Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 287, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

DIRECT AND INVERSE ROBIN-REGGE PROBLEMS

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Communicated by Tuncay Aktosun

Abstract. The Sturm-Liouville equation is considered on a bounded interval with a Robin boundary condition at the left end and a generalized Regge condition at the right end. Properties of the spectrum of this problem are derived, including the location of the spectrum in the lower half-plane and the asymptotic distribution of the eigenvalues. The inverse problem is solved, given one spectrum or parts of two spectra.

1. Introduction

It is well known that the spectra of the Neumann-Dirichlet (or the Dirichlet-Neumann) and the Dirichlet-Dirichlet boundary-value problems for the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, a], \tag{1.1}$$

with spectral parameter λ, generated by the same potential q, uniquely determine this potential in $L_2(0, a)$. It is also known that the spectra of two boundary-value problems with the same Robin boundary condition at one of the ends and different Robin conditions at the other end of the interval uniquely determine the potential and the constants in the boundary conditions. These results are due to Borg [1] (see also [12, 13, 14]). If the boundary conditions are given data, then the problem of recovering the potential appears to be overdetermined in case of Robin conditions. One does not need to know all the eigenvalues of the two spectra in this case. In a further development of this theory one spectrum together with knowledge of a part of the potential is used, see [3, 6, 7, 15, 19, 22, 23].

Another direction of generalization of the above results is to use eigenvalues of more than two spectra to determine the potential. In [4] it was shown that two thirds of the union of three spectra of boundary-value problems with the same boundary condition at one of the ends uniquely determines the potential. In [8] a similar but more general sufficient condition of unique solvability was given for the case when the known eigenvalues were taken from n different spectra, see [9] for a topical review.

It is also known that if we impose the so called generalized Regge condition

$$y'(a) + (i\lambda\alpha + \beta)y(a) = 0 \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34B09, 34B07, 34B24, 34L20, 34A55.
Key words and phrases. String problem; viscous damping; Robin boundary condition; generalized Regge boundary condition; spectral asymptotics; inverse problems.

Submitted August 8, 2017. Published November 17, 2017.

with $\alpha > 0$ at the right end of the interval, then we need roughly speaking only one spectrum instead of two spectra. This was shown first in [10, 11] for the boundary-value problem generated by the string equation. For the Sturm-Liouville problem this was shown in [20] where the condition at the left was $y(0) = 0$. In the case of the Dirichlet-Regge problem with $\alpha = 1$ and $\beta = 0$, Regge [21] described how to recover the potential from the spectrum using the Jost function and the Gelfand-Levitan equation. A detailed account of the spectral asymptotics for the Dirichlet-Regge problem, including the case $\alpha = 1$, is given in [17, Section 7.2].

In this paper we consider the Sturm-Liouville problem (1.1) with (generalized) Regge boundary condition (1.2) at the right end and Robin boundary condition

$$y'(0) - \delta y(0) = 0 \tag{1.3}$$

at the left end. Our aim in this paper is to describe the spectrum of this problem and to solve the inverse problems by one spectrum and by parts of two spectra with two different δ in the Robin condition (1.3).

In Section 2 we show that the problem of small transverse vibrations of a smooth inhomogeneous string damped at the right end can be reduced to (1.1)–(1.3) by means of the Liouville transformation. Also in that section we show that all eigenvalues of this string problem lie in the upper half-plane.

In Section 3 we investigate the spectrum of (1.1)–(1.3) admitting eigenvalues in the lower half-plane, and in particular the asymptotic distribution of eigenvalues is given.

In Section 4 we solve the corresponding inverse problem, namely we show that if a sequence of complex numbers (λ_k) satisfies certain conditions (which appear to be necessary and sufficient), then there exist a unique real $L_2(0, a)$ potential and a unique set of real parameters δ, $\alpha > 0$ and β which generate a Robin-Regge problem with the spectrum (λ_k). A corresponding existence and uniqueness result is obtained for the string problem.

In Section 5 we show that the characteristic functions of two Robin-Regge problems with the same Regge condition at the right end and different Robin conditions at the left satisfy a functional equation which was investigated in [18]. Using the results of [18] and the results of Section 4 we solve the following inverse problem: given two sets of complex numbers, given $\delta_1 - \delta_2$, find the potential and a set of real parameters δ_1, $\alpha > 0$ and β which generate a Robin-Regge problems with the same Regge condition at the right end and with different coefficients δ_2 and δ_1 in the Robin condition at the left end for which the given sets of complex numbers are parts of spectra with δ_1 and δ_2, respectively.

2. String problem

Small transversal vibrations of a smooth inhomogeneous string subject to viscous damping are described by the boundary-value problem

$$\frac{\partial^2 u}{\partial s^2} - \rho(s)\frac{\partial^2 u}{\partial t^2} = 0, \tag{2.1}$$

$$\Big(\frac{\partial u}{\partial s} - pu\Big)\Big|_{s=0} = 0, \tag{2.2}$$

$$\Big(\frac{\partial u}{\partial s} + \nu\frac{\partial u}{\partial t}\Big)\Big|_{s=l} = 0. \tag{2.3}$$

Here $u = u(s,t)$ is the transversal displacement of a point of the string which is as far as s from the left end of the string at time t, l is the length of the string, $\rho \geq \varepsilon > 0$ is its density, and the constant $p > 0$ is the Robin coefficient. We will assume $\rho \in L_\infty(0,l)$. The left end of the string is constrained elastically while the right end is free to move in the direction orthogonal to the equilibrium position of the string subject to damping with damping coefficient $\nu > 0$. Substituting $u(s,t) = v(s)e^{i\lambda t}$ we arrive at

$$\frac{\partial^2 v}{\partial s^2} + \lambda^2 \rho(s) v = 0, \tag{2.4}$$

$$\Big(\frac{\partial v}{\partial s} - pv\Big)\Big|_{s=0} = 0, \tag{2.5}$$

$$\Big(\frac{\partial v}{\partial s} + i\lambda\nu v\Big)\Big|_{s=l} = 0. \tag{2.6}$$

The eigenvalue problem (2.4)–(2.6) is described by the operator pencil

$$L(\lambda) = \lambda^2 M - i\lambda K - A, \tag{2.7}$$

where the operators A, K and M act in the Hilbert space $H = L_2(0,l) \oplus \mathbb{C}$ according to

$$A\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} -v'' \\ v'(l) \end{pmatrix}, \tag{2.8}$$

$$D(L) := D(A) = \Big\{ \begin{pmatrix} v \\ c \end{pmatrix} : v \in W_2^2(0,l),\ v'(0) - pv(0) = 0,\ c = v(l) \Big\}, \tag{2.9}$$

$$M = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix}, \tag{2.10}$$

where $W_2^2(0,l)$ is the Sobolev space of order 2 over the interval $(0,l)$.

Recall that an operator A is said to be strictly positive, written $A \gg 0$, if there is $\varepsilon > 0$ such that $(AY,Y) \geq \varepsilon(Y,Y)$ for all Y in the domain of A.

