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# CRITERIA AND ESTIMATES FOR DECAYING OSCILLATORY SOLUTIONS FOR SOME SECOND-ORDER QUASILINEAR ODES

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*Dedicated to the family Teku Kuate Kamguem Ebenizer*

Abstract. Oscillation criteria for the solutions of quasilinear second order ODE are revisited. In our early works [6, 7], we obtained basic oscillation criteria for

$$\{\phi_\alpha(u'(t))\}' + \alpha c(t)\phi_\beta(u(t)) = 0$$

by estimating of the diameters of the nodal sets of the solutions. The focus of this work is to estimate the decay of the oscillatory solutions. Let $u$ be a strongly oscillatory solution, $(t_m)$ the increasing sequence of zeros of $u'$, and $D_m$ the nodal set of $u$ that contains $t_m$. We estimate $|u(t_m)|_\infty := \max_{t\in D_m}|u(t)|$ and the diameter of $D_m$ as $m \to \infty$.

## 1. Introduction

For some constants $b, \beta, q, c_0, \alpha > 0$ and $\phi_\gamma(S) := |S|^{\gamma-1}S$ (with $\gamma > 0$), we consider problems of the type

$$\begin{gathered}\{\phi_\alpha(u'(t))\}' + \alpha c(t)\phi_\beta(u(t)) = 0, \quad t > 0;\\ u(0) = 0, \quad u'(0) = b > 0,\end{gathered} \tag{1.1}$$

where $c \in C^1(\mathbb{R}^+, (c_0, \infty))$, with $c' > 0$ and $c(t) = O(t^q)$ as $t \to \infty$. We will review some oscillation criteria for such equations and establish estimates of the decay of oscillatory solutions of (1.1).

**Definition 1.1.** A function $u$ is said to be oscillatory if it has a zeros in every exterior domain $\Omega_T := (T, \infty)$ with $T \ge 0$. A function $u$ is said to be strongly oscillatory if its zeros are isolated, or if it has nodal sets in every $\Omega_T$. A nodal set of a function $v$ is an interval $D(v) = [t, s]$ such that $v(t) = v(s) = 0$ and $v \ne 0$ in $(t, s)$. For the function $v^+(t) = \max\{0, v(t)\}$, the nodal set $D(v^+) = [t, s]$ is such that $v(t) = v(s) = 0$ with $v > 0$ in $(t, s)$.

An equation (or a problem) is said to be oscillatory in $\Omega_T$ if its bounded and non-trivial solutions belong to $C^2(\Omega_T)$ and are strongly oscillatory.

For a strongly oscillatory function $u$, $[D(u)]$ will denote the set of nodal sets of $u$. In this case there are two increasing sequences $(x_k)$ and $(t_k)$ such that $x_k < t_k < x_{k+1}$, $u(x_k) = 0$, and $u'(t_k) = 0$. We denote $D_k := D_k(u) = [x_k, x_{k+1}]$ as

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a nodal set of $u$. We denote $|u(t_k)| := \max_{D_k(u)} |u(t)|$. For $a, b \in \mathbb{R}$, we define $a \wedge b := \min\{a, b\}$.

Our main result for problem (1.1) reads as follows.

**Theorem 1.2.** *For all $b, \alpha, \beta, c_0 > 0$, any non-trivial and bounded solution of* (1.1) *is strongly oscillatory in $[0, \infty)$. With the corresponding elements as defined above. as $m \to \infty$, with $\beta_* := \alpha \wedge \beta$, we have*

$$\frac{\pi_\alpha}{[c(x_{m+1})]^{1/(\beta_*+1)}} < |x_{m+1} - x_m| < \frac{\pi_\alpha}{[c(x_m)]^{1/(\beta_*+1)}}, \tag{1.2}$$

$$|u(t_m)|_\infty \le \text{const.}[c(t_m)]^{-1/(\beta_*+1)} = \text{const.}[t_m]^{-q/(\beta_*+1)}, \tag{1.3}$$

*where*

$$\pi_\alpha := \frac{2\pi}{(\alpha+1)\sin[\pi/(\alpha+1)]}\,.$$

The result in [4] is limited to estimates (1.2) of the diameters of the nodal sets, for the case $\alpha = \beta > 0$.

Now we present some Picone-type formulae which will be used throughout this article. To start, for $w, y \in C^1(\mathbb{R}, \mathbb{R})$ and $\gamma > 0$ we define (see e.g. [1, 2])

$$\zeta_\gamma(w, y) := |w'|^{\gamma+1} - (\gamma+1) w' \phi_\gamma\big(\frac{w}{y} y'\big) + \gamma \big|\frac{w}{y} y'\big|^{\gamma+1} \tag{1.4}$$

which is strictly positive and is null only if there exists $\mu \in \mathbb{R}$ such that $w \equiv \mu y$.

