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WELL-POSEDNESS AND EXPONENTIAL STABILITY FOR A WAVE EQUATION WITH NONLOCAL TIME-DELAY CONDITION

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Abstract. Well-posedness and exponential stability of nonlocal time-delayed of a wave equation with a integral conditions of the 1st kind forms the center of this work. Through semigroup theory we prove the well-posedness by the Hille-Yosida theorem and the exponential stability exploring the dissipative properties of the linear operator associated to damped model using the Gearhart-Huang-Pruss theorem.

1. Introduction

Let $\Omega = (0,1)$ be an interval in $\mathbb{R}$, $(x,t) \in \Omega \times (0,\infty)$ and a, b be positive constants. We denote by $u = u(x,t)$ the small transversal displacements of x at the time t. The wave equation with frictional damping is modeled by

$$u_{tt} - au_{xx} + bu_t = 0. \tag{1.1}$$

Nonlocal time-delayed wave equation forms the center of this work. One of the first approach for a model with delay was given by Ludwig Eduard Boltzmann (1844-1906) who studied retarded elasticity effects. Charles Émile Picard (1856-1941) took the view that the past states are important for a realistic modelling although the Newtonian tradition claimed the opposite. The need for delays was emphasized both by Lotka and by Volterra independently of each other, Alfred J. Lotka (1880-1949) in the United States and Vito Volterra (1860-1940) in Italy. They introduced the Lotka-Volterra equations, also known as the predator-prey equations. Andrey Nikolaevich Kolmogorov (1903-1987) introduced the model which is a more general framework that can model the dynamics of ecological systems with predator-prey interactions, competition, disease, and mutualism. Anatoliy Myshkis (1920-2009) gave the first correct mathematical formulation and introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems. In fact, time delays so often arise in many physical, chemical, biological and economical phenomena (see [36] and references therein). Whenever that material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission and in this direction the

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delay is the property of a physical system by which the response to an applied restoring force is delayed in its transmission effect (see [35]). Unluckily, the delay can becomes a source of instability. In the work [11], a small delay in a boundary control could turn the well-behaved hyperbolic system into a wild one was shown. See for example [10, 16, 24, 23, 37] where an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used.

The history of nonlocal problems with integral conditions for partial differential equations is recent and goes back to [9]. In [4], a review of the progress in the nonlocal models with integral type was given with many discussions related to physical justifications, advantages, and numerical applications. For nonlocal problem for a hyperbolic equation with integral conditions of the 1st kind we cite [31]. We define the nonlocal time-delay integral of the 1st kind condition by

$$\int_0^c F(s)u_t(x,t-s)ds. \tag{1.2}$$

This kind of condition (1.2) is called nonlocal because the integral is not a pointwise relation. The nonlocal terms provoke some mathematical difficulties which makes the study of such a problem particularly interesting. For the last several decades, various types of equations have been employed as some mathematical models describing physical, chemical, biological and ecological systems. See for example the nonlocal reaction-diffusion system given in [32] and reference therein. In [20] the authors considered a nonlocal problem for a hyperbolic equation in n space variables with a different integral condition. For mixed problems with nonlocal integral conditions in one-dimensional hyperbolic equation, we cite the works [28, 7, 15, 5, 29, 30]. For a nonlocal problem for wave equation with integral condition on a cylinder, we cite [6] where the existence of a generalized solution by Galerkin procedure was proved. In [8], Cavalcanti et al recently considered a nonlinear wave equation with a degenerate and nonlocal damping term. They proved the exponential stability borrowing some ideas in [13, 12]. The generelized solution for a mixed nonlocal system of wave equation was given in [34]. Stability for coupled wave system has been considered in several works, for example in [19, 1, 2, 33, 22] among others.

For $\Omega \subset \mathbb{R}^n$ a open bounded domain with a smooth boundary, in [24] was considered the system with internal feedback

$$u_{tt} - \Delta u + \mu_0 u_t + \int_{\tau_1}^{\tau_2} a(s)\mu(s)u_t(x,t-s)ds = 0, \quad (x,t) \in \Omega \times (0,\infty)$$

and assuming

$$\mu_0 \geq \|a\|_\infty \int_{\tau_1}^{\tau_2} \mu(s)\,ds$$

was proved the exponential decay of solution by Energy Method, that consists in use of suitable multiplies to construct a Lyapunov functional, equivalent to energy functional, that decay exponentially.