Proposition 2.1. *The operators A, K and M are self-adjoint, $M \geq 0$ and $K \geq 0$ are bounded, $M + K \gg 0$, and $A \gg 0$ has a compact resolvent.*

Proof. The statements about M and K are obvious. The proof of the statements about A is similar to that of [17, Proposition 2.1.1]. To prove that A is self-adjoint, we will use [17, Theorem 10.3.5], so that we have to find U_3, U_1 and U defined in [17, (10.3.12) and (10.3.13)]. Since the differential expression in A is $-v'' = (-v')'$, the first two quasi-derivatives of v are $v^{[0]} = v$ and $v^{[1]} = -v'$, see [17, Definition 10.2.1]. It is easy to see that, in the notation of [17, Section 10.3],

$$U_1 = \begin{pmatrix} -p & -1 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & \tilde\beta & -1 \end{pmatrix},$$

with $\tilde\beta = 0$, so that

$$U_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \tilde\beta & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} -p & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & \tilde\beta & -1 & 0 & -1 \end{pmatrix}.$$

We have introduced the parameter $\tilde{\beta}$ since we will encounter the matrix V with $\tilde{\beta} \neq 0$ in the next section. Denoting the j-th standard basis eigenvector in $\mathbb{C}^n$ by e_j, it follows that

$$\begin{gathered}
N(U_1) = \operatorname{span}\{e_1 - pe_2, e_3, e_4\}, \\
U_3(N(U_1)) = \operatorname{span}\{pe_1 + e_2, e_4 - \tilde{\beta} e_5 + e_6, e_3 - e_5\}, \\
R(U^*) = \operatorname{span}\{pe_1 + e_2, e_3 - e_5, e_4 - \tilde{\beta} e_3 + e_6\},
\end{gathered}$$

so that $U_3(N(U_1)) = R(U^*)$. Hence A is self-adjoint by Theorem [17, Theorem 10.3.5]. Now it follows from [17, Theorem 10.3.8] that A has a compact resolvent and that A is bounded below.

Finally, for $Y = (v, c)^{\mathrm{T}} \in D(A)$ we conclude in view of [17, (10.2.5)] that

$$\begin{aligned}
(AY, Y) &= \int_0^a |v'(x)|^2 dx - v'(x)\overline{v(x)}|_0^l + v'(l)\overline{c} \\
&= \int_0^a |v'(x)|^2 dx - v'(l)\overline{v(l)} + v'(0)\overline{v(0)} + v'(l)\overline{c} \\
&= \int_0^a |v'(x)|^2 dx + p|v(0)|^2 \geq 0.
\end{aligned} \tag{2.11}$$

Furthermore, if $(AY, Y) = 0$, then $v' = 0$ and $v(0) = 0$, which gives $v = 0$ and then $c = v(0) = 0$. Therefore $A > 0$ has been shown. Since A has a compact resolvent and thus a discrete spectrum consisting only of eigenvalues, it follows that $A \gg 0$. □

Proposition 2.2. *All eigenvalues of* (2.4)–(2.6) *lie in the open upper half-plane.*

Proof. Let $Y = (v, c)^{\mathrm{T}} \in D(A) \cap N(K)$ and $\lambda \in \mathbb{R}$ such that $\lambda^2 MY - AY = 0$. It follows that $v'' + \lambda^2 \rho y = 0$, $v(l) = c = 0$ and $v'(l) = 0$, and the unique solution of this initial value problem is $v = 0$. An application of [17, Lemma 1.2.4, 3] completes the proof. □

For the spectral asymptotics and for the inverse problem it is more convenient to write (2.4) as a Sturm-Liouville equation. Hence we give an alternate approach under the additional assumption that $\rho \in W_2^2(0, l)$. Then the Liouville transform [2, p. 292]

$$x(s) = \int_0^s \rho^{1/2}(r)\, dr, \quad 0 \leq s \leq l, \tag{2.12}$$

$$y(\lambda, x) = \rho^{1/4}(s(x)) v(\lambda, s(x)), \quad 0 \leq x \leq a,\ \lambda \in \mathbb{C}, \tag{2.13}$$

leads to the equivalent boundary-value problem

$$y'' - q(x)y + \lambda^2 y = 0, \tag{2.14}$$

$$y'(\lambda, 0) - \delta y(\lambda, 0) = 0, \tag{2.15}$$

$$y'(\lambda, a) + (i\lambda\alpha + \beta) y(\lambda, a) = 0, \tag{2.16}$$

where

$$q(x) = \rho^{-1/4}(s(x)) \frac{d^2}{dx^2} \rho^{1/4}(s(x)), \tag{2.17}$$

$$a = \int_0^l \rho^{1/2}(r)\, dr, \tag{2.18}$$

$$\beta = -\rho^{-1/4}(s(a))\frac{d\rho^{1/4}(s(x))}{dx}\Big|_{x=a}, \tag{2.19}$$

$$\delta = p\rho^{-\frac{1}{2}}(0) + \rho^{-1/4}(0)\frac{d\rho^{1/4}(s(x))}{dx}\Big|_{x=0}, \tag{2.20}$$

$$\alpha = \rho^{-\frac{1}{2}}(s(a))\nu > 0. \tag{2.21}$$

The eigenvalue problem (2.14)–(2.16) is described by the operator pencil

$$\tilde{L}(\lambda) = \lambda^2\tilde{M} - i\lambda\tilde{K} - \tilde{A}, \tag{2.22}$$

where the operators $\tilde{A}$, $\tilde{K}$ and $\tilde{M}$ act in the Hilbert space $H = L_2(0,a)\oplus\mathbb{C}$ according to

$$\tilde{A}\begin{pmatrix} y \\ c \end{pmatrix} = \begin{pmatrix} -y'' + qy \\ y'(a) + \beta y(a) \end{pmatrix}, \tag{2.23}$$

$$D(\tilde{L}) := D(\tilde{A}) = \Big\{ \begin{pmatrix} y \\ c \end{pmatrix} : y \in W_2^2(0,a),\ y'(0) - \delta y(0) = 0,\ c = y(a)\Big\}, \tag{2.24}$$

$$\tilde{M} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}. \tag{2.25}$$

Since (2.13) establishes an isomorphism between $L_2(0,l)$ and $L_2(0,a)$ and the corresponding Sobolev spaces of order 2, it is clear that $\tilde{M}$, $\tilde{K}$ and $\tilde{A}$ have the same properties as M, K, A, respectively, stated in Proposition 2.1. In particular, $\tilde{A}$ is self-adjoint and satisfies $\tilde{A} \gg 0$.

3. Robin-Regge problem and its spectrum

3.1. **Robin-Regge problem.** In Section 2 we have seen that the string problem (2.4)–(2.6) can be transformed into a problem of the form (2.14)–(2.16). In this section we will investigate this latter problem:

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0,a] \tag{3.1}$$

$$y'(0) - \delta y(0) = 0, \tag{3.2}$$

$$y'(a) + (i\alpha\lambda + \beta)y(a) = 0, \tag{3.3}$$

where we admit general real-valued functions $q \in L_2(0,a)$ and real numbers $\alpha > 0$, β and δ. We call this problem a Robin-Regge boundary-value problem since the boundary condition (3.2) is also called a boundary condition of Robin type, see [5, p. 70], and the boundary condition (3.3) is of (generalized) Regge type, see [17, (2.1.5)].

This problem has a representation as a quadratic operator pencil $\tilde{L}$ as in (2.22) with the operators $\tilde{A}$, $\tilde{M}$, $\tilde{K}$ given by (2.23)–(2.25). The proof of Proposition 2.1, with $\tilde{\beta} = \beta$ and except for the last paragraph concerning positivity of A, also applies to these operators. Hence we have

Proposition 3.1. *The operators $\tilde{A}$, $\tilde{K}$ and $\tilde{M}$ are self-adjoint, $\tilde{M} \geq 0$ and $\tilde{K} \geq 0$ are bounded, $\tilde{M} + \tilde{K} \gg 0$, and $\tilde{A}$ is bounded below and has a compact resolvent.*

Therefore [17, Lemmas 1.1.11 and 1.2.1 and Theorem 1.3.3] give the following result.

