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EXPONENTIAL STABILITY OF SOLUTIONS OF NONLINEAR FRACTIONALLY PERTURBED ORDINARY DIFFERENTIAL EQUATIONS

EVA BRESTOVANSKÁ, MILAN MEDVEĎ

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Abstract. The main aim of this paper is to prove a theorem on the exponential stability of the zero solution of a class of integro-differential equations, whose right-hand sides involve the Riemann-Liouville fractional integrals of different orders and we assume that they are polynomially bounded. Equations of that type can be obtained e.g. from fractionally damped pendulum equations, where the fractional damping terms depend on the Caputo fractional derivatives of solutions. The set of initial values of solutions that converge to the origin is also determined. We also prove an existence and uniqueness theorem for this type of equations, which we use in the proof of the stability theorem.

1. Introduction

Recently, fractional differential equations with fractional derivatives and fractional integrals of different types have attracted many scientists from various disciplines due to their wide applications. The most known are the Riemann-Liouville and the Caputo derivatives and differential equations with these derivatives. The basic theory of fractional differential equations and many references can be found in the monographs [8, 23, 29]. These derivatives are defined as follows:

The Riemann-Liouville fractional derivative of a function $u\colon [0,\infty) \to \mathbb{R}$ of an order $\alpha \in (0,1)$ is

$$ {}^{RL}D^{\alpha}u(t) := \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}u(s)ds \tag{1.1}$$

and the Caputo derivative is

$$ {}^{C}D^{\alpha}u(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u'(s)ds, \tag{1.2}$$

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where $u'(t) = \frac{du(t)}{dt}$, $\Gamma(z) = \int_0^\infty \tau^{z-1}e^{-\tau}d\tau$ is the Euler Gamma function and the integral

$$^{RL}I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds \tag{1.3}$$

is called the Riemann-Liouville fractional integral of order α of the function $u(t)$, provided that the right-hand sides of (1.1), (1.2) and (1.3), respectively, exist.

If $u\colon [0,\infty) \to \mathbb{R}^N$, $u(t) = \big(u_1(t), u_2(t), \dots, u_N(t)\big)$, then, we define the Riemann-Liouville fractional derivative of the mapping $u(t)$ of order α as

$$^{RL}D^{\alpha}u(t) = \big(^{RL}D^{\alpha}u_1(t), ^{RL}D^{\alpha}u_2(t), \dots, ^{RL}D^{\alpha}u_N(t)\big) \tag{1.4}$$

and the Riemann-Liouville fractional integral as

$$^{RL}I^{\alpha}u(t) = \big(^{RL}I^{\alpha}u_1(t), ^{RL}I^{\alpha}u_2(t), \dots, ^{RL}I^{\alpha}u_N(t)\big). \tag{1.5}$$

An influence of viscous fluids on vibrating systems is often modeled by using the Riemann-Liouville or Caputo fractional derivative. These derivatives play the role of damping force, called the fractional damping. The well known Bargley-Torvik equation (see [2])

$$u''(t) + A\,^{C}D^{\frac{3}{2}}u(t) = au(t) + \phi(t), \tag{1.6}$$

modelling the motion of a rigid plate immersing in a viscous liquid, is one of the equations descibing the motion with the fractional damping term $A^{C}D^{\frac{3}{2}}u(t)$.

It is well known that the system of linear fractional differential equations

$$D^{\alpha}x(t) = Ax(t), \quad x(t) \in \mathbb{R}^N, \ \alpha \in (0,1), \tag{1.7}$$

where $D^{\alpha}x(t)$ is the Riemann-Liouville or the Caputo derivative of $x(t)$ of the order $\alpha \in (0,1)$ and A is a constant matrix, do not have exponentially stable solutions, but asymptoticall stable only. The equilibrium $x = 0$ of this equation is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ for all eigenvalues of the matrix A. In this case all components of $x(t)$ decay towards 0 like $t^{-\alpha}$ (see [11, 12, 20, 21, 9]).

We will show that fractional integro-differential equations of the form

$$\dot{x}(t) = Ax(t) + f\big(t, x(t), ^{RL}I^{\alpha_1}x(t), \dots, ^{RL}I^{\alpha_m}x(t)\big), \quad x(t) \in \mathbb{R}^N \tag{1.8}$$

can have exponentially stable solutions, where the solution is defined as in the next definition.

Definition 1.1. A mapping $x\colon [0,T) \to \mathbb{R}^N$, where $0 < T \le \infty$, is a solution of the equation (1.8) satisfying the initial condition $x(0) = x_0 \in \mathbb{R}^N$ if it is continuously differentiable on the interval $(0,T)$, continuous on $[0,T)$ and it satisfies the equality (1.8) for all $t \in (0,T)$. If $T = \infty$, then this solution is called global.

Equations of the form (1.8) can be obtained from the following linear multi-fractional pendulum equation

$$u''(t) + \lambda_1(t)^{C}D^{\beta_1}u(t) + \dots + \lambda_m(t)^{C}D^{\beta_m}u(t) + \lambda u'(t) + \omega^2 u(t) = 0 \tag{1.9}$$

with m fractional and one ordinary damping terms, which can be written as a system of the form (1.8) with

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\lambda \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

$$
\begin{aligned}
&f(t, x(t), {}^{RL}I^{\alpha_1}x(t), \dots, {}^{RL}I^{\alpha_m}x(t)) \\
&= \begin{pmatrix} 0 \\ -\lambda_1(t)^{RL}I^{\alpha_1}x_2(t) - \dots - \lambda_m(t)^{RL}I^{\alpha_m}x_2(t) - \lambda(t)x_2(t) \end{pmatrix},
\end{aligned}
$$

where $x_1(t) = u(t)$, $x_2(t) = u'(t)$, $\beta_i \in (0,1)$, $\alpha_i = 1 - \beta_i$, $\lambda_i(t)$ $i = 1, 2, \dots, m$ are continuous functions on $[0, \infty)$ and $\lambda > 0$, $\omega > 0$ are constants.

Let us recall an analysis of the fractional vibration equation

$$u''(t) + b^C D^\alpha u(t) + cu(t) = 0, \quad \alpha \in (0,1), \tag{1.10}$$

where $b > 0$, $c > 0$ are constants, given in the papers [10] and [18]. It is proven in [18] that if $u(t)$ is a solution of the equation (1.10) satisfying the initial conditions $u(0) = u_0$, $u'(0) = u_1$, then its Laplace transform is

$$\mathcal{L}[u(t)] = U(s) = \frac{s + bs^{\alpha-1} + c}{s^2 + bs^\alpha + c} u_0 + \frac{1}{s^2 + bs^\alpha + c} u_1,$$

the characteristic equation

$$s^2 + bs^\alpha + c = 0 \tag{1.11}$$

has a couple of complex conjugate roots

$$s_{1,2} = \beta \pm i\sigma = r^{\pm i\Theta}, \quad \beta < 0, \quad \sigma > 0, \quad r = \sqrt{\beta^2 + \sigma^2} > 0, \quad \frac{\pi}{2} < \Theta < \pi$$

and the fundamental solution $\phi_1(t)$ with

$$\mathcal{L}[\phi_1(t)] = \Phi_1(s) = \frac{s + bs^{\alpha-1} + c}{s^2 + bs^\alpha + c}$$

has the form

$$\phi_1(t) = Ce^{\beta t}\cos\sigma t + De^{\beta t}\sin\sigma t + \int_0^\infty K_\alpha(\tau)e^{-\tau t}d\tau,$$

