&\quad+\|(Z_2)^2\|^2_{L^2_\gamma((0,1))}+\int_0^1 x^2\frac{\partial Z_1}{\partial x}Z_2dx.
\end{aligned}
\tag{4.23}
$$

By using the Cauchy ε-inequality, we can transform (4.23) into

$$
\begin{aligned}
&\|S_t(x,y)\|^2_{L^2_\gamma((0,1))} + \|S_x(x,y)\|^2_{L^2_\gamma((0,1))} \\
&\le 4\Big(\|S_t\|^2_{L^2_\gamma(Q_y)} + \|S_x\|^2_{L^2_\gamma(Q_y)}\Big) \\
&\quad + 4\Big(\|(Z_1)^2_x\|^2_{L^2_\gamma((0,1))} + \|Z_2\|^2_{L^2_\gamma((0,1))} + \|f\|^2_{L^2_\gamma(Q_y)}\Big).
\end{aligned}
\tag{4.24}
$$

Consider the elementary inequality

$$
\|S(x,y)\|^2_{L^2_\gamma((0,1))} \le \|S\|^2_{L^2_\gamma(Q_y)} + \|S_t\|^2_{L^2_\gamma(Q_y)} + \|Z_1\|^2_{L^2_\gamma((0,1))}. \tag{4.25}
$$

Combination of (4.24) and (4.25) leads to

$$
\begin{aligned}
&\|S(x,y)\|^2_{W^{1,1}_{2,\gamma}(0,1)} \\
&\le 4\Big(\|S\|^2_{W^{1,1}_{2,\gamma}(Q_y)} + \|Z_1\|^2_{W^1_{2,\gamma}((0,1))} + \|Z_2\|^2_{L^2_\gamma((0,1))} + \|f\|^2_{L^2_\gamma(Q_y)}\Big).
\end{aligned}
\tag{4.26}
$$

Applying of Gronwall's lemma to inequality (4.26) and then integrating over $(0,T)$ gives

$$
\begin{aligned}
&\|S(x,y)\|^2_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))} \\
&\le 4Te^{4T}\Big(\|Z_1\|^2_{W^1_{2,\gamma}((0,1))} + \|Z_2\|^2_{L^2_\gamma((0,1))} + \|f\|^2_{L^2_\gamma(Q_y)}\Big).
\end{aligned}
\tag{4.27}
$$

It is obvious that

$$
\|\mathcal{I}u\|^2_{L^2_\gamma(Q_T)} \le C_1^2, \tag{4.28}
$$

where $C_1 > 0$.

Inequalities (4.22), (4.27) and (4.28) yield

$$
\begin{aligned}
\|v\|^2_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))} &\le 2\|W\|^2_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))} + 2\|S\|^2_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))} \\
&\le 6Te^{3T}C_1^2 + 8Te^{4T}C_2^2.
\end{aligned}
\tag{4.29}
$$

and

$$
\|v_t\|^2_{L^2_\gamma(Q_T)} \le 2\|W_t\|^2_{L^2_\gamma(Q_T)} + 2\|S_t\|^2_{L^2_\gamma(Q_T)} \le 6Te^{3T}C_1^2 + 8Te^{4T}C_2^2 \tag{4.30}
$$

If we take $A \ge 6Te^{3T}C_1^2 + 8Te^{4T}C_2^2$, we then deduce from the inequalities (4.29) and (4.30) that

$$
\|v\|^2_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))} \le A, \quad \|v_t\|^2_{L^2_\gamma(Q_T)} \le A. \tag{4.31}
$$

Hence $v \in K(A)$ and consequently $\mathcal{M}$ maps $K(A)$ into itself.

We will now show that the mapping $\mathcal{M} : K(A) \to K(A)$ is continuous. Let v_1, $v_2 \in K(A)$ and let $V_1 = \mathcal{M}(v_1)$, and $V_2 = \mathcal{M}(v_1)$.