Proposition 3.2. *The spectrum of the pencil $\tilde{L}$ consists of isolated eigenvalues of finite algebraic multiplicity located in the closed upper half-plane and on the* *imaginary axis, at most finitely many eigenvalues lie on the negative imaginary axis, and the total algebraic multiplicity of the eigenvalues on the negative imaginary axis equals the total multiplicity of the negative eigenvalues of $\tilde{A}$. The eigenvalues, with multiplicity, are symmetric with respect to the imaginary axis.*

3.2. **Characteristic function.** We denote by $\mathcal{L}^a$ the Paley-Wiener class of entire functions of exponential type $\leq a$ whose restrictions to the real axis belong to $L_2(-\infty,\infty)$.

Every solution of (3.1) which satisfies the Robin condition (3.2) is a multiple of the solution $c(\lambda,\delta,\cdot)$ of (3.1) which satisfies the initial conditions $c(\lambda,\delta,0)=1$, $c'(\lambda,\delta,0)=\delta$.

Proposition 3.3. *The function c has the representation*

$$\begin{aligned} c(\lambda,\delta,x) &= \cos\lambda x + \int_0^x B(x,t,\delta)\cos\lambda t\,dt \\ &= \cos\lambda x + B(x,x,\delta)\frac{\sin\lambda x}{\lambda} - \int_0^x B_t(x,t,\delta)\frac{\sin\lambda t}{\lambda}\,dt, \end{aligned} \tag{3.4}$$

where

$$B(x,t,\delta) = \delta + \tilde{K}(x,t) + \tilde{K}(x,-t) + \delta\int_t^x [\tilde{K}(x,\xi) - \tilde{K}(x,-\xi)]d\xi,$$
$$B_t(x,t,\delta) = \frac{\partial B(x,t,\delta)}{\partial t},$$

and $\tilde{K}(x,t)$ is the unique solution of the integral equation

$$\tilde{K}(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(\zeta)d\zeta + \int_0^{\frac{x+t}{2}}\int_0^{\frac{x-t}{2}} q(\zeta+\xi)\tilde{K}(\zeta+\xi,\zeta-\xi)\,d\xi\,d\zeta$$

in the region $\{(x,t)\in[0,a]\times[0,a] : |t|\leq x\}$. The solution $\tilde{K}(x,t)$ possesses first order partial derivatives each belonging to $L_2(0,a)$ as a function of each of its variables. Moreover,

$$B(x,x,\delta) = \delta + \frac{1}{2}\int_0^x q(t)\,dt.$$

Proof. Let $s(\lambda,\cdot)$ and $c(\lambda,0,\cdot)$ be the solutions of the initial value problems for (3.1) with the initial conditions $s(\lambda,0)=0$, $s'(\lambda,0)=1$ and $c(\lambda,0,0)=1$, $c'(\lambda,0,0)=0$, respectively. Then we obviously have

$$c(\lambda,\delta,x) = c(\lambda,0,x) + \delta s(\lambda,x). \tag{3.5}$$

From [17, Theorem 12.2.9] we know the representations

$$s(\lambda,x) = \frac{\sin\lambda x}{\lambda} + \int_0^x K(x,t)\frac{\sin\lambda t}{\lambda}\,dt,$$
$$c(\lambda,0,x) = \cos\lambda x + \int_0^x B(x,t,0)\cos\lambda t\,dt,$$

where $K(x,t)=\tilde{K}(x,t)-\tilde{K}(x,-t)$, $B(x,t,0)=\tilde{K}(x,t)+\tilde{K}(x,-t)$ and

$$B(x,x,0) = K(x,x) = \tilde{K}(x,x) = \frac{1}{2}\int_0^x q(t)\,dt.$$

Substituting these representations into (3.5) and straightforward integration by parts completes the proof. □

It is clear that

$$c'(\lambda,\delta,x) = -\lambda\sin\lambda x + B(x,x,\delta)\cos\lambda x + \int_0^x B_x(x,t,\delta)\cos\lambda t\,dt. \tag{3.6}$$

Using (3.4) and (3.6) we obtain

$$c(\lambda,\delta,a) = \cos\lambda a + B\frac{\sin\lambda a}{\lambda} + \frac{\psi(\lambda)}{\lambda}, \tag{3.7}$$

$$c'(\lambda,\delta,a) = -\lambda\sin\lambda a + B\cos\lambda a + \tilde{\psi}(\lambda), \tag{3.8}$$

where $B := B(a,a,\delta)$, $\psi \in \mathcal{L}^a$, $\tilde{\psi} \in \mathcal{L}^a$. Moreover, $\psi(0) = 0$ since $c(\cdot,\delta,a)$ is an entire function. Here we have used that by Paley-Wiener's theorem $\mathcal{L}^a$-functions are the Fourier transformations of square summable functions supported on $[0,a]$. It is clear that $\lambda \in \mathbb{C}$ belongs to the spectrum of (3.1)–(3.3) if and only if $c(\lambda,\delta,\cdot)$ satisfies the boundary condition (3.3), that is, if and only if

$$\phi(\lambda,\delta) := c'(\lambda,\delta,a) + (i\alpha\lambda + \beta)c(\lambda,\delta,a) \tag{3.9}$$

satisfies $\phi(\lambda,\delta) = 0$. The function $\phi(\cdot,\delta)$ is called the characteristic function of (3.1)–(3.3). It is well known, see e. g. [16, Section 6.3], that the algebraic multiplicity of the eigenvalue λ equals the multiplicity of the zero λ of $\phi(\cdot,\delta)$.

3.3. Spectrum in the lower half-plane.

Theorem 3.4. *The eigenvalues of the Robin-Regge problem* (3.1)–(3.3) *have the following properties:*

1. *Only a finite number of the eigenvalues lie in the closed lower half-plane.*
2. *All nonzero eigenvalues in the closed lower half-plane lie on the negative imaginary semiaxis and are simple. If their number κ is positive, they will be uniquely indexed as $\lambda_{-j} = -i|\lambda_{-j}|$, $j = 1,\dots,\kappa$, satisfying $|\lambda_{-j}| < |\lambda_{-(j+1)}|$, $j = 1,\dots,\kappa-1$.*
3. *If $\kappa > 0$, then the numbers $i|\lambda_{-j}|$, $j = 1,\dots,\kappa$, are not eigenvalues.*
4. *If $\kappa \geq 2$, then in each of the intervals $(i|\lambda_{-j}|, i|\lambda_{-(j+1)}|)$, $j = 1,\dots,\kappa-1$, the number of eigenvalues, counted with multiplicity, is odd.*
5. *If $\kappa > 0$ and 0 is not an eigenvalue, then the interval $(0, i|\lambda_{-1}|)$ contains an even number of eigenvalues, counted with multiplicity, or does not contain any eigenvalues.*
6. *If $\kappa > 0$ and 0 is an eigenvalue, then 0 is a simple eigenvalue and the interval $(0, i|\lambda_{-1}|)$ contains an odd number of eigenvalues, counted with multiplicity.*
7. *If $\alpha \neq 1$, then the Robin-Regge problem has infinitely many eigenvalues, which lie in a horizontal strip of the complex plane.*

Proof. First we are going to show that there are no eigenvalues λ with corresponding eigenvectors of the form $(y,0)^{\mathrm{T}} \in D(\tilde{A})$. Indeed, otherwise the definition of $D(\tilde{A})$ in (4.21) would imply $y(a) = c = 0$, and (3.3) then would give $y'(a) = 0$. Hence y would be a solution of the initial value problem (3.1), $y(a) = y'(a) = 0$, which has only the trivial solution, i. e., $y = 0$. But this is impossible since $(y,0)^{\mathrm{T}}$ was supposed to be an eigenvector.