Recently, using the Energy Method, Pignotti [26] studied the asymptotic and exponential stability results under suitable conditions for the abstract model of second-order evolution equations (1.3)

$$u_{tt} + Au + \mu_0 u_t - \int_0^\infty \mu(s)\mu(s)Au(x,t-s)ds = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.3}$$

where a viscoelastic damping takes the place of the standard frictional term and the delay feedback is intermittent onoff in time, being $A : D(A) \to H$ a positive self-adjoint operator with compact inverse in a real Hilbert space H.

In this work we use a different approach by semigroup technique and we prove the well-posedness and exponential stability for a wave equation with frictional damping and nonlocal time-delayed condition given by

$$u_{tt} - au_{xx} + bu_t + \int_0^c F(s)u_t(x, t-s)ds = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.5}$$

$$u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.6}$$

$$u_t(x,-s) = f_0(x,-s), \quad x \in \Omega,\ s \in (0,c), \tag{1.7}$$

and satisfying the Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0. \tag{1.8}$$

Here the initial data $u_0(x) \in H_0^1(0,1)$, $u_1(x) \in L^2(0,1)$, $f_0(x,s)$ belongs to suitable space and

$$\int_0^c F(s)ds < b.$$

We use the Sobolev spaces and semigroup theory with its properties as in [3, 25, 21]. This article is organized as follows. In section 2, we present some notation and assumptions needed to establish the well-posedness. In section 3, we prove the exponential stability using the Gearhart-Huang-Pruss theorem (see [14, 17, 27]).

2. Well-posedness

As in Nicaise and Pignotti [24] we introduce the new variable

$$z(x,\rho,t,s) = u_t(x, t-\rho s), \quad (x,\rho) \in Q = \Omega \times \Omega, \quad t > 0,\ s \in (0,c).$$

The new variable z satisfies

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0. \tag{2.1}$$

Moreover, using the approach as in [23], the equation

$$\lambda sz(x,\rho,t,s) + z_\rho(x,\rho,t,s) = f, \quad \text{with } \lambda > 0,\ f \in L^2(Q \times (0,c)) \tag{2.2}$$

has a unique solution

$$z(x,\rho,s) = z(x,0,s)e^{-\lambda\rho s} + se^{\lambda\rho s}\int_0^\rho e^{\lambda\sigma s} f(x,\sigma,s)d\sigma. \tag{2.3}$$

Consequently, problem (1.4)-(1.7) is equivalent to

$$u_{tt} - au_{xx} + bu_t + \int_0^c F(s)z(x,1,t,s)ds = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{2.4}$$

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, (x,\rho) \in Q, t > 0, s \in (0,c), \tag{2.5}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.6}$$

$$u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{2.7}$$

$$z(x,1,s) = f_0(x,-s), \quad x \in \Omega,\ s \in (0,c), \tag{2.8}$$

with the Dirichlet boundary condition (1.8) and $z(x,\rho,t,s) = 0$ on the boundary. Defining $U = (u, v, z)^T$, $v = u_t$, we formally get that U satisfies the Cauchy problem

$$\begin{gathered} U_t = \mathcal{A}U \quad t > 0, \\ U(0) = U_0 = (u_0, v_0, f_0)^T, \end{gathered} \tag{2.9}$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}U = \begin{bmatrix} v \\ au_{xx} - bv - \int_0^c F(s)z(x,1,t,s)ds \\ -s^{-1}z_\rho(x,\rho,t,s) \end{bmatrix}.$$

We introduce the energy space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(Q \times (0,c))$$

equipped with the inner product

$$\langle U, \bar{U}\rangle_{\mathcal{H}} = \int_\Omega (au_x\bar{u}_x + v\bar{v})\,dx + \int_\Omega\int_\Omega \Big[\int_0^c sF(s)z(x,\rho,s)\bar{z}(x,\rho,s)ds\Big]d\rho\,dx.$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(Q \times (0,c)).$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$ and independent of time $t > 0$. Next, we prove that the operator $\mathcal{A}$ is dissipative.