We observe that $V = V_1 - V_2$ satisfies

$$
\frac{\partial^2 V}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial V}{\partial x}\big) = \frac{d}{dt}\max(\Lambda_x(\xi v_1),0) - \frac{d}{dt}\max(\Lambda_x(\xi v_2),0), \tag{4.32}
$$

$$
V(1,t) = 0, \quad V_x(1,t) = 0, \tag{4.33}
$$

$$
V(x,0) = 0, \quad V_t(x,0) = 0 \tag{4.34}
$$

Define the function

$$
\wp(x,t) = \int_0^t V(x,\tau)d\tau, \tag{4.35}
$$

then it follows from (4.32)–(4.35) that $\wp(x,t)$ satisfies

$$
\frac{\partial^2 \wp}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\big(x\frac{\partial \wp}{\partial x}\big) = \max(\Lambda_x(\xi v_1),0) - \max(\Lambda_x(\xi v_2),0), \tag{4.36}
$$

$$\wp(1,t)=0, \quad \wp_x(1,t)=0, \tag{4.37}$$

$$\wp(x,0)=0, \quad \wp_t(x,0)=0. \tag{4.38}$$

It is obvious that

$$\|\mathcal{H}(x,t)\|^2_{L^2_\gamma(Q_T)} \leq \|v_1-v_2\|^2_{L^2_\gamma(Q_T)}, \tag{4.39}$$

where

$$\mathcal{H}(x,t)=\max(\Lambda_x(\xi v_1),0)-\max(\Lambda_x(\xi v_2),0) \tag{4.40}$$

Then we have

$$\|V\|^2_{L^2_\gamma(Q_T)} \leq d\|v_1-v_2\|^2_{L^2_\gamma(Q_T)}, \tag{4.41}$$

$$\|V_1-V_2\|_{L^2_\gamma(Q_T)} = \|\mathcal{M}(v_1)-\mathcal{M}(v_1)\|_{L^2_\gamma(Q_T)} \leq \sqrt{d}\|v_1-v_2\|_{L^2_\gamma(Q_T)}. \tag{4.42}$$

Consequently he mapping $\mathcal{M}: K(A) \to K(A)$ is continuous. The set $\overline{K(A)}$ is compact, because of the following result. □

Theorem 4.2. *Let $B_0 \subset B \subset B_1$ with compact embedding (reflexive Banach spaces) (see [13, 17]). Assume that $\alpha,\lambda \in (0,\infty)$ and $T>0$. Then*

$$W=\{\theta : \theta \in L^\alpha(0,T;B), \theta_t \in L^\lambda(0,T;B_1)\}$$

is compactly embedded in $L^\alpha(0,T;B)$, that is the bounded sets are relatively compact in $L^\alpha(0,T;B)$.

Observe that $L^2(0,T;L^2_\gamma(0,1)) = L^2_\gamma(Q_T)$, $\mathcal{M}(K(A)) \subset K(A) \subset L^2_\gamma(Q_T)$. Then by Schauder fixed point theorem the mapping $\mathcal{M}$ admits a fixed point $v \in K(A)$.

5. A priori bound for the solution

We now obtain a priori bound for the solution of problem (1.1)–(1.3). This a priori bound can be used to establish some regularity results. We may expect the solution of (1.1)–(1.3) to be in the function space $L^2(0,T;W^{1,1}_{q,\gamma}(0,1))$ with $q \leq \infty$.

Theorem 5.1. *Let $u \in L^2(0,T;W^{1,1}_{2,\gamma}(0,1))$, be a solution of problem (1.1)–(1.3), then following a priori bound holds*

$$\begin{aligned} &\|u\|^2_{L^2(0,T;W^{1,1}_{2,\gamma}(0,1))} \\ &\leq 8Te^{8T}\Big(\|Z_1\|^2_{W^1_{2,\gamma}((0,1))} + \|Z_2\|^2_{L^2_\gamma((0,1))} + \|f\|^2_{L^2_\gamma(Q_T)}\Big) \end{aligned} \tag{5.1}$$

Proof. Note that

$$\begin{aligned} &\int_0^s \frac{d}{dt}\int_0^1 x^2 u_t^2 dx \\ &= 2\int_0^s\int_0^1 x^2 u_t u_{tt}\,dx\,dt = 2\int_0^s\int_0^1 x^2 u_t\Big[\frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x}) + \mathcal{I}u\Big]\,dx\,dt \\ &= 2\int_0^s x^2(u_t(1,t)u_x(1,t) - x^2u_t(0,t)u_x(0,t))dt - 2\int_0^s\int_0^1 x^2 u_{tx}u_x\,dx\,dt \\ &\quad - 2\int_0^s\int_0^1 x u_t u_x\,dx\,dt + 2\int_0^s\int_0^1 x^2 u_t \mathcal{I}u\,dx\,dt + 2\int_0^s\int_0^1 x^2 u_t f\,dx\,dt \end{aligned} \tag{5.2}$$