In view of [17, Theorem 1.5.6, 1, and Remark 1.5.8], there are no nonzero eigenvalues of type I, i. e., eigenvalues λ such that also $-\lambda$ is an eigenvalue. Therefore all nonzero eigenvalues are of type II according to [17, Definition 1.5.2]. By [17, Remark 1.5.8] there are no nonzero real eigenvalues of type II, and it follows that there are no nonzero real eigenvalues at all. Therefore statements (1) through (5) follow from [17, Theorem 1.5.7].

To prove statement (6) we have to consider the case that 0 is an eigenvalue. Then the first paragraph of this proof shows that $N(\tilde{A}) \not\subset N(\tilde{K})$, and 0 is not an eigenvalue of type II by [17, Remark 1.5.3, 2]. Furthermore, $N(\tilde{M}) \cap N(\tilde{A}) \subset N(M) \cap D(\tilde{A}) = \{0\}$, and the geometric multiplicity of the eigenvalue is 1 by the first paragraph of this proof. Next we are going to show that the eigenvalue is simple, that is, there are no associated vectors. By proof of contradiction, assume there are $Y_0 = (y_0, c_0)^{\mathrm{T}} \neq 0$ and $Y_1 = (y_1, c_1)^{\mathrm{T}}$ in $D(A)$ such that

$$\tilde{A}Y_0 = 0 \quad \text{and} \quad -i\tilde{K}Y_0 - \tilde{A}Y_1 = 0.$$

We recall that $c_0 \neq 0$. It follows that

$$-y_j'' + qy_j = 0,\ y_j'(0) - \delta y_j(0) = 0, \quad j = 0, 1,$$

so that y_1 is a multiple of y_0, say $y_1 = dy_0$. But then also $Y_1 = dY_0$, and

$$(0, c_0)^{\mathrm{T}} = \frac{1}{\alpha}\tilde{K}Y_0 = i\frac{1}{\alpha}\tilde{A}Y_1 = i\frac{d}{\alpha}\tilde{A}Y_0 = 0,$$

which is impossible. Hence statement (6) also follows from [17, Theorem 1.5.7].

We have established in the previous subsection that the spectrum coincides with the zeros of $\phi(\cdot, \delta)$, and an application of [17, Lemma 7.1.3] shows that statement (7) holds. □

3.4. **Spectral asymptotics.** For $\alpha \neq 1$ the Robin-Regge problem has infinitely many eigenvalues, and their the asymptotic representation is given in the following theorem.

Theorem 3.5. (1) *If $\alpha \in (0, 1)$ then the eigenvalues $(\xi_k)_{k=-\infty,\ k\neq 0}^{\infty}$ of problem* (3.1)–(3.3) *can be indexed such that $\xi_{-k} = -\overline{\xi_k}$ for all not pure imaginary ξ_k and behave asymptotically as follows:*

$$\xi_k = \frac{\pi(k-1)}{a} + \frac{i}{2a}\ln\Big(\frac{\alpha+1}{1-\alpha}\Big) + \frac{P}{k} + \frac{\gamma_k}{k}, \quad k \in \mathbb{N}, \tag{3.10}$$

where

$$P = \frac{1}{\pi}\Big(B - \frac{\beta}{\alpha^2 - 1}\Big), \tag{3.11}$$

and where $(\gamma_k)_{k=1}^{\infty} \in l_2$.
(2) *If $\alpha \in (1, \infty)$ then the eigenvalues $(\xi_k)_{k=-\infty}^{\infty}$ of problem* (3.1)–(3.3) *can be indexed such that $\xi_{-k} = -\overline{\xi_k}$ for all not pure imaginary ξ_k and behave asymptotically as follows:*

$$\xi_k = \frac{\pi\left(k - \frac{1}{2}\right)}{a} + \frac{i}{2a}\ln\left(\frac{\alpha+1}{\alpha-1}\right) + \frac{P}{k} + \frac{\tilde{\gamma}_k}{k}, \quad k \in \mathbb{N}, \tag{3.12}$$

where P is given by (3.11) *and where $(\tilde{\gamma}_k)_{k=1}^{\infty} \in l_2$.*

Proof. Using (3.7), (3.8) and (3.9) we obtain that the characteristic function ϕ of (3.1)–(3.3) given in (3.9) satisfies

$$-i\phi(\lambda, \delta) = \lambda[\alpha\cos\lambda a + i\sin\lambda a] - i(B+\beta)\cos\lambda a + \alpha B\sin\lambda a + \hat{\psi}(\lambda), \tag{3.13}$$

where $\hat{\phi} \in \mathcal{L}^a$. The right hand side of (3.13) is of the form [17, (7.1.4)] with $M = \alpha B$ and $N = B + \beta$. Then the number P defined in [17, Lemma 7.1.3] becomes

$$P = \frac{N - \alpha M}{\pi(1-\alpha^2)} = \frac{B + \beta - \alpha^2 B}{\pi(1-\alpha^2)} = \frac{B}{\pi} + \frac{\beta}{\pi(1-\alpha^2)}\,, \tag{3.14}$$

and (3.10) and (3.12) follow from [17, Lemma 7.1.3]. $\square$

4. Inverse problem

Definition 4.1. (1) The function θ is said to be a Nevanlinna function if:

- (i) θ is analytic in the half-planes $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$;
- (ii) $\theta(\overline{\lambda}) = \overline{\theta(\lambda)}$ if $\operatorname{Im}\lambda \neq 0$;
- (iii) $\operatorname{Im}\lambda \operatorname{Im}\theta(\lambda) \geq 0$ for $\operatorname{Im}\lambda \neq 0$.

(2) The class $\mathcal{N}^{\mathrm{ep}}$ of essentially positive Nevanlinna functions is the set of all Nevanlinna functions which are analytic in $\mathbb{C}\setminus[0,\infty)$ with the possible exception of finitely many poles.
(3) The class $\mathcal{N}^{\mathrm{ep}}_+$ is the set of all functions $\theta \in \mathcal{N}^{\mathrm{ep}}$ such that for some $\gamma \in \mathbb{R}$ we have $\theta(\lambda) > 0$ for all $\lambda \in (-\infty, \gamma)$.

Definition 4.2. Let P and Q be entire functions with no common zeros such that P and Q are real on the real axis and such that $\frac{Q}{P}$ belongs to $\mathcal{N}^{\mathrm{ep}}_+$. Then the function ω defined by $\omega(\lambda) = P(\lambda^2) + i\lambda Q(\lambda^2)$ is said to belong to the class of symmetric shifted Hermite-Biehler functions. The class of all symmetric shifted Hermite-Biehler functions is denoted by SSHB. If the number of negative zeros of P is κ, then we say that ω belongs to SSHB_κ.