where C, D are functions of the variables r, Θ. The function

$$f_1(t) = Ce^{\beta t}\cos\sigma t + De^{\beta t}\sin\sigma t$$

represents a decaying oscillation along the t-axis, where the amplitude decays exponentially. The function $\mathcal{L}[\phi_1(t)] = \Phi_1(s)$ has the asymptotic representation $\Phi_1(s) \sim \frac{b}{c}s^{\alpha-1}$ as $s \to 0$ and hence the function $f_2(t) = \int_0^\infty K_\alpha(\tau)e^{-\tau t}d\tau$ has the asymptotic representation $f_2(t) \sim \frac{b}{c}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ as $t \to \infty$. The derivative $u'(t)$ has a similar asymptotic representation $u'(t) \sim \frac{b}{c}\frac{(-\alpha)t^{-\alpha-1}}{\Gamma(1-\alpha)}$ as $t \to \infty$. We conclude that the solution $u(t)$ of the equation (1.10) decays towards 0 as $t \to \infty$ like $t^{-\alpha}$ and $u'(t)$ has similar asymptotic properies. This means that the equilibrium $x = (x_1, x_2) = (u, u') = (0, 0)$ of the system of equations, corresponding to the the solution $u(t)$ of the equation (1.10), is asymptotically stable, but not exponentially.

We were motivated by the paper [25], where an existence and uniqueness result for the initial value problem

$$Au'' + \sum_{k=1}^N B_k\, {}^cD^{\alpha_k}u(t) = f(t), \quad u(0) = u_0, \quad u'(0) = c_1 \tag{1.12}$$

with $0 < \alpha_k < 2$, $k = 1, 2, \dots, N$ is proved. The Caputo fractional derivatives in the equation (1.12) play there the role of damping terms.

In the paper [5] the initial value problem

$${}^{RL}D^\alpha x(t) = f(t, x(t)), \quad t > 0, \quad \lim_{t\to 0} t^{1-\alpha}x(t) = b, \quad \alpha \in (0,1), \quad b \in \mathbb{R}, \tag{1.13}$$

where $f\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\phi\colon R^+ \to R^+$ are continuous functions, is studied. It is assumed there that

$$|f(t,x)| \le t^\mu \phi(t) e^{-\sigma t} |u|^m, \quad \text{for all } (t,u) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.14}$$

where $\mu \ge 0$, $m > 1$, $\sigma > 0$. Using the desingularization method, proposed in [14] and the Bihari inequality, it is proven there that if

$$\|\phi\|_q = \int_0^\infty \phi(s)^q ds < L := \frac{\Gamma(\alpha)}{2^{1+m-\alpha}|b|^{m-1}} \Big(\frac{2^m}{m-1}\Big)^{1/q} \Big[\frac{(\sigma p)^{\lambda_1}}{\Gamma(\lambda_1)(1+\frac{\lambda_1}{\lambda_2})}\Big]^{1/p},$$

where $pq = p+q$, $\lambda_1 = 1 + p(\mu - (1-\alpha)m]$, $\lambda_2 = 1 + p(\alpha - 1)$, then any solution $x(t)$ of the initial value problem (1.13) is global and there exists a constant $c > 0$ such that $|x(t)| \le \frac{c}{t^{1-\alpha}}$ for all $t \in (0,\infty)$. This means that the trivial solution $x_0(t) \equiv 0$ is asymptotically stable. It is obvious that similar results for more general type of power nonlinearities are extraordinary complicated. In the paper [15] the equation

$$\dot{x}(t) = Ax(t) + f\big(t, x(t), {}^{RL}I^{\alpha_1}[g_1 x](t), \dots, {}^{RL}I^{\alpha_m}[g_m x](t)\big), \quad x(t) \in \mathbb{R}^N, \tag{1.15}$$

where $f\colon \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping, $g_i\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $(t,x) \mapsto g_i(t,x)$, $i = 1,2,\dots,m$ are continuous mappings and

$${}^{RL}I^{\alpha_i}[g_i x](t) := \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} g_i(s, x(s)) ds, \quad 0 < \alpha_i < 1, \; i = 1,2,\dots,m \tag{1.16}$$

is studied. A sufficient condition for the exponential stability of the trivial solution $x(t) \equiv 0$ of this equation is proven there. In the paper [16] a sufficient condition for the non-existence of blow-up solutions for a fractional functional-differential equations of the form

$$\begin{gathered} \dot{x}(t) = Ax(t) + h\Big(t, x(t), x_t, (I^{\alpha_1}[g_1 x])(t), \dots, (I^{\alpha_m}[g_m x])(t)\Big), \quad t > 0, \\ x(t) = \Phi(t), \quad t \in [-r, 0], \end{gathered} \tag{1.17}$$

where $r > 0$, $\Phi \in C_r := C([-r,0], X)$, X is a Banach space, $x(t) \in X$, $x_t \in C$, $x_t(\Theta) := x(t+\Theta)$ $t > 0$, $\Theta \in [-r, 0]$, A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\ge 0}$, $S(t) := e^{At}$, $h\colon \mathbb{R}_+ \times X \times C_r \times X^m \to X$, $X^m := X \times \dots \times X$ (m times) is a continuous map, $\mathbb{R}_+ = [0,\infty)$, $g_i\colon \mathbb{R}_+ \times X \to X$, $(t,x) \mapsto g_i(t,x)$, $i = 1,2,\dots,m$ are continuous maps, is proved

In this paper, we study equation (1.15) with $g_i(t,x) \equiv x(t)$, i.e., we have the Riemann-Liouville fractional integrals of $x(t)$ in the equation (1.8) instead of the nonlinear functions $g_i(t, x(t))$. Moreover, the mapping f is more general than in [15]. The aim is to give some conditions under which the trivial solution of this equation is exponentially stable.

2. Existence and uniqueness result

In this section, we prove a local existence and uniqueness result concerning the initial value problem

$$\begin{gathered} \dot{x}(t) = Ax(t) + f\Big(t, x(t), {}^{RL}I^{\alpha_1} x(t), \dots, {}^{RL}I^{\alpha_m} x(t)\Big), \quad t > 0, \\ x(t) \in R^N, \quad x(t_0) = x_0. \end{gathered} \tag{2.1}$$

Many papers are devoted to the fractional initial value problem

$${}^{RL}D^\alpha x(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.2}$$

where the right-hand side is independent of fractional derivatives and fractional integrals, respectivelly. Some basic existence results for the problem (2.2) can be found in the monograph [8], where they are proved by using the classical Picard method of successive approximations. Many local and global existence results for various classes of fractional differential equations are proved by using fixed point theorems (see, e.g., the monograph [29], [6] and the paper [30]). Some existence results for the second-order abstract differential equations on Banach spaces, involving several fractional derivatives on their right-hand sides are proved in the papers [7, 27, 28]. In its proof, we use the method of Picard successive approximations. There is a problem to apply the Banach fixed point theorem without the assumption of the global boundedness of the mapping f, because there are fractional integrals of the unknown function $x(t)$ in its arguments.