Let $C, C_1, \alpha, \beta > 0$ and $w, z, u \in C^1(\mathbb{R})$, respectively, be solutions in $\mathbb{R}^+$, for

$$\begin{gathered} \big\{\phi_\alpha(u'(t))\big\}' + c(t)\alpha\phi_\beta(u(t)) = 0; \\ \big(\phi_\alpha(z')\big)' + C\alpha\phi_\alpha(z) = 0, \\ \big(\phi_\alpha(w')\big)' + C_1\alpha\phi_\beta(w) = 0. \end{gathered}$$

Using that for $\gamma > 0$ and $S, T \in \mathbb{R}$,

$$S\phi'_\gamma(S) = \gamma\phi_\gamma(S), \quad S\phi_\gamma(S) = |S|^{\gamma+1}, \quad \phi_\gamma(ST) = \phi_\gamma(S)\phi_\gamma(T),$$

wherever $u \neq 0$, we have

$$\begin{aligned} \big[z\phi_\alpha(z') - z\phi_\alpha(\frac{z}{u}u')\big]' &= \zeta_\alpha(z, u) + \alpha|z|^{\alpha+1}\big\{c(t)|u|^{\beta-\alpha} - C\big\}; \\ \big[w\phi_\alpha(w') - w\phi_\alpha(\frac{w}{u}u')\big]' &= \zeta_\alpha(w, u) + \alpha|w|^{\beta+1}\big\{c(t)\big|\frac{u}{w}\big|^{\beta-\alpha} - C_1\big\} \\ &= \zeta_\alpha(w, u) + \alpha|w|^{\alpha+1}\big\{c(t)\big|u\big|^{\beta-\alpha} - C_1\big|w\big|^{\beta-\alpha}\big\}. \end{aligned} \tag{1.5}$$

Note that:

(1) For $\mu > 0$, if the function $Z(t) := \mu z(t)$ is used in (1.5)(i), $\big(\phi_\alpha(Z')\big)' + C\alpha\phi_\alpha(Z) = 0$ and (1.5)(i) remains the same with $Z$ replacing $z$.

(2) But if $W(t) := \mu w(t)$ then

$$\big(\phi_\alpha(W')\big)' + \mu^{\alpha-\beta}C_1\alpha\phi_\beta(W) = 0$$

and (1.5)(ii) with $W$ holds with $C_1$ replaced by $\mu^{\alpha-\beta}C_1$.

(3) The Picone-type formulae in (1.5) will be the main tools in this work. In fact as the formulae make sense only wherever $u \neq 0$ ($w \neq 0$), if the right-hand side of the formula happens to be strictly positive in a set $D$ then the integration over $D$ would give 0 at the left and a strictly positive value at the right if $u \neq 0$ ($w \neq 0$) in $D$ and $u|_{\partial D} = 0$. Therefore if the right-hand

side of (1.5) is strictly positive on a set $D$, we cannot have $u\ (w) \neq 0$ inside $D$ and $u|_{\partial D} = 0$, implying that $u$ has to have a zero inside such a $D$.

Now we study equations with positive constant coefficients.

**Theorem 1.3.** *For each $k, \theta, \beta > 0$, any bounded and non-trivial solution $u$ of the problem*

$$\big\{\phi_\theta(u')\big\}' + k\theta\phi_\beta(u) = 0, \quad t > 0; \quad u(0) = 0; \quad u'(0) = A > 0 \tag{1.6}$$

*is oscillatory and*

$$\begin{gathered}(\beta+1)|u'(t)|^{\theta+1} + k(\theta+1)|u(t)|^{\beta+1} = (\beta+1)A^{\theta+1} \quad \forall t > 0,\\ u(T) = 0 \Rightarrow |u'(T)| = A \quad \forall T > 0\\ u'(S) = 0 \Rightarrow |u(S)| = \big[\frac{(\beta+1)A^{\theta+1}}{k(\theta+1)}\big]^{\frac{1}{\beta+1}} \quad \forall S > 0,\\ \text{which implies } \max_{\mathbb{R}^+}|u| = \big[\frac{(\beta+1)A^{\theta+1}}{k(\theta+1)}\big]^{\frac{1}{\beta+1}} \text{ and } \max_{\mathbb{R}^+}|u'| = A.\end{gathered} \tag{1.7}$$

*When $\beta = \theta > 0$, (1.7)(iv) reads*

$$\max_{\mathbb{R}^+}|u| = \big[\frac{1}{k}\big]^{\frac{1}{\beta+1}}A \quad \text{and} \quad \max_{\mathbb{R}^+}|u'| = A.$$

*Proof.* That this problem is oscillatory has been established in [6, 7] but for self-contained purpose we show it using a method relevant to the present work. Let $u \in C^2(\mathbb{R}^+)$ be a non-trivial and bounded solution of (1.4). Then

$$\begin{aligned}\big(\phi_\theta(u')\big)' &= \Big(\big[u'^2\big]^{(\theta-1)/2}u'\Big)'\\ &= u''\Big(\big[u'^2\big]^{(\theta-1)/2}\Big) + u'\Big((u'^2)^{(\theta-1)/2}\Big)'\\ &= u''\Big(|u'|^{\theta-1}\Big) + (\theta-1)u''|u'|^{\theta-1}\\ &= \theta u''|u'|^{\theta-1}\end{aligned}$$

and

$$u'\big(\phi_\theta(u')\big)' = \frac{\theta}{2}(u'^2)'\big(u'^2\big)^{(\theta-1)/2} = \frac{\theta}{(\theta+1)}\big(|u'|^{\theta+1}\big)'.$$