By using boundary and initial conditions (3.2) and (3.3), then from (5.2) one has

$$
\begin{aligned}
&\int_0^1 x^2u_t^2(x,s)dx + \int_0^1 x^2u_x^2(x,s)dx \\
&= \int_0^1 x^2Z_2^2(x)dx + \int_0^1 x^2(\frac{\partial Z_1}{\partial x})^2dx - 2\int_0^s\int_0^1 xu_tu_x\,dx\,dt \\
&\quad + 2\int_0^s\int_0^1 x^2u_t\mathcal{I}u\,dx\,dt + 2\int_0^s\int_1^1 x^2u_tf\,dx\,dt.
\end{aligned} \tag{5.3}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^s\int_0^1 \Big[x^2u_xu_{tt} - \frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})(x\frac{\partial u}{\partial x})\Big]\,dx\,dt \\
&= \int_0^1 x^2u_x(x,s)u_t(x,s)dx + \int_0^s\int_0^1 xu_t^2\,dx\,dt \\
&= \int_0^s\int_0^1 x^2u_x\mathcal{I}u\,dx\,dt + \int_0^s\int_1^0 x^2u_xf\,dx\,dt,
\end{aligned} \tag{5.4}
$$

and

$$
\begin{aligned}
&2\int_)^s\int_0^1 \Big[xu_tu_{tt} - \frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})u_t\Big]dx \\
&= \int_0^1 xu_t^2(x,T)dx + \int_0^1 xu_x^2(x,T)dx \\
&= 2\int_0^s\int_0^1 xu_t\mathcal{I}u\,dx\,dt + 2\int_0^s\int_0^1 xu_tf\,dx\,dt + \int_0^1 xZ_2^2(x)dx \\
&\quad + \int_0^1 x(\frac{\partial Z_1}{\partial x})^2dx.
\end{aligned} \tag{5.5}
$$

Equalities (5.3)–(5.5) and the elementary equality

$$
\begin{aligned}
&\int_0^1 (x+x^2)u^2(x,s)dx \\
&\le \int_0^s\int_0^1 (x+x^2)u^2\,dx\,dt + \int_0^s\int_0^1 (x+x^2)u_t^2\,dx\,dt \\
&\quad + \int_0^1 (x+x^2)Z_1^2(x)dx,
\end{aligned} \tag{5.6}
$$

lead to

$$
\begin{aligned}
&\int_0^1 x^2u_t^2(x,s)dx + \int_0^1 x^2u_x^2(x,s)dx + \int_0^s\int_0^1 xu_t^2\,dx\,dt \\
&+ \int_0^1 xu_t^2(x,s)dx + \int_0^1 xu_x^2(x,s)dx \\
&= \int_0^1 (x^2+x)Z_2^2(x)dx + \int_0^1 (x^2+x)(\frac{\partial Z_1}{\partial x})^2dx - 2\int_0^s\int_0^1 xu_tu_x\,dx\,dt \\
&\quad + 2\int_0^s\int_0^1 x^2u_t\mathcal{I}u\,dx\,dt + 2\int_0^s\int_0^1 x^2u_tf\,dx\,dt + \int_0^1 (x+x^2)Z_1^2(x)dx
\end{aligned}
$$

$$+\int_0^s\int_0^1 x^2u_x\mathcal{I}u\,dx\,dt+\int_0^s\int_0^1 x^2u_xf\,dx\,dt+\int_0^1 x^2u_x(x,s)u_t(x,s)dx$$
$$+2\int_0^s\int_0^1 xu_t\mathcal{I}u\,dx\,dt+2\int_0^s\int_0^1 xu_tf\,dx\,dt.$$

Young's inequality and a Poincaré type of inequality [17], transforms the above inequality into

$$\begin{aligned}
&\int_0^1(x+x^2)u^2(x,s)dx+\int_0^1(x+x^2)u_t^2(x,s)dx+\int_0^1(x+x^2)u_x^2(x,s)dx\\
&\le 8\Big(\int_0^s\int_0^1(x+x^2)u^2\,dx\,dt+\int_0^s\int_0^1(x+x^2)u_t^2\,dx\,dt+\int_0^s\int_0^1(x+x^2)u_x^2\,dx\,dt\Big)\\
&\quad+8\Big(\int_0^1(x^2+x)Z_2^2(x)dx+\int_0^1(x+x^2)\Big(Z_1^2(x)+\Big(\frac{\partial Z_1}{\partial x}\Big)^2\Big)dx\\
&\quad+\int_0^s\int_0^1(x+x^2)f^2\,dx\,dt\Big).
\end{aligned}$$