Theorem 2.1. *Let $G \subset \mathbb{R} \times \mathbb{R}^N$ be a region, $H_m \subset \mathbb{R}^m$ is a region with $0 \in H_m$ and $f \in C(G \times H_m, \mathbb{R}^N)$ be a continuous locally Lipschitz mapping. Then for any $(t_0, x_0) \in G, t_0 \geq 0$, there exists a $\delta > 0$ such that the initial value problem (2.1) has a unique solution $x(t)$ on the interval $I_\delta = [t_0, t_0 + \delta)$.*

Proof. Let

$$\begin{aligned} G_0 = \big\{&(t, x, u_1, \dots, u_m) \in G \times H_m : t_0 \leq t \leq t_0 + a, t_0 \geq 0, \\ &\|x - x_0\| \leq b,\ \|u_i\| \leq \|x_0\| + b,\ i = 1, 2, \dots, m\big\}, \end{aligned} \tag{2.3}$$

for some $a > 0$, $b > 0$. Let

$$M_1 = \max_{\|x - x_0\| \leq b} \|Ax\|, \quad M_2 = \max_{(t,x,u_1,\dots,u_m) \in G_0} \|f(t, x, u_1, \dots, u_m)\|$$

and the mapping f satisfies the condition

$$\|f(t, x, u_1, u_2, \dots, u_m) - f(t, y, v_1, v_2, \dots, v_m)\| \leq L_0\|x - y\| + \sum_{i=1}^{m} L_i\|u_i - v_i\| \tag{2.4}$$

for all $(t, x, u_1, u_2, \dots, u_m), (t, y, v_1, v_2, \dots, v_m) \in G_0$. Let

$$0 < \delta = \min\big\{a, \frac{b}{M_1 + M_2},\ c,\ \frac{1}{\|A\| + L_0 + \sum_{i=1}^m L_i}\big\},$$

where $c = \min_{1 \leq i \leq m} \big[\Gamma(\alpha_i)\alpha_i\big]^{\frac{1}{\alpha_i}}$. Let $C_\delta := C(I_\delta, \mathbb{R}^N)$ be the Banach space of continuous mappings from I_δ into $\mathbb{R}^N$ endowed with the metric $d(h, g) := \|h - g\| := \max_{t \in I_\delta} \|h(t) - g(t)\|$. Let us define the successive approximations $\{x_n\}_{n=0}^\infty, x_n \in C_\delta := C(I_\delta, \mathbb{R}^N)$, $I_\delta = [t_0, t_0 + \delta]$, by

$$\begin{aligned} &x_0(t) \equiv x_0, \\ &x_{n+1}(t) \\ &= x_0 + \int_{t_0}^t Ax_n(s)ds + \int_{t_0}^t f\Big(s, x_n(s), \frac{1}{\Gamma(\alpha_1)} \\ &\quad \times \int_0^s (s - \tau)^{\alpha_1 - 1} x_n(\tau)d\tau, \dots, \frac{1}{\Gamma(\alpha_m)} \int_0^s (s - \tau)^{\alpha_m - 1} x_n(\tau)d\tau\Big)ds, \\ &n = 0, 1, 2, \dots\ t \in I_\delta. \end{aligned} \tag{2.5}$$

First, let us prove that $\|x_n(t)-x_0\| \le b$ for all $n \ge 1$, $t \in I_\delta$. From the definition of the number c it follows that

$$\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}ds \le \frac{1}{\Gamma(\alpha_i)}\frac{\delta^{\alpha_i}}{\alpha_i} \le \frac{1}{\Gamma(\alpha_i)}\frac{c^{\alpha_i}}{\delta^{\alpha_i}} \le \frac{1}{\Gamma(\alpha_i)}\frac{\Gamma(\alpha_i)\alpha_i}{\alpha_i} = 1 \tag{2.6}$$

for $i = 1, 2, \dots, m$; and so, we have

$$\|\frac{1}{\Gamma(\alpha_i)}\int_{t_0}^t (t-\tau)^{\alpha_i-1}x_0 d\tau\| \le \frac{1}{\Gamma(\alpha_i)}\frac{\delta^{\alpha_i}}{\alpha_i}[\|x_0\|+b] \le \|x_0\|+b, \tag{2.7}$$

for $i = 1, 2, \dots, m$, $t \in I_\delta$, and $i = 1, 2, \dots m$. Hence, the first approximation $x_1(t)$ is well defined and

$$\|x_1(t)-x_0\| \le M_1\delta + M_2\delta = (M_1+M_2)\delta \le (M_1+M_2)\frac{b}{M_1+M_2} = b,$$

for $t \in I_\delta$. This yields the inequality

$$\|x_1(t)\| \le \|x_0\| + b \quad \text{for all } t \in I_\delta.$$

and thus

$$\Big(t, x_1(t), \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\tau)^{\alpha_1-1}x_1(\tau)d\tau, \dots, \frac{1}{\Gamma(\alpha_m)}\int_0^t (t-\tau)^{\alpha_m-1}x_1(\tau)d\tau\Big) \in G_0$$

for all $t \in I_\delta$. Now, we find by using the Lipschitz condition (2.4) and inequality (2.6) that

$$\begin{aligned}
\|x_2(t)-x_1(t)\| &\le \delta(\|A\|+L_0)\|x_1(t)-x_0(t)\| \\
&\quad + \sum_{i=1}^m \frac{L_i}{\Gamma(\alpha_i)}\int_{t_0}^t\int_0^s (s-\tau)^{\alpha_i-1}\|x_1(\tau)-x_0(\tau)\|d\tau ds \\
&\le \delta(\|A\|+L_0)\|x_1-x_0\| \\
&\quad + \sum_{i=1}^m \frac{L_i}{\Gamma(\alpha_i)}\Big(\int_{t_0}^t\int_0^s (s-\tau)^{\alpha_i-1}d\tau ds\Big)\|x_1-x_0\| \\
&\le \delta k\|x_1-x_0\|,
\end{aligned} \tag{2.8}$$

where $k = \|A\| + L_0 + \sum_{i=1}^m L_i$ and so, we get

$$\|x_2-x_1\| \le k\delta\|x_1-x_0\|.$$

Now assume that the estimate

$$\|x_n(t)-x_{n-1}(t)\| \le (k\delta)^{n-1}$$

holds for $n > 2$. Then, using this inequality, the Lipschitz condition (2.4) and the inequality (2.6), one can get

$$\|x_{n+1}(t)-x_n(t)\| \le (k\delta)^n\|x_1-x_0\|$$

and so, we have

$$\|x_{n+1}-x_n\| \le (k\delta)^n\|x_1-x_0\|.$$

Since

$$x_n(t) = x_0(t) + \sum_{i=1}^n [x_i(t)-x_{i-1}(t)] \quad \text{with} \quad x_0(t) \equiv x_0, \tag{2.9}$$

we obtain

$$\begin{aligned} \|x_0(t)+\sum_{i=1}^{n}[x_i(t)-x_{i-1}(t)]\| &\le \|x_0\|+\sum_{i=1}^{n}\|x_i(t)-x_{i-1}(t)\| \\ &\le \Big(\|x_0\|+\sum_{i=1}^{n}(k\delta)^i\Big)\|x_1-x_0\|, \end{aligned} \tag{2.10}$$

for all $t\in I_\delta$. From the definition of δ it follows that $k\delta<1$, and so the series $\|x_0\|+\sum_{i=1}^{\infty}(k\delta)^i$ is convergent. This yields the uniform convergence of the sequence $\{x_n(t)\}_{i=0}^{\infty}$ on the interval I_δ to a continuous mapping $x\in C_\delta$. This implies

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{\Gamma(\alpha_i)}\int_0^s(s-\tau)^{\alpha_i-1}x_n(\tau)d\tau \\ &=\frac{1}{\Gamma(\alpha_i)}\int_0^s(s-\tau)^{\alpha_i-1}x(\tau)d\tau \quad \text{for } i=1,2,\dots,m,\ s\in I_\delta \end{aligned} \tag{2.11}$$

and therefore

$$\begin{aligned} &\lim_{n\to\infty}f\Big(s,x_n(s),\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-\tau)^{\alpha_1-1}x_n(\tau)d\tau, \\ &\dots,\frac{1}{\Gamma(\alpha_m)}\int_0^s(s-\tau)^{\alpha_m-1}x_n(\tau)d\tau\Big)ds \\ &=f\Big(s,x(s),\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-\tau)^{\alpha_1-1}x(\tau)d\tau, \\ &\dots,\frac{1}{\Gamma(\alpha_m)}\int_0^s(s-\tau)^{\alpha_m-1}x(\tau)d\tau\Big)ds \quad \text{for all } s\in I_\delta. \end{aligned} \tag{2.12}$$