Similarly

$$\theta k u'\phi_\beta(u) = \theta k u' u|u|^{\beta-1} = \frac{\theta}{2}k(u^2)'\big(u^2\big)^{(\beta-1)/2} = \frac{\theta}{(\beta+1)}k\big(|u|^{\beta+1}\big)'.$$

The two inequalities above lead to

$$\big\{(\beta+1)|u'|^{\theta+1} + k(\theta+1)|u|^{\beta+1}\big\}' = 0 \tag{1.8}$$

and (1.7)(i) follows. (1.7)(ii) to (1.7)(v) follow immediately.

Assume that $u > \nu > 0$ in some $\Omega_T$. Then with $k$ replacing $c(t)$, in (1.5)(i), $k|u|^{\beta-\alpha} - C \geq k\nu^{\beta-\alpha} - C > 0$ if we take $C$ small enough. With this, the integration over $D(z^+) \subset \Omega_T$ would lead to a contradiction as the left hand side would be zero and the right strictly positive.

If in such an $\Omega_T$ $u > 0$ and $u \searrow 0$ then (i) or (ii) would be violated. Therefore $u$ has to have a zero in any $\Omega_T$. □

**Corollary 1.4.** *Let $A_1, A_2, \theta, \beta > 0$ $\theta \geq \beta$. Let $u_1$ and $u_2$, respectively, be oscillatory solutions for*

$$\big\{\phi_\theta(u_i')\big\}' + k_i\theta\phi_\beta(u_i) = 0, \quad t > 0; \quad u(0) = 0; \quad u'(0) = A_i > 0 \tag{1.9}$$

*with*

$$\frac{A_1^{\theta+1}}{k_1} < \frac{A_2^{\theta+1}}{k_2}.$$

*Let $D(u_i^+)$ denote a nodal set of $u_i^+$, and assume that $D(u_1^+) \cap D(u_2^+) \neq \emptyset$. If $R \in D := D(u_1^+) \cap D(u_2^+)$ with $u_1'(R) = u_2'(R) = 0$, then*

$$\max_{D(u_1^+)} u_1 := u_1(R) > \max_{D(u_2^+)} u_2 := u_2(R). \tag{1.10}$$

*Let $u_1, u_2, u_3$, respectively, be non-trivial oscillatory solutions for*

$$\big\{\phi_\theta(u_i')\big\}' + k_i\theta\phi_\beta(u_i) = 0,\ t > 0; \quad u(0) = 0; \quad u'(0) = A > 0, \tag{1.11}$$

*where $k_1 > k_2 > k_3 > 0$. Then if there is $S > 0$ such that for some $D(u_1^+)$, $D(u_2^+)$ and $D(u_3^+)$,*

$$S \in D(u_1^+) \cap D(u_2^+) \cap D(u_3^+), \quad D(u_i'(S) = 0, \quad \textit{for } i = 1, 2, 3,$$

*then*

$$\max_{D(u_3^+)} u_3^+(t) = u_3(S) \leq \max_{D(u_2^+)} u_2^+(t) = u_2(S) \leq \max_{D(u_1^+)} u_1^+(t) = u_1(S).$$

The proof of the above corollary follows straight from (1.7)(iv).

**Remark 1.5.** (1) It is easy to show that when the coefficient of $\phi_\beta$ is a positive constant, the solutions are periodic.

(2) There are two transformations which could be used in some proofs:

- (i) For any oscillatory function $u$, and $\lambda > 0$, the associated function $u_\lambda(t) := \lambda u(t)$ is also oscillatory, having exactly the same zeros as $u$ but with $|u_\lambda|_\infty = \lambda|u|_\infty$ and $|u_\lambda'|_\infty = \lambda|u'|_\infty$.
- (ii) For $\xi \in \mathbb{R}$, the translated function $U_\xi(t) := u(t+\xi)$ would be also oscillatory as $u$ and the curve $(t, U_\xi(t))$ would be that of $u$, slit alongside the $t$-axis forward (if $\xi < 0$) or backward (if $\xi > 0$).

(3) Let $u$ and $v$, respectively, be oscillatory solutions of

$$\big(\phi_\alpha(u')\big)' + c(t)\phi_\beta(u) = 0; \quad t > 0;$$
$$\big(\phi_\alpha(v')\big)' + C\phi_\beta(v) = 0,\ t > 0; \quad v(0) = 0, \quad v'(0) = b > 0.$$

If some of their nodal sets satisfy $D(u^+) \cap D(v^+) \neq \emptyset$, and $R \in D(u^+)$ satisfies $u'(R) = 0$, then $\xi$ can be chosen such that the transformed $W(t) := v(t + \xi)$ has the same singularity $R$ in the resulted $D(W^+)$ i.e. $W'(R) = u'(R) = 0$.