Proposition 2.1. *For $U = (u, v, z) \in D(\mathcal{A})$ we have*

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le -(b - \int_0^c F(s)ds) \int_\Omega v^2\,dx. \tag{2.10}$$

Proof. We have

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \int_\Omega \Big\{av_xu_x + \Big[au_{xx} - bv - \int_0^c F(s)z(x,1,s)ds\Big]v\Big\}\,dx \\ &\quad - \int_\Omega\int_\Omega \Big[\int_0^c F(s)z_\rho(x,\rho,s)z(x,\rho,s)ds\Big]d\rho\,dx. \end{aligned}$$

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= a\int_\Omega v_xu_x\,dx + a\int_\Omega u_{xx}v\,dx - b\int_\Omega v^2\,dx \\ &\quad - \int_\Omega\int_0^c F(s)z(x,1,s)v\,ds\,dx \\ &\quad - \int_\Omega\int_\Omega \Big[\int_0^c F(s)z_\rho(x,\rho,s)z(x,\rho,s)ds\Big]d\rho\,dx. \end{aligned}$$

Integrating by parts in Ω,

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -b\int_\Omega v^2\,dx \\ &\quad - \int_\Omega\int_0^c F(s)\,z(x,1,s)\,v\,ds\,dx \\ &\quad - \int_\Omega\int_\Omega \Big[\int_0^c F(s)z_\rho(x,\rho,s)z(x,\rho,s)ds\Big]d\rho\,dx. \end{aligned}$$

Using $z(x,0,s)=u_t(x,t)=v$ note that

$$\begin{aligned}
&\int_\Omega\int_\Omega\Big[\int_0^c F(s)z_\rho(x,\rho,s)z(x,\rho,s)ds\Big]d\rho\,dx\\
&=\int_\Omega\int_0^c\Big[\int_\Omega F(s)\frac{1}{2}\frac{d}{d\rho}|z(x,\rho,s)|^2d\rho\Big]ds\,dx\\
&=\frac{1}{2}\int_\Omega\int_0^c F(s)|z(x,1,s)|^2ds\,dx-\frac{1}{2}\int_\Omega\int_0^c F(s)v^2ds\,dx.
\end{aligned}$$

Then we have

$$\begin{aligned}
\langle\mathcal{A}U,U\rangle_{\mathcal{H}}&=-b\int_\Omega v^2\,dx-\int_\Omega\int_0^c F(s)\,z(x,1,s)\,v\,ds\,dx\\
&\quad-\frac{1}{2}\int_\Omega\int_0^c F(s)|z(x,1,s)|^2ds\,dx+\frac{1}{2}\int_0^c F(s)ds\int_\Omega v^2\,dx.
\end{aligned}$$

Applying Young's inequality,

$$\begin{aligned}
\langle\mathcal{A}U,U\rangle_{\mathcal{H}}&=-b\int_\Omega v^2\,dx+\frac{1}{2}\int_\Omega\int_0^c F(s)|z(x,1,s)|^2ds\,dx+\frac{1}{2}\int_0^c F(s)ds\int_\Omega v^2\,dx\\
&\quad-\frac{1}{2}\int_\Omega\int_0^c F(s)|z(x,1,s)|^2ds\,dx+\frac{1}{2}\int_0^c F(s)ds\int_\Omega v^2\,dx.
\end{aligned}$$

From where it follows that

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}\le-(b-\int_0^c F(s)ds)\int_\Omega v^2\,dx.$$

□

The well-posedness of (2.4)-(2.8) is ensured by the following theorem.