Using Gronwall's lemma [14, Lemma 4.1] this implies

$$\|u(x,s)\|^2_{W^{1,1}_{2,\gamma}(0,1))}\le 8Te^{8T}\Big(\|Z_1\|^2_{W^1_{2,\gamma}((0,1))}+\|Z_2\|^2_{L^2_\gamma((0,1))}+\|f\|^2_{L^2_\gamma(Q_T)}\Big)$$

Integrating with respect to s over the interval $[0,T]$ gives the a priori bound (5.1). □

Conclusion. The well posedness of a one point initial boundary value problem for a nonlinear hyperbolic singular integro-differential equation is proved. The Schauder fixed point theorem is the main tool used to establish the existence of solution. The solvability of the problem is proved when the given data are related to a weighted Sobolev space. A priori bound for the solution may allow to study the regularity of the solution is obtained.

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References

[1] K. Adolfsson, M. Enelund, S. Larsson, M. Racheva; *Discretization of integro-differential equations modeling dynamic fractional order viscoelasticity*, LNCS 3743 (2006), 76–83.

[2] G. A. Baker; *Error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal. 13 (1976), 564–576.

[3] K. Balachandran, J. Y. Park; *Existence of mild solution of a functional integro-differential equation with nonlocal condition.* Bulletin of the Korean Mathematical Society, 2001, 38: 175–182

[4] G. Barles, C. Imbert; *Second-order elliptic integro-differential equations: viscosity solutions' theory revisited*, Ann. l'Inst. H. Poincaré. Anal. Non Linéaire., 25, No. 3 (2008), 567–585.

[5] S. Berrimi, S. A. Messaoudi; *Existence and decay of solutions of a viscoelastic equation with a nonlinear source*, Nonlinear Anal. 64 (2006) 2314–2331.

[6] M. M. Bokalo, O. V. Domanska; *Diruchlet problem for stationary anisotropic higher order partial integro-differential equations with variable exponents of nonlinearity*, Journal of Mathematical Sciences, Vol. 201, No. 1, August, 2014.

[7] M. M. Bokalo, V. M. Dmytriv; *Boundary-value problems for integro-differential equations in anisotropic spaces*, Visn. Lviv. Univ. Ser. Mekh.-Mat., Issue 59, 84–101 (2001).

[8] L. Caffarelli, L. Silvestre; *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math., 62, No. 5 (2009), 597–638.
[9] P. Cannarsa, D. Sforza; *Semilinear integro-differential equations of hyperbolic type: existence in the large*, Mediterr. J. Math. 1 (2004) 151–174.
[10] M. Chipot, S. Guesmia; *On a class of integro-differential problems*, Comm. Pure Appl. Anal., 9, No. 5 (2010), 1249–1262.
[11] R. Cont, E. Voltchkova; *Integro-differential equations for option prices in exponential Lévy models*, Finance Stochast., 9, No. 3 (2005), 299–325.
[12] A. A. Elbeleze, A. Kilicman, B. M. Taib; *Application of homotopy perturbation and variational iteration method for Fredholm integro-differential equation of fractional order*. Abstract & Applied Analysis, 2012, 2012: 14 pages.
[13] J.-L. Lions; *Équations Différentielles Opérationnelles et Problèmes aux Limites*, vol. 111 of Die Grundlehren der mathematischen Wissenschaften, Springer, Berlin, Germany, 1961.
[14] S. Mesloub; *A nonlinear nonlocal mixed problem for a second order parabolic equation*. J. Math. Anal. Appl. 316, (2006), 189-209.
[15] M. Renardy, W. J. Hrusa, J. A. Nohel; *Mathematical Problems in Viscoelasticity*, Pitman Monogr. Pure Appl. Math., vol. 35, Longman Sci. Tech., Harlow, Essex, 1988.
[16] M. Renardi, W. J. Hrusa, J. A. Nohel; *Mathematical problems in viscoelasticity*, Pitman Monographs and Surveys Pure Appl. Math. No. 35, Wiley, 1987.
[17] J. Simon; *Compact sets in the space* $L^p(0,T;B)$, Annali di Matematica Pura ed Applicata, vol. 146, no. 1, pp. 65–96, 1987.
[18] I. H. Sloan, V. Thomee; *Time discretization of an integro-differential equation of parabolic type*, SIAM J. Numer. Anal. 23 (1986), 1052–1061.
[19] Weiqing Xie, A class of nonlinear parabolic integro-differential equations, Differential and integral equations, Volume 6, Number 3, (1993), 627-642.

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