Definition 4.3. An entire function ω of positive exponential type is said to be a sine type function if

- (i) there is $h > 0$ such that all zeros of ω lie in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < h\}$,
- (ii) there are $h_1 \in \mathbb{R}$ and positive numbers $m < M$ such that $m \leq |\omega(\lambda)| \leq M$ holds for $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda = h_1$,
- (iii) the type of ω in the lower half-plane coincides with that in the upper half-plane.

Lemma 4.4. *Let $(\xi_k)_{k=-\infty}^{\infty}$ be a sequence of complex numbers satisfying the following conditions:*

- (i) *$\xi_{-k} = -\overline{\xi_k}$ for all not pure imaginary ξ_k,*
- (ii) *the sequence satisfies the asymptotic condition*

$$\xi_k = \frac{\pi(k-1/2)}{a} + ig + \frac{h}{k} + \frac{\gamma_k}{k}, \quad k \in \mathbb{N}, \tag{4.1}$$

where $a, g > 0$, $h \in \mathbb{R}$, $(\gamma_k)_{k=1}^{\infty} \in l_2$.

Then there exist $M_1 \in \mathbb{R}$, $M_2 \in \mathbb{R}$ and $\psi \in \mathcal{L}^a$ such that the sequence of zeros, counted with multiplicity, of the entire function

$$\phi(\lambda) = -\lambda \sin\lambda a + i\lambda \coth(ga)\cos\lambda a + M_1 \cos\lambda a + iM_2 \sin\lambda a + \psi(\lambda) \tag{4.2}$$

coincides with the given sequence $(\xi_k)_{k=-\infty}^{\infty}$, where

$$\frac{1}{\pi(\coth^2(ga) - 1)}(\coth(ga)M_2 - M_1) = h. \tag{4.3}$$

Furthermore, there exists a number $C > 0$ such that

$$\phi(\lambda) = C \prod_{k=-m}^{m} (i(\lambda - \xi_k)) \prod_{k=m+1}^{\infty} \Big(1 - \frac{\lambda}{\xi_k}\Big)\Big(1 - \frac{\lambda}{\xi_{-k}}\Big), \tag{4.4}$$

where we have assumed that the pure imaginary terms (including 0) of the sequence carry the indices $-m, \dots, m$.

Proof. This lemma easily follows from the first part of the proof of [17, Theorem 8.1.4]. Defining

$$\chi(\lambda) = \tilde{C} \prod_{\substack{k=-m\\k\neq 0}}^{m} (i(\lambda - \xi_k)) \prod_{k=m+1}^{\infty} \Big(1 - \frac{\lambda}{\xi_k}\Big)\Big(1 - \frac{\lambda}{\xi_{-k}}\Big)$$

with $\tilde{C} \in \mathbb{C} \setminus \{0\}$, this function with $m = 1$ is of the form [17, (8.1.20)]. It is easy to see that, for a suitably chosen $\tilde{C}$, also in case $m > 1$ the representation

$$\chi(\lambda) = \cos\lambda a + i\alpha \sin\lambda a + \frac{\pi h - T\alpha}{\lambda} \sin\lambda a + \frac{i(T - \pi h\alpha)}{\lambda} \cos\lambda a + \frac{\tilde{\psi}(\lambda)}{\lambda}\,, \tag{4.5}$$

in [17, (8.1.28)] holds, where $\alpha = \tanh(ga)$, $T \in \mathbb{R}$ and $\tilde{\psi} \in \mathcal{L}^a$. Setting

$$\phi(\lambda) = i\coth(ga)(\lambda - \xi_0)\chi(\lambda) \tag{4.6}$$

it is now clear that ϕ is of the form (4.4) with $C \in \mathbb{C} \setminus \{0\}$, and a straightforward expansion of ϕ proves (4.2). To verify that $C > 0$ we observe that the leading term $i\lambda\coth(ga)\cos\lambda a$ of the representation (4.2) is negative on the positive imaginary axis. □

Lemma 4.5. *Let $(\xi_k)_{k=-\infty,k\neq 0}^{\infty}$ be a sequence of complex numbers satisfying the following conditions:*

(i) *$\xi_{-k} = -\overline{\xi_k}$ for all not pure imaginary ξ_k,*
(ii) *the sequence satisfies the asymptotics*

$$\xi_k = \frac{\pi(k-1)}{a} + ig + \frac{h}{k} + \frac{\gamma_k}{k}, \quad k \in \mathbb{N}, \tag{4.7}$$

where $a, g > 0$, $h \in \mathbb{R}$, $\{\gamma_k\}_{-\infty}^{\infty} \in l_2$.

Then there exist $M_1 \in \mathbb{R}$, $M_2 \in \mathbb{R}$ and $\psi \in \mathcal{L}^a$ such that the sequence of zeros, counted with multiplicity, of the entire function

$$\phi(\lambda) = -\lambda\sin\lambda a + i\lambda\tanh(ga)\cos\lambda a + M_1\cos\lambda a + iM_2\sin\lambda a + \psi(\lambda) \tag{4.8}$$

coincides with the given sequence $(\xi_k)_{k=-\infty,k\neq 0}^{\infty}$, where

$$\frac{1}{\pi(\tanh^2(ga) - 1)}(\tanh(ga)M_2 - M_1) = h. \tag{4.9}$$

Furthermore, there exists $C > 0$ such that

$$\phi(\lambda) = C \prod_{k=1}^{m} (i(\lambda - \xi_k))(i(\lambda - \xi_{-k})) \prod_{k=m+1}^{\infty} \Big(1 - \frac{\lambda}{\xi_k}\Big)\Big(1 - \frac{\lambda}{\xi_{-k}}\Big), \tag{4.10}$$

where we have assumed that the pure imaginary terms (including 0), if any, of the sequence carry the indices $\pm 1, \dots, \pm m$.

Proof. Define the sequence $(\zeta_k)_{k=-\infty}^{\infty}$ by $\zeta_k = \xi_k + \frac{\pi}{2a}$ for $k > 0$ and $\zeta_k = \xi_{k-1} + \frac{\pi}{2a}$ for $k \le 0$. For $k \in \mathbb{N}$ we therefore have

$$\zeta_k = \frac{\pi(k-1/2)}{a} + ig + \frac{h}{k} + \frac{\gamma_k}{k}, \quad k \neq 0,$$

and, for $k \ge m+1$,

$$\begin{aligned}\overline{\zeta_{-k}} + \zeta_k &= \overline{\xi_{-k-1}} + \xi_k + \frac{\pi}{a} = -\xi_{k+1} + \xi_k + \frac{\pi}{a}\\ &= \frac{h}{k} - \frac{h}{k+1} + \frac{\gamma_k}{k} - \frac{\gamma_{k+1}}{k+1} = \frac{\tilde{\gamma}_k}{k},\end{aligned}$$

where $(\tilde{\gamma}_k)_{k=m+1}^{\infty} \in l_2$. From the proof of [17, Theorem 8.1.4] we see that this condition is sufficient to arrive at a slightly weakened form of the statement of Lemma 4.4. Indeed, the relation [17, (8.1.23)] between the numbers ζ_k, $k \neq 0$, and the eigenvalues of the auxiliary problem [17, (8.1.21)–(8.1.23)], is still valid, and (4.5) becomes

$$\tilde{\chi}(\lambda) = \cos\lambda a + i\alpha\sin\lambda a + \frac{\pi h - T\alpha}{\lambda}\sin\lambda a + \frac{i(T-\pi h\alpha)}{\lambda}\cos\lambda a + \frac{\tilde{\psi}(\lambda)}{\lambda}, \quad (4.11)$$

where T is a complex number which is not necessarily real. Then it follows, see (4.6), that the sequence $(\zeta_k)_{k=-\infty}^{\infty}$ is the sequence of the zeros of the function $\tilde{\phi}$ defined by

$$\tilde{\phi}(\lambda) = (\lambda - \zeta_0)\tilde{\chi}(\lambda). \quad (4.12)$$