Therefore from (2.5) it follows that $x(t)$ is a solution of the initial value problem (2.1), defined on the interval I_δ. Now let us prove its uniqueness. Assume that there are two different solutions $x,y\in C_\delta$ of the initial value problem (2.1). Let $w(t):=\|x(t)-y(t)\|$, $t\in I_\delta$ and $W=\max_{t\in I_\delta}w(t)$. Then, by using the Lipschitz condition (2.4) and the inequality (2.6) we obtain

$$\begin{aligned} w(t)&\le(\|A\|+L_0)\int_{t_0}^t w(s)ds+\sum_{i=0}^{m}\frac{L_i}{\Gamma(\alpha_i)}\int_{t_0}^t\int_0^s(s-\tau)^{\alpha_i-1}\|x(\tau)-y(\tau)\|d\tau ds \\ &\quad+\delta\Big(\|A\|+L_0+\sum_{i=0}^{m}\frac{L_i}{\Gamma(\alpha_i)}\int_{t_0}^t(t-\tau)^{\alpha_i-1}d\tau\Big)W \\ &\le\delta\Big(\|A\|+L_0+\sum_{i=0}^{m}L_i\Big)W \\ &=(\delta k)W \quad \text{for all } t\in I_\delta \end{aligned} \tag{2.13}$$

and this yields the inequality $W\le(k\delta)W<W$. This is a contradiction and hence, we have $x(t)=y(t)$ for all $t\in I_\delta$. □

3. Stability Theorem

In this section, we prove a result on the exponential stability of the trivial solution $x(t)\equiv 0$ of the equation (1.8). In its proof, we apply a desingularization method, proposed in the paper [14], where it is applied in the study of nonlinear integral inequalities with weakly singular kernels.

We assume that the following conditions are satisfied:

(A1)
$$\|e^{At}x\| \le Ke^{-at}\|x\| \quad \text{for all } t \ge 0,\ x \in \mathbb{R}^N,$$
where $K > 0$, $a > 0$ are constants;

(A2) The mapping $f\colon \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{mN} \to \mathbb{R}^N$ is continuous and it satisfies the condition
$$\begin{aligned} \|f(t,x,u_1,u_2,\dots,u_m)\| \le \mu_1(t)e^{-\gamma_1 t}\|x\| + \sum_{i=1}^m \lambda_{i1}(t)e^{-\gamma_{i1}t}\|u_i\| \\ + \sum_{j=2}^l \mu_j(t)\|x\|^{k_j} + \sum_{j=2}^l\sum_{i=1}^m \lambda_{ij}(t)\|u_i\|^{k_j} \end{aligned} \tag{3.1}$$
for all $(t,x,u_1,u_2,\dots,u_m) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{mN}$, where $\mu_j(t), \lambda_{ij}(t)$, $i = 1,2,\dots,m$, $j = 1,2,\dots,l$ are nonnegative continuous functions on $[0,\infty)$, $\gamma_1 > 0$, $\gamma_{i1} > a+1$, $i = 1,2,\dots,m$, $1 = k_1 < k_2 < \dots < k_l$, $\|z\| = \max\{|z_1|,|z_2|,\dots,|z_N|\}$;

(A3) There exist numbers $p_i > 1$, $i = 1,2,\dots,m$ such that
$$p_i(\alpha_i - 1) + 1 > 0, \quad i = 1,2,\dots,m$$
and
$$\omega_j := \int_0^\infty \mu_j(s)^q ds < \infty, \quad j = 1,2,3,\dots,l, \tag{3.2}$$
where $q = q_1q_2\dots q_m$, $q_i = \frac{p_i}{p_i-1}$, $i = 1,2,\dots,m$;

(A4)
$$\begin{gathered} \eta_{i1} := \int_0^\infty e^{-[\gamma_{i1}-(a+1)]s}\lambda_{i1}(s)ds < \infty, \\ \eta_{ij} := \int_0^\infty e^{(a+k_j)s}s^{\frac{k_j-1}{q_i}}\lambda_{ij}(s)ds < \infty, \quad i = 1,2,\dots,m,\ j = 2,3,\dots,l, \end{gathered} \tag{3.3}$$
where $q_1,q_2,\dots,q_m$ are defined as in (A3).

(A5) The mapping $f(t,x,u_1,u_2,\dots,u_m)$ is locally lipschitz with respect to the variables $x,u_1,\dots,u_m$.

In the proof of the main result, we use the following corollary of the Pinto's inequality (see [22, Theorem 1], [1, Theorem 10.2] and [26, Example 5]). We present it in the form of the next lemma also with its proof, because we did not find this formulation in literature.

Lemma 3.1. *Let $c > 0$ be a constant, $\Psi_j(t)$, $j = 1,2,\dots,l$ be continuous, nonnegative functions on $[a,\infty)$ and $u(t)$ be a continuous nonnegative function satisfying the integral inequality*
$$u(t) \le c + \sum_{j=1}^l \int_a^t \Psi_j(s)u(s)^{k_j}ds, \quad t \in [a,\infty),$$
where $a \in \mathbb{R}$, $1 = k_1 \le k_2 < \dots \le k_l$. Let the following conditions be satisfied:
$$(k_j-1)(cD_j)^{k_j-1}\int_a^\infty \Psi_j s)ds < 1, \quad j = 2,3,\dots,l, \quad \int_0^\infty \Psi_1(s)ds < \infty, \tag{3.4}$$

where

$$
\begin{gathered}
D_1 = e^{\int_0^\infty \Psi_1(s)ds}, \\
D_j = \Big(1-(k_j-1)(D_{j-1}c)^{k_j-1}\int_a^\infty \Psi_j(s)ds\Big)^{-\frac{1}{k_j-1}}, \quad j=2,3,\dots,l.
\end{gathered} \tag{3.5}
$$

Then

$$
u(t) \le cD_1D_2\dots D_l, \quad \text{for all } t\in[a,\infty). \tag{3.6}
$$

Proof. By [22, Theorem 1] (see also [26, Example 5]),

$$
u(t) \le W_l^{-1}\Big[W_l(c_{l-1}(t)) + \int_a^t \Psi_l(s)ds\Big], \tag{3.7}
$$

where

$$
\begin{gathered}
c_0(t)\equiv c, \quad c_i(t) = W_i^{-1}\Big[W_i(c_{i-1}(t)) + \int_a^t \Psi_i(s)ds\Big], \\
W_i(z) = \int_{u_i}^z \frac{dy}{y^{k_i}}, \quad y\ge u_i>0, \quad i=1,2,\dots,l.
\end{gathered} \tag{3.8}
$$

One can calculate that

$$
W_1(y) = \frac{1}{1-k_1}\big[y^{1-k_1}-c^{1-k_1}\big], \quad W_1^{-1}(u) = \big[c^{1-k_1}-(k_1-1)u\big]^{-\frac{1}{k_1-1}}
$$

and

$$
c_1(t) = W_1^{-1}\Big[W_1(c) + \int_a^t \Psi_1(s)ds\Big] \le cD_1. \tag{3.9}
$$