Theorem 2.2. *For $U_0\in\mathcal{H}$, there exists a unique weak solution U of (2.9) satisfying*

$$U\in C((0,\infty);\mathcal{H}). \tag{2.11}$$

Moreover, if $U_0\in D(\mathcal{A})$, then

$$U\in C((0,\infty);D(\mathcal{A}))\cap C^1((0,\infty);\mathcal{H}). \tag{2.12}$$

Proof. We will use the Hille-Yosida theorem. Since $\mathcal{A}$ is dissipative and $D(\mathcal{A})$ is dense in $\mathcal{H}$, it is sufficient to show that $\mathcal{A}$ is maximal; that is, $I-\mathcal{A}$ is surjective. Given $G=(g_1,g_2,g_3)\in\mathcal{H}$, we prove that there exists $U=(u,v,z)\in D(\mathcal{A})$ satisfying $(I-\mathcal{A})U=G$ which is equivalent to

$$u-v=g_1\in H_0^1(\Omega), \tag{2.13}$$

$$v-au_{xx}+bv+\int_0^c F(s)z(x,1,s)ds=g_2\in L^2(\Omega), \tag{2.14}$$

$$sz(x,\rho,s)+z_\rho(x,\rho,s)=sg_3\in L^2(\Omega\times(0,c)). \tag{2.15}$$

From (2.2),(2.3) it follows that equation (2.15) has a unique solution given by

$$z(x,\rho,s)=ve^{-\rho s}+se^{\rho s}\int_0^\rho e^{\sigma s}g_3(x,\sigma,s)d\sigma. \tag{2.16}$$

From this and (2.14) we obtain

$$(1+b)u-au_{xx}=g\in L^2(\Omega) \tag{2.17}$$

where

$$g = g_1 + g_2 - \int_0^c F(s)z(x,1,s)ds.$$

We can reformulate (2.17) as follows

$$\int_\Omega ((1+b)u - au_{xx})\omega\, dx = \int_\Omega g\omega\, dx \text{for all } \omega \in H_0^1(\Omega).$$

Integrating by parts,

$$(1+b)\int_\Omega u\omega\, dx + a\int_\Omega u_x\omega_x\, dx = \int_\Omega g\omega\, dx \quad \text{for all } \omega \in H_0^1(\Omega), \tag{2.18}$$

that can be written as the variational problem

$$\phi(u,\omega) = L(\omega), \quad \text{for all } \omega \in H_0^1(\Omega).$$

By the properties of the $H_0^1(\Omega)$, we have that ϕ is continuous and coercive. Naturally L is continuous. Applying the Lax-Milgram Theorem, problem (2.18) admits a unique solution

$$u \in H_0^1(\Omega), \quad \text{for all } \omega \in H_0^1(\Omega).$$

By elliptical regularity [18, Theorem 3.3.3, page 135.], it follows from (2.17) that $u \in H^2(\Omega)$, and then

$$u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Note that from (2.13) and (2.16), it implies $v \in H_0^1(\Omega)$ and $z \in L(Q \times (0,c))$ respectively and then $(u,v,z) \in D(\mathcal{A})$. Thus the operator $(I - \mathcal{A})$ is surjective. As consequence of the Hille-Yosida theorem [21, Theorem 1.2.2, page 3], we have that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. From semigroup theory, $U(t) = e^{t\mathcal{A}}U_0$ is the unique solution of (2.9) satisfying (2.11) and (2.12). The proof is complete. □

3. Exponential stability

The necessary and sufficient conditions for the exponential stability of the C_0-semigroup of contractions on a Hilbert space were obtained by Gearhart [14] and Huang [17] independently, see also Pruss [27]. We will use the following result due to Gearhart.

Theorem 3.1. *Let $\rho(\mathcal{A})$ be the resolvent set of the operator $\mathcal{A}$ and $S(t) = e^{t\mathcal{A}}$ be the C_0-semigroup of contractions generated by $\mathcal{A}$. Then, $S(t)$ is exponentially stable if and only if and only if*

$$i\mathbb{R} = \{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A}), \tag{3.1}$$

$$\limsup_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\| < \infty. \tag{3.2}$$

The main result of this manuscript is the following theorem.