A straightforward calculation shows that

$$\tilde{\phi}(\lambda) = \lambda\cos\lambda a + i\lambda\tanh(ga)\sin\lambda a + \tilde{M}_1\sin\lambda a - i\tilde{M}_2\cos\lambda a + \hat{\psi}(\lambda),$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are complex numbers satisfying

$$\frac{1}{\pi(\tanh^2(ga)-1)}(\tanh(ga)\tilde{M}_2 - \tilde{M}_1) = h.$$

By definition of the sequence $(\zeta_k)_{k=-\infty}^{\infty}$ it follows that the sequence $(\xi_k)_{k=-\infty,k\neq 0}^{\infty}$ is the sequence of the zeros of the function ϕ defined by $\phi(\lambda) = \tilde{\phi}(\lambda + \frac{\pi}{2a})$. Therefore the representations (4.8) and (4.9) hold with complex numbers M_1 and M_2. Since $\tilde{\chi}$ is a sine type function, (4.10) follows from [17, Lemma 12.2.29] with some nonzero complex number C. The infinite product in (4.10) is positive for sufficiently large λ on the positive imaginary axis, whereas in (4.8) the leading term on the positive imaginary axis, $-\lambda\sin\lambda a$, is positive there. It follows that $C > 0$, and then (4.10) shows that $\phi(-\overline{\lambda}) = \overline{\phi(\lambda)}$ for all $\lambda \in \mathbb{C}$. In particular, ϕ is real on the imaginary axis, which finally gives that M_1 and M_2 are real. □

Theorem 4.6. *Consider a sequence of complex numbers which is indexed as*

(i) $(\xi_k)_{k=-\infty}^{\infty}$, *or*
(ii) $(\xi_k)_{-\infty,k\neq 0}^{\infty}$

and which satisfies the following conditions:

(1) $\xi_{-k} = -\overline{\xi_k}$ *for all not pure imaginary* ξ_k.
(2) *The numbers* ξ_k *which lie in the closed lower half-plane and on the positive imaginary semiaxis satisfy the properties (1)–(6) of Theorem 3.4, and if* $\kappa > 0$, *then the interval* $(i|\lambda_\kappa|, i\infty)$ *contains at least one of the numbers* ξ_k.

(3) *In case (i), the sequence satisfies the asymptotic condition*

$$\xi_k = \frac{\pi(k-1/2)}{a} + ig + \frac{h}{k} + \frac{\gamma_k}{k}, \quad k \in \mathbb{N}, \tag{4.13}$$

and in case (ii), the sequence satisfies the asymptotic condition

$$\xi_k = \frac{\pi(k-1)}{a} + ig + \frac{h}{k} + \frac{\gamma_k}{k}, \quad k \in \mathbb{N}, \tag{4.14}$$

where $a, g > 0$, $h \in \mathbb{R}$ and $(\gamma_k)_{k=1}^\infty \in l_2$.

Then there exists a unique collection of a real-valued function $q \in L_2(0,a)$, $\alpha \in (0,1) \cup (1,\infty)$, $\beta \in \mathbb{R}$ and $\delta \in \mathbb{R}$ such that the spectrum of problem (3.1)–(3.3) coincides with (ξ_k). Here $\alpha \in (1,\infty)$ in case (i) and $\alpha \in (0,1)$ in case (ii).

Proof. We know from Lemmas 4.4 and 4.5 that there is an entire function ϕ with the representation

$$\phi(\lambda) = -\lambda \sin \lambda a + i\lambda\alpha \cos \lambda a + M_1 \cos \lambda a + iM_2 \sin \lambda a + \psi(\lambda) \tag{4.15}$$

whose zeros coincide with the given sequence (ξ_k), where

$$\alpha = \coth(ga) \text{ in case (i) and } \alpha = \tanh(ga) \text{ in case (ii)}, \tag{4.16}$$

where M_1, M_2 are real numbers with

$$\frac{1}{\pi(\alpha^2-1)}(\alpha M_2 - M_1) = h,$$

and where $\psi \in \mathcal{L}^a$. In view of the assumptions made in part 2, we conclude from [17, Theorem 5.2.16] that ϕ is of SSHB class. Hence there are entire functions $\tilde{P}$ and $\tilde{Q}$ which are real on the real axis and which do not have common zeros so that

$$\phi(\lambda) = \tilde{P}(\lambda^2) + i\lambda\tilde{Q}(\lambda^2)$$

and $\frac{\tilde{Q}}{\tilde{P}}$ is of class $\mathcal{N}_+^{\mathrm{ep}}$. We define

$$\tilde{\beta} = \frac{\alpha M_1 - M_2}{\alpha^2}, \tag{4.17}$$

$$Q(\lambda) = \frac{1}{\alpha}\tilde{Q}(\lambda), \tag{4.18}$$

$$P(\lambda) = \tilde{P}(\lambda) - \tilde{\beta}\tilde{Q}(\lambda). \tag{4.19}$$

From [17, Lemma 5.1.22] we conclude that

$$\frac{Q}{P} = \frac{1}{\alpha}\,\frac{\tilde{Q}}{\tilde{P} - \tilde{\beta}\tilde{Q}} = \frac{1}{\alpha}\,\frac{1}{\frac{\tilde{P}}{\tilde{Q}} - \tilde{\beta}}$$

is a meromorphic Nevanlinna function. Since $\tilde{P}$ and $\tilde{Q}$ do not have common zeros, also P and Q do not have common zeros, and since zeros and poles of meromorphic Nevanlinna functions are real, simple and interlace, see [17, Lemma 11.1.3], it follows that the zeros of P and Q are real, simple and interlace. From (4.15), (4.18) and (4.19) we obtain

$$Q(\lambda^2) = \cos \lambda a + \frac{M_2}{\alpha}\frac{\sin \lambda a}{\lambda} + \frac{\psi_1(\lambda)}{\lambda}, \tag{4.20}$$

$$P(\lambda^2) = -\lambda \sin \lambda a + \frac{M_2}{\alpha}\cos \lambda a + \psi_2(\lambda), \tag{4.21}$$

where ψ_1 and ψ_2 belong to the Paley-Wiener class $\mathcal{L}^a$. In (4.21) we have used that

$$M_1 - \tilde{\beta}\alpha = \frac{M_2}{\alpha}$$

in view of (4.17).

The functions P and Q satisfy the assumptions of [18, Theorem 2.6]. Hence there are entire functions S_0 and S_1 which are real on the real axis, whose zeros are real, simple and interlace, and which are of the form

$$S_0(\lambda^2) = \frac{\sin\lambda a}{\lambda} - A\frac{\cos\lambda a}{\lambda^2} + \frac{\hat{\psi}_3(\lambda)}{\lambda^2}\,, \tag{4.22}$$

$$S_1(\lambda^2) = \cos\lambda a + A\frac{\sin\lambda a}{\lambda} + \frac{\hat{\psi}_4(\lambda)}{\lambda}\,, \tag{4.23}$$

where $A \in \mathbb{R}$ and $\hat{\psi}_3, \hat{\psi}_4 \in \mathcal{L}^a$, such that

$$QS_1 - PS_0 = 1. \tag{4.24}$$

In view of [17, Lemma 12.3.3 and Theorem 12.6.2], there is a unique function $q \in L_2(0,a)$ such that $S_0(\lambda^2) = s(\lambda,a)$ and $S_1(\lambda^2) = s'(\lambda,a)$, where $s(\lambda,\cdot)$ is the solution of the initial value problem (3.1), $s(\lambda,0)=0$, $s'(\lambda,0)=1$.