In summary if (a) $D(u^+) \cap D(v^+) \neq \emptyset$ and (b) $u$ has a zero inside $D(v^+)$, then there is $(\xi, \lambda) \in \mathbb{R} \times \mathbb{R}^+$ such that for some $R \in D(u^+)$, then the function $V(t) := \lambda v(t + \xi)$ satisfies $V'(R) = u'(R) = 0$ and $|V|_\infty = \lambda|v|_\infty$.

## 2. Equations with increasing and unbounded coefficients

It is known that if $c(t)$ is increasing and unbounded then if $(x_n)_{n\in\mathbb{N}}$ denotes the increasing successive zeros of the oscillatory solution $z$ of

$$\big(\phi_\alpha(z')\big)' + \alpha c(t)\phi_\alpha(z) = 0, \tag{2.1}$$

then

$$|x_{n+1} - x_n| = O\big(\pi_\alpha[c(x_n)]^{-1/(\alpha+1)}\big)$$

for large $n$. In fact as for large $m \in \mathbb{N}$, $c(x_m) \le c(t) \le c(x_{m+1})$, inside $D_m := [x_m, x_{m+1}]$; from [4], with $C(x) := [c(x)]^{1/(\alpha+1)}$, we have

$$\frac{\pi_\alpha}{C(x_{m+1})} \le |x_{m+1} - x_m| \le \frac{\pi_\alpha}{C(x_m)}. \tag{2.2}$$

**Lemma 2.1.** *For some $c_0 > 0$ let $c \in C^1(\mathbb{R}, (c_0, \infty))$ be increasing, and let $\alpha, \beta > 0$. Then any non-trivial and bounded solution $u$ of*

$$\begin{gathered} \big\{\phi_\alpha(u'(t))\big\}' + c(t)\alpha\phi_\beta(u(t)) = 0, \quad t > 0; \\ u(0) = 0, \quad u'(0) = b > 0 \end{gathered}$$

*is oscillatory.*

*Proof.* The oscillatory character of the equations have been established in our early papers [4, 7] but for later use purpose, we provide some slightly different proofs using Picone-type formulae.

(1) Assume that $\alpha \ge \beta > 0$. Let $u$ be such a solution and with some $C > 0$. Let $z$ be an oscillatory solution of

$$\big(\phi_\alpha(z')\big)' + C\alpha\phi_\alpha(z) = 0; \quad t > 0.$$

If we suppose that $u > \mu > 0$ in some $\Omega_S$ then $c(t)|u(t)|^{\beta-\alpha} > c(t)\mu^{\beta-\alpha}$ for $t > S$ and the right-hand side of (1.5)(i) is eventually strictly positive in $\Omega_S$.

Assume that $u > 0$ in some $\Omega_T$ for some $T > 0$ and $u \searrow 0$ as $t \to \infty$. Still because $0 < \beta \le \alpha$, the function $c(t)|u(t)|^{\beta-\alpha}$ is unbounded in $\Omega_T$ and the right-hand side of (1.5)(i) is eventually strictly positive in $\Omega_S$ for large $S > T$. In those cases, the right-hand side of (1.5)(i) is strictly positive in any such a $D(z^+) \subset \Omega_T$. Thus the assumption cannot stand; $u$ has a zero in any $\Omega_T$.

(2) Assume that $\beta > \alpha > 0$. For a constant $C > 0$ and an oscillatory solution $z$ of

$$\big(\phi_\alpha(w')\big)' + \alpha C\phi_\beta(w) = 0,\ t > 0; \quad w(0) = 0, \quad w'(0) = b > 0$$

wherever $u \ne 0$ in some interval $D$, (1.5)(ii) holds (with $C$ instead of $C_1$ ).

As $C$ is constant, from (1.7), $w^+$ has a constant maximum value in any nodal set $D(w^+)$ which is

$$|w|_\infty := |w|_{C(D(w^+))} = \max_{D(w^+)} |w| = \big[\frac{(\beta+1)b^{\alpha+1}}{(\alpha+1)C}\big]^{\frac{1}{(\beta+1)}}.$$

We see that the smaller $b := w'(0)$ is, the smaller $|w|_\infty$ will be.

If there exists $\nu > 0$ such that $u > \nu$ in $\Omega_R$ then as $c$ is unbounded, the right-hand side of (1.5)(ii) is eventually strictly positive in any nodal set $D(w^+) \subset \Omega_S$ for large enough $S > R$ as we would have

$$\big\{\frac{c(t)}{C}\big|\frac{u}{w}\big|^{\beta-\alpha} - 1\big\} > \big\{\frac{c(t)}{C}\big|\frac{u}{|w|_\infty}\big|^{\beta-\alpha} - 1\big\}$$

with an unbounded $c(t)$.