From the assumption (3.4) it follows that $0<D_1<\infty$. Using the inequality (3.9), we obtain

$$
\begin{aligned}
c_2(t) &\le W_2^{-1}\big[W_2(c_1(t)) + \int_a^t \Psi_2(s)ds\big] \le W_2^{-1}\big[W_2(cD_1) + \int_a^t \Psi_2(s)ds\big] \\
&\le \Big[(D_1c)^{1-k_2} - (k_2-1)\int_a^t \Psi_2(s)ds\Big]^{-\frac{1}{k_2-1}} \le cD_1D_2,
\end{aligned} \tag{3.10}
$$

where

$$
D_2 = \Big[1-(k_2-1)(cD_1)^{k_2-1}\int_a^\infty \Psi_2(s)ds\Big]^{-\frac{1}{k_2-1}}.
$$

Now, assume that

$$
c_{l-1}(t) \le cD_1D_2\cdots D_{l-1}.
$$

Using the same arguments as above one can prove inequality (3.6). □

Theorem 3.2. *Let conditions* (A1)–(A5) *be satisfied and* $\|x_0\|<\rho$*, where* $\rho=\infty$*, if* $l=1$ *and if* $l>1$*, then*

$$
\rho = \sup\Big\{z\in\mathbb{R} : |C(z)D_{j-1}(z)| < \Big[\frac{1}{(k_j-1)G_j}\Big]^{\frac{1}{k_j-1}},\ i=2,3,\dots,l\Big\}, \tag{3.11}
$$

where

$$
\begin{gathered}
C(z) = d^qK^qz^q, \quad d=m(l+1)+2, \quad D_1(z)\equiv G_1, \\
D_2(z) = \big[1-(k_2-1)[C(z)D_1(z)]^{k_2-1}G_2\big]^{-\frac{1}{k_2-1}}, \\
D_j(z) = \big[1-(k_j-1)\big[C(z)D_{j-1}(z)\big]^{k_j-1}G_j\big]^{-\frac{1}{k_j-1}}, \quad j=3,\dots,l,
\end{gathered} \tag{3.12}
$$

$$
\begin{aligned}
G_1 &= d^{q-1}K^q\Big\{\Big(\frac{1}{\gamma_1 p}\Big)^{q/p}\omega_1 + \sum_{i=1}^m \Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^q \eta_{i1}^q \Big(\frac{1}{aq_i\hat{p}_i}\Big)^{\frac{1}{\hat{p}_i}}\frac{1}{q}\Big\}, \\
G_j &= d^{q-1}K^q\Big\{\Big[\frac{1}{(k_j-1)ap}\Big]^{q/p}\omega_j + \sum_{i=1}^m \Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^{k_jq} \eta_{ij}^q \Big(\frac{1}{aq_i\hat{p}_ik_j}\Big)^{\frac{1}{\hat{p}_i}}\frac{1}{k_jq}\Big\},
\end{aligned} \tag{3.13}
$$

$j = 2,3,\dots,l$, $p_i > 1$, $q_i = \frac{p_i}{p_i-1}$, $i = 1,2,\dots,m$, $q = q_1q_2\dots q_m$, $p = \frac{q}{q-1}$, $\hat{q}_i = q_1\dots q_{i-1}q_{i+1}\dots q_m$, $\hat{p}_i = \frac{\hat{q}_i}{\hat{q}_i-1}$,

$$
Q_i = \Big(\frac{\Gamma(p_i(\alpha_i-1)+1)}{p_i^{p_i(\alpha_i-1)+1}}\Big)^{1/p_i}. \tag{3.14}
$$

Then for the solution $x(t)$ of the equation (1.8), satisfying the condition $x(0) = x_0$, the following inequality holds:

$$
\|x(t)\| \le \Psi(\|x_0\|)e^{-at} \quad \text{for all } t \in [0,\infty), \tag{3.15}
$$

where the function $\Psi(w)$ is defined as

$$
\Psi(w) = \Big[C(w)D_1(w)D_2(w)D_2(w)\dots D_l(w)\Big]^{1/q}, \quad |w| < \rho,
$$

for which $\lim_{w\to 0}\Psi(w) = 0$ and if $l > 1$, then $\lim_{w\to\rho^-}\Psi(w) = \infty$.

Proof. Let $x(t)$ be the maximal solution of the equation (1.8), defined on the interval $[0,d)$, satisfying the condition $x(0) = x_0 \in \mathbb{R}^N$. From Theorem 2.1 it follows that this solution exists. Then

$$
x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f\big(s, x(s), {}^{RL}I^{\alpha_1}x(s), \dots, {}^{RL}I^{\alpha_m}x(s)\big)ds, \quad t \in [0,d)
$$

and the conditions (A1), (A2) yield

$$
\begin{aligned}
\|x(t)\| &\le Ke^{-at}\|x_0\| + Ke^{-at}\int_0^t e^{-(\gamma_1-a)s}\mu_1(s)\|x(s)\|ds \\
&\quad + Ke^{-at}\sum_{i=1}^m \int_0^t e^{-(\gamma_{i1}a)s}\lambda_{i1}(s)\|{}^{RL}I^{\alpha_i}x(s)\|ds \\
&\quad + Ke^{-at}\int_0^t e^{as}\Big(\sum_{j=2}^l \mu_j(s)\|x(s)\|^{k_j} + \sum_{j=2}^l\sum_{i=1}^m \lambda_{ij}(s)\|{}^{RL}I^{\alpha_i}x(s)\|^{k_j}\Big)ds.
\end{aligned} \tag{3.16}
$$

If $u(t) = e^{at}\|x(t)\|$, then $\|x(t)\| = e^{-at}u(t)$, $\|x(t)\|^{k_j} = e^{-ak_jt}u(t)^{k_j}$,

$$
\begin{aligned}
\|{}^{RL}I^{\alpha_i}x(s)\| &= \frac{1}{\Gamma(\alpha_i)}\Big\|\int_0^t (t-s)^{\alpha_i-1}x(s)ds\Big\| \\
&\le \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}\|x(s)\|ds \\
&\le \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}e^{-as}u(s)ds,
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
\|{}^{RL}I^{\alpha_i}x(s)\|^{k_j} &= \frac{1}{\Gamma(\alpha_i)^{k_j}}\Big\|\int_0^t (t-s)^{\alpha_i-1}x(s)ds\Big\|^{k_j}\\
&\le \frac{1}{\Gamma(\alpha_i)^{k_j}}\Big(\int_0^t (t-s)^{\alpha_i-1}\|x(s)\|ds\Big)^{k_j} \qquad (3.18)\\
&\le \frac{1}{\Gamma(\alpha_i)^{k_j}}\Big(\int_0^t (t-s)^{\alpha_i-1}e^{-as}u(s)ds\Big)^{k_j}.
\end{aligned}
$$

Now, we apply the desingularization method as follows:

$$
\begin{aligned}
&\int_0^t (t-s)^{\alpha_i-1}e^{s}e^{-s}e^{-as}u(s)ds\\
&\le \Big(\int_0^t (t-s)^{p_i(\alpha_i-1)}e^{p_is}ds\Big)^{1/p_i}\Big(\int_0^t e^{-q_is}e^{-aq_is}u(s)^{q_i}ds\Big)^{1/q_i} \qquad (3.19)
\end{aligned}
$$

with $q_i=\frac{p_i}{p_i-1}$, $p_i(\alpha_i-1)+1>0$. By the following inequality, proved in [14] (see also [17]),

$$
\Big(\int_0^t (t-s)^{p_i(\alpha_i-1)}e^{p_is}ds\Big)^{1/p_i}\le Q_ie^{t},
$$

where the number Q_i is given by (3.14),we obtain the inequality

$$
\Big(\int_0^t (t-s)^{\alpha_i-1}\|x(s)\|ds\Big)^{k_j}\le Q_i^{k_j}e^{k_jt}\Big(\int_0^t e^{-q_is}e^{-aq_is}u(s)^{q_i}ds\Big)^{k_j/q_i}. \qquad (3.20)
$$