Theorem 3.2. *The semigroup $S(t) = e^{t\mathcal{A}}$ generated by $\mathcal{A}$ is exponentially stable.*

Proof. It is sufficient to verify (3.1) and (3.2). If (3.1) is not true, it means that there is a $\beta \in \mathbb{R}$ such that $\beta \neq 0$, β is in the spectrum de $\mathcal{A}$. From the compact immersion of $D(\mathcal{A})$ in $\mathcal{H}$, there is a vector function

$$U = (u,v,z) \in D(\mathcal{A}), \text{ with } \|U\|_{\mathcal{H}} = 1$$

such that $\mathcal{A}U = i\beta U$, which is equivalent to

$$i\beta u - v = 0, \tag{3.3}$$

$$i\beta v - au_{xx} + bv + \int_0^c F(s)z(x,1,s)\,ds = 0, \tag{3.4}$$

$$i\beta sz(x,\rho,s) + z_\rho(x,\rho,s) = 0. \tag{3.5}$$

Using (3.3) we obtain $v_x = i\beta u_x$. Multiplying by v_x, integrating and using Young's inequality we have

$$\int_\Omega |v_x|^2\,dx = i\beta \int_\Omega u_x v_x\,dx \le -\frac{1}{2}\beta^2 \int_\Omega |u_x|^2\,dx + \frac{1}{2}\int_\Omega |v_x|^2\,dx,$$

from where it follows that

$$\frac{1}{2}\beta^2 \int_\Omega |u_x|^2\,dx + \frac{1}{2}\int_\Omega |v_x|^2\,dx \le 0. \tag{3.6}$$

Applying Poincaré's inequality in (3.6) we obtain $u = v = 0$ a.e. in $L^2(\Omega)$.

Note that (2.3) gives us $z = ve^{-i\beta\rho s}$ as the unique solution of (3.5). Using the Euler formula for complex numbers we have

$$z^2 = v^2[\cos(2\beta\rho s) - i\sin(2\beta\rho s)].$$

Taking the real part, integrating on $\Omega \times \Omega \times (0,c)$ and remember that $v = u_t(x,t)$ we obtain

$$\int_\Omega\int_\Omega\int_0^c z^2(x,\rho,s)\,d\rho\,ds\,dx \le \int_\Omega\int_\Omega\int_0^c v^2\,dx\ leqc \int_\Omega v^2\,dx \le 0,$$

which implies $z = 0$ a.e. in $L^2(Q \times (0,c))$. But $u = v = z = 0$ is a contradiction with $\|U\|_{\mathcal{H}} = 1$ and then (3.1) holds.

To prove (3.2) we use contradiction argument again. If (3.2) is not true, there exists a real sequence β_n, with $\beta_n \to \infty$ and a sequence of vector functions $V_n \in \mathcal{H}$ that satisfies

$$\frac{\|(\lambda_n I - \mathcal{A})^{-1}V_n\|_{\mathcal{H}}}{\|V_n\|_{\mathcal{H}}} \ge n, \quad \text{where } \lambda_n = i\beta_n.$$

Hence

$$\|(\lambda_n I - \mathcal{A})^{-1}V_n\|_{\mathcal{H}} \ge n\|V_n\|_{\mathcal{H}}. \tag{3.7}$$

Since $\lambda_n \in \rho(\mathcal{A})$ it follows that there exists a unique sequence $U_n = (u_n, v_n, z_n) \in D(\mathcal{A})$ with unit norm in $\mathcal{H}$ such that

$$(\lambda_n I - \mathcal{A})^{-1}V_n = U_n.$$

Denoting $\xi_n = \lambda_n U_n - \mathcal{A}U_n$ we have from (3.7) that

$$\|\xi_n\|_{\mathcal{H}} \le \frac{1}{n}$$

and then $\xi_n \to 0$ strongly in $\mathcal{H}$ as $n \to \infty$.

Taking the inner product of ξ_n with U_n we have

$$\lambda_n\|U_n\|^2_{\mathcal{H}} - \langle \mathcal{A}U_n, U_n\rangle_{\mathcal{H}} = \langle \xi_n, U_n\rangle_{\mathcal{H}}.$$

Using proposition 2.1

$$\lambda_n\|U_n\|^2_{\mathcal{H}} + (b - \int_0^c F(s)ds)\int_\Omega v_n^2\,dx = \langle \xi_n, U_n\rangle_{\mathcal{H}}$$

and taking the real part we have

$$\Big(b-\int_0^c F(s)ds\Big)\int_\Omega v_n^2\,dx=\operatorname{Re}\langle\xi_n,U_n\rangle_{\mathcal{H}}.$$