Assume that $u > 0$ and $u$ decreases to zero at $\infty$ in some $\Omega_T$, with $T > 0$. Then for any $R > T$ and $J_R := [R, 2R]$, we define $\nu := u(2R) := \min_{J_R}[u^+]$. We take $C := c(R) := C_1$ and $R > T$ so big that $w(R) = O(R^{-q/(\beta+1)})$. With such a large $c(R)$, $w^+$ has many nodal sets $D(w^+)$ in $J_R$ and with $b$ small enough, $|w|_\infty < \nu$ and $\big\{\frac{c(t)}{C(R)}\big|\frac{\nu}{w}\big|^{\beta-\alpha} - |\big\} > 0$ in many of them.

The integration over such a $D(w^+)$ of (1.5)(ii) would lead to a contradiction as the left hand side would give 0 and the right strictly positive. Thus $u > 0$ cannot hold in any $\Omega_T$. This, as above, completes the oscillatory character of $u$. □

**Theorem 2.2.** *(1) Let $u$ and $z$, respectively, be oscillatory solutions of*

$$\big\{\phi_\alpha(u'(t))\big\}' + c(t)\alpha\phi_\beta(u(t)) = 0, \quad \big(\phi_\alpha(z')\big)' + m\alpha\phi_\alpha(z) = 0, \quad t > 0$$

*where for some $c_0 > 0$, $c \in C^1(\mathbb{R}, (c_0, \infty))$ is an increasing and unbounded function and $\alpha \ge \beta > 0$.*

*Assume that there are two overlapping nodal sets $D(z^+)$ and $D(u^+)$ such that*

- (i) *thee exists $R \in D(z^+) \cap D(u^+)$ such that $z'(R) = u'(R) = 0$;*
- (ii) *$u$ has a zero inside $D(z^+)$ and $\big\{c(t)|u|^{\beta-\alpha} - m\big\} > 0$ in $D(z^+)$.*

*Then $D(u^+) \subset D(z^+)$ whence*

$$\operatorname{diam}\big[D(u^+)\big] \le \operatorname{diam}\big[D(z^+)\big] = O\big(\big[\frac{1}{m}\big]^{1/(\alpha+1)}\big). \tag{2.3}$$

*(2) Also if $0 < \alpha < \beta$ instead of $z$ the solution $w$ of*

$$\big(\phi_\alpha(w')\big)' + m\alpha\phi_\beta(w) = 0, \quad t > 0$$

*is used, then under the conditions (i) and (ii) the results hold with $w$ replacing $z$ with the following changes: $\big\{c(t)\big|\frac{u}{w}\big|^{\beta-\alpha} - m\big\} > 0$ in $D(w^+)$ and we have*

$$\operatorname{diam}\big[D(u^+)\big] \le \operatorname{diam}\big[D(w^+)\big] = O\big(\big[\frac{1}{m}\big]^{1/(\beta+1)}\big). \tag{2.4}$$

*Proof.* Let $D(z^+) := [t_1, t_2]$ and $D(u^+) := [x_1, x_2]$ with $t_1 < x_1 < R < t_2$. We claim that $R < x_2 < t_2$.

Otherwise if $u > 0$ in $(R, t_2)$ the integration of (1.5)(i) (where $m = C$) over $(R, t_2)$ leads to an absurdity as unlike the right-hand side, the left would be zero. Thus $x_2$ has to be between $R$ and $t_2$ and using (2.2), it leads to (2.3).

For the case of $w$ we just use (1.5)(ii) instead of (1.5)(i). □

As a prelude for the next results we have the following Lemma;

**Lemma 2.3.** *For the strongly oscillatory solution $u$ of*

$$\big(\phi_\alpha(u')\big)' + \alpha c(t)\phi_\beta(u) = 0, \quad t > 0; \quad u(0) = 0, u'(0) = b > 0 \tag{2.5}$$

*define the increasing sequences $(T_k)$ and $S_k)$ such that*

1. *for all $n \in \mathbb{N}$, $[T_n, T_{n+1}] := D_n \in \big[D(u^+)\big]$, $S_n \in D_n$; $u'(S_n) = 0$;*
2. *$c_n(t) = c(t)$ for $t \in (0, T_n]$ and $c_n(t) = c(T_n)$ for $t \ge T_n$.*

*For any $n$, let $u_n$ and $z_n$, respectively, be the solutions of*

$$\big(\phi_\alpha(u')\big)' + \alpha c_n(t)\phi_\beta(u) = 0,$$

$$\big(\phi_\alpha(z')\big)' + \alpha c(T_n)\phi_\beta(z) = 0; \quad z(0) = 0, \quad z'(0) = u'(T_n).$$

*Then* $u_n \equiv z_n$ *in* $\Omega_{T_n}$ *and with* $\beta_* := \max\{\alpha, \beta\}$, *as* $n \to \infty$,

$$|u_n|_{D(u_n^+)} = z_n(S_n) = \Big[\frac{(\beta+1)u'(T_n)^{\theta+1}}{c(T_n)(\theta+1)}\Big]^{\frac{1}{\beta_*+1}} = O\big([T_n]^{-q/(\beta_*+1)}\big). \tag{2.6}$$

*Proof.* The identity $u_n \equiv z_n$ in $\Omega_{T_n}$ is due to the fact that the two satisfy the same initial values at $T_n$. In fact if $w$ and $v$ are two $C^2(\Omega_T)$ solutions for

$$\big(\phi_\alpha(u')\big)' + \alpha c(t)\phi_\beta(u) = 0; \quad u(T) = 0, \quad u'(T) = b > 0$$

then without loss of generality we assume that $u' > v' > 0$ in some $(T, \tau)$.