Then the Hölder inequality yields

$$
\Big(\int_0^t e^{-q_is}e^{-aq_is}u(s)^{q_i}ds\Big)^{k_j}\le t^{k_j-1}\int_0^t e^{-q_ik_js}e^{-aq_ik_js}u(s)^{k_jq_i}ds,\quad j>2
$$

and hence, we have the inequality

$$
\|{}^{RL}I^{\alpha_i}x(s)\|^{k_j}\le\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^{k_j}e^{k_js}s^{\frac{k_j-1}{q_i}}\Big(\int_0^s e^{-q_ik_j\tau}e^{-aq_ik_j\tau}u(\tau)^{k_jq_i}d\tau\Big)^{1/q_i}.
$$

Thus, we obtain the inequality

$$
\begin{aligned}
u(t)&\le K\|x_0\|+K\int_0^t e^{-\gamma_1s}\mu_1(s)u(s)ds\\
&\quad+K\sum_{i=1}^m\frac{Q_i}{\Gamma(\alpha_i)}\int_0^t e^{-[\gamma_{i1}-(a+1))s]}\lambda_{i1}(s)\Big(\int_0^s e^{-q_i\tau}e^{-aq_i\tau}u(\tau)^{q_i}d\tau\Big)^{1/q_i}ds\\
&\quad+K\sum_{j=2}^l\int_0^t e^{-(k_j-1)a)s}\mu_j(s)u(s)^{k_j}\Big)ds+K\sum_{j=2}^l\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^{k_j}\\
&\quad\times\int_0^t\lambda_{ij}(s)e^{(a+k_j)s}s^{\frac{k_j-1}{q_i}}\Big(\int_0^s e^{-q_ik_j\tau}e^{-aq_ik_j\tau}u(\tau)^{k_jq_i}d\tau\Big)^{1/q_i}ds.
\end{aligned}
$$

If $q=q_1q_2\dots q_m$ and $d=m(l+1)+2$, then the inequality $(z_1+z_2+\dots+z_d)^q\le d^{q-1}(z_1^q+z_2^q+\dots+z_m^q)$, valid for any $z_1,z_2,\dots,z_d\ge0$, yields

$$
\begin{aligned}
&u(t)^q\\
&\le d^{q-1}K^q\|x_0\|^q+d^{q-1}K^q\Big(\int_0^t e^{-\gamma_1s}\mu_1(s)u(s)ds\Big)^q\\
&\quad+d^{q-1}K^q\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^q\Big[\int_0^t e^{-[\gamma_{i1}-(a+1)]s}\lambda_{i1}(s)\Big(\int_0^s e^{-q_i\tau}e^{-aq_i\tau}u(\tau^{q_i}d\tau\Big)ds\Big]^q
\end{aligned}
$$

$$
\begin{aligned}
&+ d^{q-1}K^q \sum_{j=2}^{l} \Big[\int_0^t e^{-(k_j-1)as} \mu_j(s) u(s)^{k_j} ds \Big]^q + d^{q-1}K^q \sum_{j=2}^{l}\sum_{i=1}^{m} \Big(\frac{Q_i}{\Gamma(\alpha_i)} \Big)^{k_j q} \\
&\times \Big[\int_0^t \lambda_{ij}(s) e^{(a+k_j)s} s^{\frac{k_j-1}{q_i}} \Big(\int_0^s e^{-aq_ik_j\tau} e^{-k_jq_i\tau} u(\tau)^{k_jq_i} d\tau \Big)^{1/q_i} ds \Big]^q \\
\le\ & d^{q-1}K^q \|x_0\|^q + d^{q-1}K^q \Big(\frac{1}{\gamma_1 p} \Big)^{q/p} \int_0^t \mu_1(s)^q u(s)^q ds \\
&+ d^{q-1}K^q \sum_{i=1}^{m} \Big(\frac{Q_i}{\Gamma(\alpha_i)} \Big)^q \eta_{i1} \Big(\int_0^t e^{-q_i\tau} e^{-aq_i\tau} u(\tau)^{q_i} d\tau \Big)^{\hat{q}_i} \\
&+ d^{q-1}K^q \sum_{j=2}^{l}\sum_{i=1}^{m} \Big(\frac{Q_i}{\Gamma(\alpha_i)} \Big)^{k_j q} \eta_{ij}^q \Big(\int_0^t e^{-q_ik_j\tau} e^{-aq_ik_j\tau} u(\tau)^{q_ik_j} d\tau \Big)^{\hat{q}_i},
\end{aligned}
$$

where $\hat{q}_i = q_1 q_2 \dots q_{i-1} q_{i+1} \dots q_m$.

Using Hölder's inequality with $\hat{q}_i, \hat{p}_i = \frac{\hat{q}_i}{\hat{q}_i - 1}$ and with p, q, we obtain

$$
\begin{aligned}
\Big(\int_0^t e^{-\gamma_1 s} \mu_1(s) u(s) ds \Big)^q &\le \Big(\int_0^t e^{-\gamma_1 p s} ds \Big)^{q/p} \int_0^t \mu_1(s)^q u(s)^q ds \\
&\le \Big(\frac{1}{\gamma_1 p} \Big)^{q/p} \int_0^t \mu_1(s)^q u(s)^q ds,
\end{aligned} \tag{3.21}
$$

$$
\Big(\int_0^t e^{-[\gamma_{i1}-(a+1)]s} \lambda_{i1}(s) u(s) ds \Big)^q \le \Big(\frac{1}{[\gamma_{i1}-(a+1)]p} \Big)^{q/p} \int_0^t \mu_{i1}(s)^q u(s)^q ds,
$$

$$
\begin{aligned}
\Big(\int_0^t e^{as} \mu_j(s) e^{-k_j as} u(s)^{k_j} ds \Big)^q &\le \Big(e^{-(k_j-1)aps} ds \Big)^{q/p} \int_0^t \mu_j(s)^q u(s)^{k_j q} ds \\
&\le \Big[\frac{1}{(k_j-1)ap} \Big]^{q/p} \int_0^t \mu_j(s)^q u(s)^{k_j q} ds,
\end{aligned}
$$

$$
\begin{aligned}
&\Big[\int_0^t \lambda_{ij}(s) e^{(a+k_j)s} s^{\frac{k_j-1}{q_i}} \Big(\int_0^s e^{-aq_ik_j\tau} e^{-k_jq_i\tau} u(\tau)^{k_jq_i} d\tau \Big)^{1/q_i} ds \Big]^q \\
&\le \Big(\int_0^t \lambda_{ij}(s) e^{(a+k_j)s} s^{\frac{k_j-1}{q_i}} ds \Big)^q \Big(\int_0^t e^{-aq_ik_j\tau} e^{-k_jq_i\tau} u(\tau)^{k_jq_i} d\tau \Big)^{\hat{q}_i},
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&\Big(\int_0^t e^{-aq_ik\tau} e^{-k_jq_i\tau} u(\tau)^{kq_i} d\tau \Big)^{\hat{q}_i} \\
&\le \Big(\int_0^t e^{-aq_i\hat{p}_ik\tau} d\tau \Big)^{\frac{1}{\hat{p}_i}} \Big(\int_0^t e^{-k_jq\tau} u(\tau)^{k_jq} d\tau \Big) \\
&\le \Big(\frac{1}{aq_i\hat{p}_ik_j} \Big)^{\frac{1}{\hat{p}_i}} \int_0^t e^{-k_jq\tau} u(\tau)^{k_jq} d\tau.
\end{aligned} \tag{3.23}
$$