Let $c(\lambda,0,\cdot)$ be the solution of the initial value problem (3.1), $c(\lambda,0,0) = 1$, $c'(\lambda,0,0)=0$. By the Lagrange identity, the Wronskian

$$c(\lambda,0,x)s'(\lambda,x) - c'(\lambda,0,x)s(\lambda,x), \quad \lambda \in \mathbb{C},$$

is independent of x, and its value at $x=0$ is 1. Together with (4.24) we conclude that

$$(Q(\lambda^2) - c(\lambda,0,a))s'(\lambda,a) - (P(\lambda^2) - c'(\lambda,0,a))s(\lambda,a) = 0, \quad \lambda \in \mathbb{C}. \tag{4.25}$$

Since the zeros of $s(\cdot,a)$ and $s'(\cdot,a)$ are distinct, we have $Q(\lambda^2) - c(\lambda,0,a) = 0$ whenever $s(\lambda,a)=0$. Hence there is an entire function g such that

$$Q(\lambda^2) - c(\lambda,0,a) = g(\lambda)s(\lambda,a).$$

Because $\lambda \mapsto Q(\lambda^2) - c(\lambda,0,a)$ belongs to the Payley Wiener class $\mathcal{L}^a$ in view of Proposition 3.3 and because $\lambda \mapsto \lambda s(\lambda,a)$ is a sine type function of exponential type a, it follows that $\lambda^{-1}g(\lambda) \to 0$ as $|\lambda| \to \infty$ outside a countable set of disjoint discs, see [17, Remark 11.2.21 and Lemma 12.2.4], and therefore $\lambda^{-1}g(\lambda) = o(\lambda)$ as $|\lambda| \to \infty$. It follows by Liouville's theorem that g is constant, say $g(\lambda) = \delta$ for some $\delta \in \mathbb{C}$ and all $\lambda \in \mathbb{C}$. Substitution into (4.25) gives

$$\begin{gathered}
Q(\lambda^2) = c(\lambda,0,a) + \delta s(\lambda,a) = c(\lambda,\delta,a), \quad \lambda \in \mathbb{C}, \\
P(\lambda^2) = c'(\lambda,0,a) + \delta s'(\lambda,a) = c'(\lambda,\delta,a), \quad \lambda \in \mathbb{C}.
\end{gathered}$$

Therefore,

$$\begin{aligned}
\phi(\lambda) &= \tilde{P}(\lambda^2) + i\lambda\tilde{Q}(\lambda^2) \\
&= P(\lambda^2) + \tilde{\beta}\alpha Q(\lambda^2) + i\lambda\alpha Q(\lambda^2) \\
&= c'(\lambda,\delta,a) + (i\lambda\alpha + \beta)c(\lambda,\delta,a),
\end{aligned}$$

where $\beta = \tilde{\beta}\alpha$.

Thus we have shown that ϕ is the characteristic function of a problem of the form (3.1)–(3.3). Next we are going to prove the uniqueness of this problem.

From the spectral asymptotics (3.10), (3.12) it follows that α is uniquely determined by (4.13) and (4.14). In view of (3.13), the numbers

$$B = \delta + \frac{1}{2}\int_0^a q(x)\,dx \tag{4.26}$$

and $B+\beta$, and therefore β, are uniquely determined.

Finally, let $q_1 \in L_2(0,a)$ and $\delta_1 \in \mathbb{R}$ be arbitrary such that the eigenvalue problem (3.1)–(3.3) with q_1 and δ_1 has the given sequence (ξ_k) as eigenvalues. Then its characteristic function is

$$\phi_1(\lambda,\delta_1) := c_1'(\lambda,\delta_1,a) + (i\alpha\lambda+\beta)c_1(\lambda,\delta_1,a),$$

see (3.9), where $c_1(\lambda,\delta_1,\cdot)$ is the solution of (3.1) with potential q_1 and with the initial conditions $c_1(\lambda,\delta_1,0) = 1$, $c_1'(\lambda,\delta_1,0) = \delta_1$. The functions $\lambda \mapsto (\lambda-\xi_1)^{-1}\phi_1(\lambda,\delta_1)$ and $\lambda \mapsto (\lambda-\xi_1)^{-1}\phi(\lambda)$ are sine type functions whose zeros coincide and which have the same leading terms. Therefore they are equal, see [17, Lemma 11.2.29], and hence $\phi_1(\cdot,\delta_1) = \phi$. Since c, c_1, c', c_1' are even functions in the variable λ, it follows that

$$c_1(\lambda,\delta_1,a) = c(\lambda,\delta,a), \quad c_1'(\lambda,\delta_1,a) = c'(\lambda,\delta,a).$$

Because the Wronskian of (3.1) is constant, (3.5) shows that

$$\begin{gathered} c_1(\lambda,\delta_1,a)s_1'(\lambda,a) - c_1'(\lambda,\delta_1,a)s(\lambda,a) = 1, \\ c(\lambda,\delta,a)s'(\lambda,a) - c'(\lambda,\delta,a)s(\lambda,a) = 1, \end{gathered}$$

and therefore

$$c(\lambda,\delta,a)(s_1'(\lambda,a) - s'(\lambda,a)) - c'(\lambda,\delta,a)(s_1(\lambda,a) - s(\lambda,a)) = 0.$$

Since $c(\cdot,\delta,a)$ and $c'(\cdot,\delta,a)$ do not have common zeros,

$$\chi(\lambda) := \frac{s_1(\lambda,a) - s(\lambda,a)}{c(\lambda,\delta,a)} = \frac{s_1'(\lambda,a) - s'(\lambda,a)}{c'(\lambda,\delta,a)}$$

defines an entire function χ. But $c(\cdot,\delta,a)$ is a sine type function of type a, whereas $s_1(\cdot,a) - s(\cdot,a) \in \mathcal{L}^a$. Therefore $\chi = 0$, see [18, Remark 2.3], i.e., $s_1(\lambda,a) = s(\lambda,a)$ and $s_1'(\lambda,a) = s'(\lambda,a)$. The uniqueness statement in [17, Theorem 12.6.2] now shows that $q_1 = q$, and finally $\delta_1 = \delta$ by (4.26). ☐

Corollary 4.7. *Consider a sequence of complex numbers which is indexed as*

(i) $(\xi_k)_{k=-\infty}^{\infty}$, *or*
(ii) $(\xi_k)_{-\infty,k\neq 0}^{\infty}$

and which satisfies the following conditions:

(1) $\xi_{-k} = -\overline{\xi_k}$ *for all not pure imaginary* ξ_k.
(2) *All numbers numbers* ξ_k *lie in the open upper half-plane.*
(3) *In case (i), the sequence satisfies the asymptotic condition*

$$\xi_k = \frac{\pi(k-1/2)}{a} + ig + \frac{h}{k} + \frac{\gamma_k}{k}, \quad k \in \mathbb{N}, \tag{4.27}$$

and in case (ii), the sequence satisfies the asymptotics

$$\xi_k = \frac{\pi(k-1)}{a} + ig + \frac{h}{k} + \frac{\gamma_k}{k}, \quad k \in \mathbb{N}, \tag{4.28}$$

where $a, g > 0$, $h \in \mathbb{R}$ *and* $(\gamma_k)_{k=1}^{\infty} \in l_2$.