From $\phi_\alpha(w')' = \alpha\frac{w''}{w'}\phi_\alpha(w')$ (as $S\phi_\alpha'(S) = \alpha\phi_\alpha(S)$), and from their equations

$$v'u'' - u'v'' = c(t)|u'v'|^{1-\alpha}\big[v^\beta|u'|^{\alpha-1} - u^\beta|v'|^{\alpha-1}\big] := c(t)|u'v'|^{1-\alpha}\Gamma(u,v),$$

$\Gamma(u,v) = 0$ at $T$ and remains strictly positive as long as $v' > 0$. Therefore as long as $v' > 0$, $\frac{u'}{v'}$ is increasing as $v'u'' - u'v'' = (v')^2\big(\frac{u'}{v'}\big)'$. But from these formulae, $v'$ should not be zero while $u' > 0$. Thus $v'$ and $u'$ have the same first zero after $T$ which is a contradiction. (2.6) follows from (2.3) and (2.4). $\square$

## 3. Estimates for some decaying oscillatory solutions

Now we take for oscillatory functions $z := z_R$ which will a fortiori depend upon the function $u$ through their bounded coefficients. Namely we will use $z$, a solution of

$$\big\{\phi_\alpha(z')\big\}' + \alpha C\phi_\beta(z) = 0;\ t > 0; \quad z(0) = 0; \quad u'(0) = b > 0$$

where $C$ will be the value of $c$ at some point $R > 0$.

**Theorem 3.1.** *Let* $R, c_0, \beta, \alpha > 0$ *and* $c \in C^1(\mathbb{R}^+, (c_0, \infty))$ *be unbounded and increasing. Then if* $u$ *and* $z := z_R$ *are, respectively, two non-trivial oscillatory solutions of*

$$\begin{gathered}\big\{\phi_\alpha(u'(t))\big\}' + c(t)\alpha\phi_\beta(u(t)) = 0,\\ \big(\phi_\alpha(z')\big)' + c(R)\alpha\phi_\beta(z) = 0, \quad t > 0; \quad z(0) = 0;\ z'(0) = b > 0.\end{gathered} \tag{3.1}$$

*Then there is* $R_1 > 0$ *such that* $u$ *has a zero inside any nodal set* $D(z^+) \subset \Omega_R$ *for all* $R > R_1$.

*Proof.* Let $u$ and $z$ be such oscillatory solutions. We saw that any multiplication of $z$ by a positive $\lambda > 0$ would not affect any $D(z)$ but only that $|\lambda z|_\infty = \lambda|z|_\infty$. Also for all $T > 0$, there are a multitude of $D(z^+)$ and $D(u^+)$ inside $\Omega_T$.

(1) Suppose that $\beta > \alpha > 0$. Let $T_1 > 0$ be such that $c(t) > 1$ for all $t > T_1$. Assume that there exists $T > T_1$ such that for all $R > T$ there is a nodal set $D(z_R^+) := D_1(z^+) \subset \Omega_R$ such that $u > 0$ in $D_1(z^+)$.

We take $T_1$ big enough for $J_R := [R, 2R]$ to contain many nodal sets of $z^+$ including $D_1(z^+)$ which is guaranteed by the fact that bigger $R$ is, the smaller $\operatorname{diam}(D(z_R^+))$ is.

If for some $\nu > 0$, $|u|^{\beta-\alpha} > \nu^{\beta-\alpha} > 0$ in $D_1(z^+)$, then, in $D_1(z^+) := D(Z^+)$, the function $Z(t) =: \nu z(t)$ satisfies

$$\begin{gathered}\big(\phi_\alpha(Z')\big)' + \nu^{\beta-\alpha}c(R)\alpha\phi_\beta(Z) = 0, \quad t > 0,\\ \big[Z\phi_\alpha(Z') - Z\phi_\alpha(\frac{Z}{u}u')\big]' = \zeta_\alpha(Z,u) + \alpha|Z|^{\alpha+1}\big\{c(t)|u|^{\beta-\alpha} - \nu^{\beta-\alpha}c(R)\big\} > 0.\end{gathered} \tag{3.2}$$

The integration over $D(Z^+)$ of (3.2) provides a contradiction. Therefore the assumption cannot be true and $u$ has to have a zero in $D_1(z^+)$.