The above inequalities yield

$$
\begin{aligned}
&u(t)^q \\
&\le d^{q-1}K^q\|x_0\|^q + d^{q-1}K^q\Big(\frac{1}{\gamma_1 p}\Big)\frac{q}{p}\int_0^t \mu_1(s)^q u(s)^q ds \\
&\quad + d^{q-1}K^q\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^q\eta_{i1}^q\Big(\frac{1}{aq_i\hat{p}_i}\Big)^{\frac{1}{\hat{p}_i}}\int_0^t e^{-q\tau}u(s)^q ds \\
&\quad + d^{q-1}K^q\sum_{j=2}^l\Big[\frac{1}{(k_j-1)ap}\Big]^{q/p}\int_0^t \mu_j(s)^q u(s)^{k_jq}ds \\
&\quad + d^{q-1}K^q\sum_{j=2}^l\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^{k_jq}\eta_{ij}^q\Big(\frac{1}{aq_i\hat{p}_ik_j}\Big)^{\frac{1}{\hat{p}_i}}\int_0^t e^{-k_jq\tau}u(\tau)^{k_jq}d\tau.
\end{aligned}
\tag{3.24}
$$

Therefore $v(t)=u(t)^q$ satisfies the integral inequality

$$
v(t)\le d^{q-1}K^q\|x_0\|^q+\sum_{j=1}^l\int_0^t F_j(s)v(s)^{k_j}ds, \tag{3.25}
$$

where

$$
F_1(t)=d^{q-1}K^q\Big\{\Big(\frac{1}{\gamma_1 p}\Big)^{q/p}\mu_1(t)^q+\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^q\eta_{i1}^q\Big(\frac{1}{aq_i\hat{p}_i}\Big)^{\frac{1}{\hat{p}_i}}e^{-qt}\Big\},
$$

$$
F_j(t)=d^{q-1}K^q\Big\{\Big[\frac{1}{(k_j-1)ap}\Big]^{q/p}\mu_j(s)^q+\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^{k_jq}\eta_{ij}^q\Big(\frac{1}{aq_i\hat{p}_ik_j}\Big)^{\frac{1}{\hat{p}_i}}e^{-k_jq\tau}\Big\}.
$$

Obviously,

$$
\int_0^\infty F_1(s)ds=d^{q-1}K^q\Big\{\Big(\frac{1}{\gamma_1 p}\Big)^{q/p}\omega_1+\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^q\eta_{i1}^q\Big(\frac{1}{aq_i\hat{p}_i}\Big)^{\frac{1}{\hat{p}_i}}\frac{1}{q}\Big\},
$$

$$
\int_0^\infty F_j(s)ds=d^{q-1}K^q\Big\{\Big[\frac{1}{(k_j-1)ap}\Big]^{q/p}\omega_j+\sum_{i=1}^m\Big(\frac{Q_i}{\Gamma(\alpha_i)}\Big)^{k_jq}\eta_{ij}^q\Big(\frac{1}{aq_i\hat{p}_ik_j}\Big)^{\frac{1}{\hat{p}_i}}\frac{1}{k_jq}\Big\},
$$

i.e., $G_j=\int_0^\infty F_j(s)ds<\infty$, $j=1,2,\dots,l$.

From Lemma 3.1 it follows that if $\|x_0\|<\rho$, where $\rho>0$ is defined by (3.11), then

$$
v(t)\le\Psi_0\|x_0\|)=C(\|x_0\|)D_1(\|x_0\|)D_2(\|x_0\|)D_2(\|x_0\|)\dots D_l(\|x_0\|), \tag{3.26}
$$

for $t\in[0,d)$, where

$$
\begin{gathered}
C(\|x_0\|)=d^qK^q\|x_0\|^q, D_1(\|x_0\|)=\int_0^\infty F_1(s)ds,\\
D_2(\|x_0\|)=\Big[1-(k_1-1)[C(\|x_0\|D_1(\|x_0\|)]^{k_1-1}\int_0^\infty F_2(s)ds\Big]^{-\frac{1}{k_1-1}},\\
D_j(\|x_0\|)=\Big[1-(k_j-1)\big[C(\|x_0\|)D_{j-1}(\|x_0\|)\big]^{k_j-1}\int_0^\infty F_j(s)ds\Big]^{-\frac{1}{k_j-1}},
\end{gathered}
\tag{3.27}
$$

for $j=3,\dots,l$.

Obviously, the right hand side of the inequality (3.26) is finite if $\|x_0\| < \rho$ and it is going to ∞ for $\|x_0\| \to \rho$, if $l > 1$. If $l = 1$, then it is defined for all $x_0 \in \mathbb{R}^N$. Hence, we have the estimate

$$v(t) = u(t)^q = \big(\|x(t)\|e^{at}\big)^q \le \Psi_0(\|x_0\|),\ t \in [0, d),$$

i.e.,

$$\|x(t)\| \le \Psi(\|x_0\|)e^{-at},\ t \in [0, d). \tag{3.28}$$

From this inequality it follows that $\lim_{t\to d} x(t) = d_- < \infty$ and by Theorem 2.1 there is an $\epsilon > 0$ such that the inial value problem (2.1) has a unique solution $w(t)$, defined on the interval $[0, d+\epsilon)$. This is a contradiction with the maximality of the solution $x(t)$. Hence, the inequality (3.28) holds for all $t \in [0,\infty)$ and the proof is complete. □

4. Illustrative example

Let us apply Theorem 3.2 to the fractionally perturbed pendulum equation

$$v''(t) + 2v'(t) + 4v(t) + 3e^{-4t\,C}D^{1/3}v(t) + 5e^{-4t}\Big[{}^{C}D^{1/2}v(t)\Big]^2 = 0. \tag{4.1}$$

We can write this equation in the form of system (1.8), where $x(t) = (x_1(t), x_2(t)) = (v(t), v'(t))$, $m = 2$, $\alpha_1 = 1 - \frac{1}{3} = \frac{2}{3}$, $\alpha_2 = 1 - \frac{1}{2} = \frac{1}{2}$,

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix}, \quad f(t, x, u_1, u_2) = \begin{pmatrix} 0 \\ -3e^{-4t}u_{12} - 5e^{-4t}u_{22}^2, \end{pmatrix},$$

where $u_1 = (u_{11}, u_{12})$, $u_2 = (u_{21}, u_{22})$,

$$f(t, x(t), {}^{RL}I^{\alpha_1}x(t), {}^{RL}I^{\alpha_2}x(t)) = \begin{pmatrix} 0 \\ -3e^{-4t\,RL}I^{\frac{2}{3}}x_2(t) - 5e^{-4t}\Big[{}^{RL}I^{1/2}x_2(t)\Big]^2. \end{pmatrix},$$

Obviously,

$$\|f(t, x, u_1, u_2)\| \le 3e^{-4t}\|u_1\| + 5e^{-4t}\|u_2\|^2. \tag{4.2}$$

The system has the form

$$\dot{x}(t) = Ax(t) + \begin{pmatrix} 0 \\ -3e^{-4t\,RL}I^{\frac{2}{3}}x_2(t) - 5e^{-4t}\Big[{}^{RL}I^{1/2}x_2(t)\Big]^2 \end{pmatrix}. \tag{4.3}$$