As U_n is bounded and $\xi_n\to 0$ we obtain

$$v_n\to 0\quad\text{as } n\to\infty. \tag{3.8}$$

Now, for $\xi_n=(\xi_n^1,\xi_n^2,\xi_n^3)$, $\xi_n=\lambda_nU_n-\mathcal{A}U_n$ is equivalent to

$$i\beta_nu_n-v_n=\xi_n^1\to 0\quad\text{in } H_0^1(\Omega), \tag{3.9}$$

$$i\beta_nv_n-au_{n_{xx}}+bv_n+\int_0^c F(s)z_n(x,1,s)ds=\xi_n^2\to 0\quad\text{in } L^2(\Omega), \tag{3.10}$$

$$i\beta_nsz_n(x,\rho,s)+z_{n,\rho}(x,\rho,s)=\xi_n^3\to 0\quad\text{in } L^2(Q\times(0,c)). \tag{3.11}$$

By (2.3),

$$z_n(x,\rho,s)=v_ne^{-i\beta_n\rho s}+se^{-i\beta_n\rho s}\int_0^\rho e^{i\beta_n\sigma s}\xi_n^3(x,\sigma,s)\,d\sigma. \tag{3.12}$$

Using the Euler formula for complex numbers in (3.12) we obtain

$$\begin{aligned}z_n&=[\cos^2(\beta_n\rho s)-\sin^2(\beta_n\rho s)]s\int_0^\rho \xi_n^3(x,\sigma,s)\,d\sigma\\&\quad -i[2\cos(\beta_n\rho s)\sin(\beta_n\rho s)]s\int_0^\rho \xi_n^3(x,\sigma,s)\,d\sigma.\end{aligned}$$

Taking the real part we obtain

$$|z_n|\le 2\int_0^\rho \xi_n^3(x,\sigma,s)d\sigma$$

deducing that

$$z_n\to 0\quad\text{as } n\to\infty. \tag{3.13}$$

As an immediate consequence of (3.12),

$$\int_0^\rho F(s)z_n(\rho,1,s)\,ds\to 0\quad\text{as } n\to\infty. \tag{3.14}$$

Now we prove that $u_n\to 0$. Using (3.9) and (3.10) we have

$$-\beta_n^2-au_{n_{xx}}=f_n(x),\quad\text{where } f_n(x)=\xi_n^2+b\xi_n^1-\int_0^c F(s)z_n(x,1,s)\,ds. \tag{3.15}$$

Multiplying (3.15) by u_n, integrating by parts and applying Poincaré's inequality, we obtain

$$\frac{a}{C_p}\int_0^1|u_n|^2\,dx\le\int_0^1(\beta_nu_n)^2\,dx+\int_0^1 f_n(x)u_n\,dx.$$

Writing

$$\int_0^1 f_n(x)u_n\,dx=\int_0^1\big[\frac{\sqrt{C_p}}{\sqrt{a}}f_n(x)\big]u_n\big[\frac{\sqrt{a}}{\sqrt{C_p}}u_n\big]\,dx$$

and applying Young's inequality we obtain

$$\frac{a}{C_p}\int_0^1|u_n|^2\,dx\le\int_0^1(\beta_nu_n)^2\,dx+\frac12\frac{C_p}{a}\int_0^1|f_n(x)|^2\,dx+\frac12\frac{a}{C_p}\int_0^1|u_n|^2\,dx.$$

So, we have

$$\frac{1}{2}\frac{a}{C_p}\int_0^1 |u_n|^2\,dx \le \int_0^1 (\beta_n u_n)^2\,dx + \frac{1}{2}\frac{C_p}{a}\int_0^1 |f_n(x)|^2\,dx. \tag{3.16}$$

From (3.8) and (3.9) we have

$$\beta_n u_n \to 0 \tag{3.17}$$

and by (3.14)

$$f_n(x) \to 0. \tag{3.18}$$

Using (3.17) and (3.18) in (3.16) we obtain

$$u_n \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

Finally, (3.8), (3.14) and (3.19) give us a contradiction with $\|U_n\|_{\mathcal{H}} = 1$. The proof is complete. □

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