(2) Assume that $\alpha \geq \beta > 0$. For this case (1.5)(i) is used instead of (3.2), and the same conclusion is obtained. □

**Corollary 3.2.** *(1) Let $u$ and $z$ be the two solutions in (3.1) where $C > 0$ is arbitrary. Let two of their nodal sets, Let $D(u^+)$ and $D(z^+)$, be such that $u$ has a zero in $D(z^+)$ and $S \in D(u^+)$ is the singularity of $u^+$ therein. Then there is $\xi \in \mathbb{R}$ such that the translated function $Z(t) := z(t + \xi)$ satisfies*

$$Z'(S) = u'(S) = 0, \quad D(u^+) \subset D(Z^+), \quad \operatorname{diam} D(u^+) \leq \operatorname{diam} D(z^+).$$

*(2) Moreover, for $t$ large enough,*

$$\max_{D(u^+)} u^+ := |u|_{D(u^+)} \leq \max_{D(Z^+)} Z^+ := |Z^+|_{D(Z^+)} = |z^+|_{D(z^+)}. \tag{3.3}$$

*Proof.* (1) This follows from Theorem 2.2 and Theorem 3.1. (2) follows from Lemma 2.3. □

*Proof of the Theorem 1.2.* Any such a solution of (1.1) is strongly oscillatory by Lemma 2.1 and [4, 7]. The estimates follow from Theorem 2.2, Theorem 3.1 and Corollary 3.2. □

## 4. An application

For a restoring $h \in C(\mathbb{R})$ (i.e. $\forall y \in \mathbb{R} \setminus \{0\}, yh(y) > 0$) consider the problem

$$\big\{\phi_\alpha(u')\big\}' + \alpha c(t)h(u) = 0,\ t > 0; \quad u(0) = 0, \quad u'(0) = b > 0, \tag{4.1}$$

where $\alpha, \beta, q > 0$ and $c$ being as before and for small $S > 0$, $h((s) = O\big(S^\beta\big)$.

For the strongly oscillatory solution $z$ of $\big\{\phi_\alpha(z')\big\}' + \alpha C\phi_\alpha(z) = 0$, $t > 0$, and $w$ of $\big\{\phi_\alpha(w')\big\}' + \alpha C\phi_\beta(w) = 0$ , wherever $u \neq 0$, we have

$$\begin{gathered}\big[z\phi_\alpha(z') - z\phi_\alpha(\frac{z}{u}u')\big]' = \zeta_\alpha(z, u) + \alpha C|z|^{\alpha+1}\big\{\frac{c(t)h(u)}{C\phi_\alpha(u)} - 1\big\},\\ \big[w\phi_\alpha(w') - w\phi_\alpha(\frac{w}{u}u')\big]' = \zeta_\alpha(w, u) + \alpha|w|^{\alpha+1}\big\{\frac{c(t)h(u)}{C\phi_\alpha(u)} - |w|^{\beta-\alpha}\big\}\end{gathered} \tag{4.2}$$

As $h$ is a restoring function, we can define the function $h_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ by $h(S) := h_1(S^2)S$ for all $S \in \mathbb{R}$ and define $H_1(t) := \int_0^t sh_1(s^2)ds$ such that equation (4.1) reads

$$\big(\phi_\alpha(u')\big)' + \alpha c(t)h_1(u^2)u = 0,\ t > 0; \quad u(0) = 0, \quad u'(0) = b > 0. \tag{4.3}$$

Thus, similar to Theorem 1.3, we have the following result.

**Lemma 4.1.** *With $h_1$ defined in (4.3), $\forall C, \alpha, b, \beta > 0$ the problem*

$$\big(\phi(u')\big)' + \alpha Ch_1(u^2)u = 0,\ t > 0; \quad u(0) = 0, \quad u'(0) = b$$

*is strongly oscillatory. furthermore and for its solution $u$, and all $t > 0$, we have*

$$2|u'(t)|^{\alpha+1} + (\alpha + 1)CH_1(u^2(t)) = 2b^{\alpha+1},$$

$$u(S) = 0 \text{ and } u'(T) = 0 \implies |u'(S)| = b \text{ and } |u(T)| = \big[H_1^{-1}\big(\frac{2b^{\alpha+1}}{(\alpha + 1)C}\big)\big]^{1/2}. \tag{4.4}$$

*Proof.* From $\big(\phi(u')\big)' + \alpha Ch_1(u^2)u = 0$, $u'u''\phi_\alpha'(u') + \alpha Ch_1(u^2)uu' = \alpha u''\phi_\alpha(u') + \alpha\frac{C}{2}(u^2)'h_1(u^2) = 0$ thus

$$\frac{1}{2}(u'^2)'\big(u'^2\big)^{\frac{\alpha-1}{2}} + \frac{C}{2}(u^2)'h_1(u^2) = \big[\frac{1}{\alpha+1}|u'|^{\alpha+1} + \frac{C}{2}H_1(u^2)\big]' = 0$$

leading to (4.4)(i). Then (4.4)(ii) follows as well. The oscillation of the solution is obtained as for the Theorem 1.3. $\square$