Now, let us find the constants $a > 0$ and $K > 0$ from condition (A1). The Jordan block of the matrix A has the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha, \end{pmatrix},$$

where $\alpha = -1$, $\beta = 1$. One can prove by using the Putzer's method (see [24]) that

$$e^{At}x = e^{\alpha t}\Big(\cos(\beta t)I + \frac{1}{\beta}\sin(\beta t)(A - I)\Big)x, \tag{4.4}$$

where I is the unit matrix. This yields the estimate

$$\|e^{At}x\| \le e^{-t}\big(\|I\| + \|A - I\|\big)\|x\|, \tag{4.5}$$

where we use the norm $\|C\| = \max\{|c_{11}| + |c_{12}|, |c_{21}| + |c_{22}|\}$ of a 2×2 matrix $C = (c_{ij})$. For this norm the inequality $\|Cy\| \le \|C\|\|y\|$ is valid for any $y = (y_1, y_2)$ with the norm $\|y\| = \max\{|y_1|, |y_2|\}$. Since $\|I\| = 1, \|A - I\| = 7$, from (4.4) we obtain

$$\|e^{At}x\| \le 8e^{-t}\|x\| \quad \text{for all } x \in \mathbb{R}^2, \tag{4.6}$$

i.e., we have $a = 1$ and $K = 8$. This means that the condition (A1) is satisfied with $a = 1, K = 8$.

From (4.2) it follows that the mapping f has the form

$$\|f(t, x, u_1, u_2)\| \le \lambda_{11}(t)e^{-\gamma_{11}t}\|u_1\| + \lambda_{12}(t)\|u_2\|^{k_2} \tag{4.7}$$

with $\lambda_{11}(t) = 3$, $\gamma_{11} = 4$, $\lambda_{12}(t) = e^{-4t}, k_2 = 2$. This means that the condition (A2) is satisfied . The condition (A3) is trivially satisfied because all ω_j are equal zero. The condition (A4) is also satisfied because

$$\eta_{11} = \int_0^\infty e^{-[\gamma_{11}-(a+1)]s}\lambda_{11}(s)ds = 3\int_0^\infty e^{-2s}ds = \frac{3}{2}, \tag{4.8}$$

$$\begin{aligned} \eta_{12} &= \int_0^\infty e^{[a+k_2]s}s^{\frac{k_2-1}{q}}\lambda_{12}(s)ds = \int_0^\infty e^{3s}s^{1/q}e^{-4s}ds \\ &= \int_0^\infty s^{1/q}e^{-s} = \Gamma\Big(1+\frac{1}{q}\Big) \end{aligned} \tag{4.9}$$

with $q = q_1q_2$, where we define $q_1 = 2$, $q_2 = 3$, i.e., $q = 6$. For the numbers $p_1 = \frac{q_1}{q_1-1} = 2$ and $p_2 = \frac{q_2}{q_2-1} = \frac{3}{2}$, $\hat{q}_1 = q_2 = 3$, $\hat{q}_2 = q_1 = 2, \hat{p}_1 = \frac{\hat{q}_1}{\hat{q}_1-1} = \frac{3}{2}$, $\hat{p}_2 = \frac{\hat{q}_2}{\hat{q}_2-1} = 2$, we have

$$p_1(\alpha_1 - 1) + 1 = \frac{1}{3}, \quad p_2(\alpha_2 - 1) + 1 = \frac{1}{4}.$$

Since the mapping f is smooth in all its variables, the condition (A5) follows from the Lagrange mean value theorem.

Now let us calculate the numbers Q_1, Q_2, G_1, G_2 from the assumptions of Theorem 3.2:

$$Q_1 = \Big[\frac{\Gamma(p_1(\alpha_1-1)+1)}{p_1^{p_1(\alpha_1-1)+1}}\Big]^{1/p_1} = \Big[\frac{\Gamma(\frac{1}{3})}{2^{1/3}}\Big]^{1/2} = \frac{\Gamma(\frac{1}{3})^{1/2}}{2^{1/6}}, \tag{4.10}$$

$$Q_2 = \Big[\frac{\Gamma(p_2(\alpha_2-1)+1)}{p_2^{p_2(\alpha_2-1)+1}}\Big]^{1/p_2} = \Big[\frac{\Gamma(\frac{1}{4})}{2^{\frac{1}{4}}}\Big]^{1/3} = \frac{\Gamma(\frac{1}{4})^{1/3}}{3^{1/12}} \tag{4.11}$$

and since $m = 2$, $l = 2$, $d = m(l+1)+2 = 8$, $\eta_{11} = \frac{3}{2}$, $\eta_{12} = \Gamma\big(1+\frac{1}{q}\big) = \Gamma\big(\frac{7}{6}\big)$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{2}$, we have

$$G_1 = 8^5\cdot 8^6\Big(\frac{\Gamma(\frac{1}{3})^{1/2}}{2^{1/6}}\frac{1}{\Gamma(\frac{2}{3})}\Big)^6\Big(\frac{3}{2}\Big)^6\Big(\frac{1}{3\cdot 2\cdot\frac{3}{2}}\Big)^{\frac{3}{2}}\frac{1}{6}, \tag{4.12}$$

$$G_2 = 8^5\cdot 8^6\Big(\frac{\Gamma(\frac{1}{4})^{1/3}}{3^{1/2}}\frac{1}{\Gamma(\frac{1}{2})}\Big)^{12}\Gamma\Big(\frac{7}{6}\Big)^6\Big(\frac{1}{3\cdot 2, 2}\Big)^{1/2}\frac{1}{2\cdot 6}, \tag{4.13}$$

$$D_2(z) = \frac{1}{1-[C(z)D_1(z)]G_2} = \frac{1}{1-d^qK^qz^qG_1G_2} = \frac{1}{1-8^6\cdot 8^6z^qG_1G_2}, \tag{4.14}$$

where $D_1(z) \equiv G_1$. Therefore

$$\rho = \sup\big\{z\in\mathbb{R} : d^qK^q|z|^qG_1 < \frac{1}{G_2}\big\} = \sup\big\{z\in\mathbb{R} : 8^6\cdot 8^6|z|^6\cdot G_1 < \frac{1}{G_2}\big\},$$

i.e.,

$$\rho = \sup\big\{z\in\mathbb{R} : |z| < \frac{1}{(8^6\cdot 8^6\cdot G_1\cdot G_2)^{\frac{1}{6}}}\big\},$$

where G_1, G_2 are given by (4.12) and (4.13), respectively, and

$$\Psi(w) = [C(w)D_1(w)D_2(w)]^{1/q} = \frac{dKw}{[1 - d^q K^q w^q G_1 G_2]^{1/q}}.$$

By Theorem 3.2,

$$\|x(t)\| \le \frac{dK\|x_0\|}{[1 - d^q K^q \|x_0\|^q G_1 G_2]^{1/q}} e^{-t} = \frac{8 \cdot 8\|x_0\|}{[1 - 8^6 \cdot 8^6 \|x_0\|^q G_1 G_2]^{1/6}} e^{-t} \tag{4.15}$$

for the solution $x(t)$ of the equation (4.3) satisfying the initial condition $x(0) = x_0$ with $\|x_0\| < \rho$.

This means that for the solution $v(t)$ of the equation (4.1), which is the first coordinate of the solution $x(t) = (v(t), v'(t))$, the inequality (4.15) holds.

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Eva Brestovanská

Department of Economics and Finance, Faculty of Management, Comenius University, Odbojárov 10, 831 04 Bratislava, Slovakia

E-mail address: `Eva.Brestovanska@fm.uniba.sk`

Milan Medveď (corresponding author)

Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, 842 48 Bratislava, Slovakia

E-mail address: `Milan.Medved@fmph.uniba.sk`