**Theorem 4.2.** *For $c_0, \alpha, \beta, q > 0$, let $h_1 \in C(\mathbb{R}, [0,\infty))$ with $h_1(S^2)S = O(s^\beta)$ for small $S > 0$ and $c \in C^1(\mathbb{R}, (c_0,\infty))$ with $c' > 0$ and $c(t) = O(t^q)$ as $t \to \infty$. Then any non-trivial and bounded solution of*

$$\big(\phi_\alpha(u')\big)' + \alpha c(t)h_1(u^2)u = 0,\ t > 0; \quad u(0) = 0; \quad u'(0) = b > 0 \tag{4.5}$$

*is strongly oscillatory.*

*(1) Moreover for any $R > 0$ let $z := z_R$ be a non-trivial oscillatory solution of*

$$\big(\phi_\alpha(z')\big)' + c(R)\alpha\phi_\alpha(z) = 0; \quad t > 0.$$

*Then for $S > 0$ large enough, the oscillatory solution $u$ of (4.5) has a zero in any nodal set $D(z_R^+) \subset \Omega_S$ for $R > S$.*

*(2) Consequently as $t \to \infty$, for $\beta_* := \alpha\wedge\beta$ the solution in (4.5) has the estimates*

$$\begin{gathered} |u(t)| \le \text{const.}[t]^{\frac{-q}{\beta+1}} := \text{const.}\big[\frac{1}{c(t)}\big]^{1/(\beta_*+1)}, \\ \operatorname{diam}(D(u^+)) = O\Big(\big[\frac{1}{c(t)}\big]^{1/(\beta_*+1)}\Big). \end{gathered} \tag{4.6}$$

*Proof.* (1) For some $C > 0$ let $z$ be a strongly oscillatory solution to

$$\big\{\phi_\alpha(z')\big\}' + \alpha C\phi_\alpha(z) = 0.$$

Then (4.2)(i) with $h(u)$ replaced by $h_1(u^2)u$ becomes

$$\big[z\phi_\alpha(z') - z\phi_\alpha(\frac{z}{u}u')\big]' = \zeta_\alpha(z,u) + \alpha C|z|^{\alpha+1}\big\{\frac{c(t)h_1(u^2)u}{C\phi_\alpha(u)} - 1\big\}.$$

If we assume that $u > \nu > 0$ in some $\Omega_R$, then

$$\zeta_\alpha(z,u) + \alpha C|z|^{\alpha+1}\big\{\frac{c(t)h_1(u^2)u}{C\phi_\alpha(u)} - 1\big\} > \zeta_\alpha(z,u) + \alpha C|z|^{\alpha+1}\big\{c(t)G(\nu) - 1\big\}$$

with

$$G(\nu) := \inf_{u\ge\nu}\frac{c(t)h_1(u^2)u}{C\phi_\alpha(u)}.$$

Because $c(t)$ is unbounded, $\{c(t)G(\nu) - 1\}$ is eventually strictly positive. Assume that that $u > 0$ and decreases to zero in some $\Omega_S$.

(a) Case where $\alpha > \beta > 0$. For very large $R > S$, as $u \searrow 0$,

$$\big\{\frac{c(t)h_1(u^2)u}{C\phi_\alpha(u)} - 1\big\} > \text{const.}\big[\frac{c(t)}{C}|u|^{\beta-\alpha} - 1\big] > 0$$

eventually and the integration over $D(z)$ of (4.2)(i) leads to a contradiction.

(b) Let $\beta \ge \alpha > 0$ and $w$ the oscillatory solution in (4.2)(ii). Assume that $u > 0$ in some $\Omega_S$. We use $h_1(u^2)u$ instead of $h(u)$ there. For $T > 0$, We define $J_T := (T, 2T)$ and $\nu := \nu(T) = \inf_{J_T}\frac{h_1(u^2)u}{C\phi_\alpha(u)}$. We take $R > S$ so large that $w^+$

has many nodal sets in $J_R$ and $c(t) > C$ there. We choose $b = w'(0)$ such that $(w^+)^{\beta-\alpha} < \nu(R)$. Then in $J_R$,

$$\Big\{\frac{c(t)h_1(u^2)u}{C\phi_\alpha(u)} - |w|^{\beta-\alpha}\Big\} > 0,$$

and integration over $D(w)$ of (4.2)(ii) leads to a contradiction. Therefore $u$ cannot remain positive throughout any $\Omega_T$.

Assume that there is $T > 0$ such that for all $R > T$, there is a nodal set $D(z_R^+) := D_R \subset J_R$ such that for some $\mu > 0$, $u > \mu$ on $D_R$. We remind that $c(t) \geq c(R)$ for all $t > R$. Then similar to (a) and (b) above, we see that as we can make $z_R^+$ arbitrary small in $J_R$, we cannot find $T$ and $\mu > 0$ such that the assumption holds.

(2) The estimates are obtained through the Corollary 3.2, keeping in mind that as $h_1(\tau^2) \leq \text{const}.\tau^\beta$, we have $H_1(\tau) \leq \text{const}.\tau^{\beta+1}$. $\square$

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