Then for each $l > 0$ there exists a unique collection of a real-valued function $\rho \gg 0$ in $W_2^2(0,a)$ and positive real numbers p and ν such that the spectrum of problem (2.4)–(2.6) coincides with (ξ_k).

Proof. The assumptions of Theorem 4.6 are satisfied, and hence there are unique real numbers $a > 0$, $\alpha \in (0,1) \cup (1,\infty)$, δ and β and a unique real function $q \in L_2(0,a)$ such that (ξ_k) is the spectrum of the corresponding eigenvalue problem (3.1)–(3.3). As in the proof of [17, Therorem 8.3.4] we can construct a unique function $\rho \in W_2^2(0,l)$ which satisfies (2.17)–(2.19) and which is positive on $(0,l]$. The only necessary modification in the reasoning is that the Robin boundary condition does not give $\rho(0) \neq 0$ directly. Rather, the reference function ψ considered in the proof of [17, Therorem 8.3.4] has to satisfy the Robin condition $\psi'(0) - \delta\psi(0) = 0$ here, and this gives $\psi(0) \neq 0$ because $\psi \neq 0$. Then the Sturm comparison theorem, see [24, Theorem 13.1] and the continuity of the Prüfer angle show that also $\rho(0) \neq 0$.

The numbers $\nu > 0$ and $p \in \mathbb{R}$ are uniquely determined by (2.20) and (2.21). To show that $p > 0$ assume that $p \leq 0$. Since (λ_k) is the spectrum of the operator pencil L defined in (2.7), it follows from [17, Theorem 1.3] that $A \geq 0$ and therefore $A \gg 0$ since the invertibility of $L(0)$ implies that A is invertible. Now choose a real function $v \in W_2^2(0,l)$ and $\eta \in (0, \frac{l}{2})$ such that $v(0) = 2$, $v'(0) = pv(0)$, $|v'(x)| \leq 2|p|$ for $x \in [0,\eta]$, $v'(x) = 0$ for $x \in (\eta, l)$ and $v(l) \geq 1$. For example, for $x \in [0,\eta]$ we can take $v(x) = 2 + 2\frac{p}{\theta}\sin(\theta x)$ with $\theta = \frac{3\pi}{2\eta}$. Then $Y = (v, v(l))^{\mathrm{T}}$ satisfies $\|Y\|^2 \geq 1$ and, see (2.11),

$$(AY,Y) = \int_0^a |v'(x)|^2\,dx + p|v(0)|^2 \leq 4\eta|p|^2 + 4p \leq 4\eta|p|^2.$$

Since $\eta \in (0, \frac{l}{2})$ is arbitrary, this contradicts $A \gg 0$. Hence $p > 0$ follows. $\square$

5. Inverse problems by parts of spectra

Consider the two problems

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0,a], \tag{5.1}$$

$$y'(0) - \delta_j y(0) = 0, \quad j = 1,2, \tag{5.2}$$

$$y'(a) + (i\alpha\lambda + \beta)y(a) = 0, \tag{5.3}$$

with $\delta_j \in \mathbb{R}$. The corresponding characteristic functions are

$$\phi(\lambda,\delta_j) = c'(\lambda,\delta_j,a) + (i\alpha\lambda + \beta)c(\lambda,\delta_j,a), \tag{5.4}$$

see (3.9).

Proposition 5.1. *For all $\lambda \in \mathbb{C}$ and all $\delta_1, \delta_2 \in \mathbb{R}$ the following identity holds*

$$\phi(\lambda,\delta_1)\phi(-\lambda,\delta_2) - \phi(-\lambda,\delta_1)\phi(\lambda,\delta_2) = 2i\alpha\lambda(\delta_2 - \delta_1). \tag{5.5}$$

Proof. Using (5.4) and taking into account that $c(\lambda,\delta_j,a)$ and $c'(\lambda,\delta_j,a)$ are even functions of λ we obtain

$$\begin{aligned} &\phi(\lambda,\delta_1)\phi(-\lambda,\delta_2) - \phi(-\lambda,\delta_1)\phi(\lambda,\delta_2) \\ &= 2i\alpha\lambda[c(\lambda,\delta_1,a)c'(\lambda,\delta_2,a) - c'(\lambda,\delta_1,a)c(\lambda,\delta_2,a)]. \end{aligned} \tag{5.6}$$

Now using the Lagrange identity and the initial conditions $c(\lambda,\delta_j,0) = 1$, and $c'(\lambda,\delta_j,0) = 1$, see Section 4, we obtain

$$c(\lambda,\delta_1,a)c'(\lambda,\delta_2,a) - c'(\lambda,\delta_1,a)c(\lambda,\delta_2,a)$$

$$
\begin{aligned}
&= c(\lambda,\delta_1,0)c'(\lambda,\delta_2,0) - c'(\lambda,\delta_1,0)c(\lambda,\delta_2,a)\\
&= \delta_2 - \delta_1,
\end{aligned}
$$

which proves (5.5). □

Definition 5.2. Let $\alpha > 0$.
(1) For $M_1, M_2 \in \mathbb{R}$ let Φ_{α,M_1,M_2} be the class of entire functions ϕ which satisfy the following conditions:

- (i) if $\lambda \in \mathbb{C} \setminus \{0\}$ and $\phi(\lambda) = 0$, then $\phi(-\lambda) \neq 0$,
- (ii) 0 is at most a simple zero of ϕ,
- (iii) $\phi(-\overline{\lambda}) = \overline{\phi(\lambda)}$ for all $\lambda \in \mathbb{C}$,
- (iv) there is $\tau \in \mathcal{L}^a$ such that

$$
\phi(\lambda) = -\lambda \sin \lambda a + i\lambda\alpha \cos \lambda a + M_1 \cos \lambda a + iM_2 \sin \lambda a + \tau(\lambda). \tag{5.7}
$$

(2) Define

$$
\Phi_\alpha = \cup_{M_1,M_2\in\mathbb{R}} \Phi_{\alpha,M_1,M_2}.
$$

Theorem 5.3. *Let $\kappa \in \mathbb{N}$, $N_2 = \{\pm 1, \pm 2, \dots, \pm\kappa\}$, $N_1 = \{0, \pm(\kappa+1), \pm(\kappa+2), \dots\}$. Let $(\lambda_k)_{k=-\infty}^{\infty}$ be a sequence of complex numbers such that*

1. *$\operatorname{Im}\lambda_k > 0$ for all $k \in \mathbb{Z}$,*
2. *$\lambda_{-k} = -\overline{\lambda_k}$ for all not pure imaginary λ_{-k},*
3. *$\lambda_k \neq \lambda_j$ for all $k \in N_1$ and $j \in N_2$.*

Let the sequence (λ_k) satisfy (4.1) where g and h are real constants, $(\beta_k) \in l_2$.

Then for any given $\Delta < 0$ there exist a real potential $q \in L_2(0,a)$, real numbers $\alpha > 1$, β and δ_1 such that $(\lambda_k)_{k\in N_1}$ are eigenvalues of problem (5.1)–(5.3) with $j = 1$, and $(\lambda_k)_{k\in N_2}$ are eigenvalues of problem (5.1)–(5.3) with $j = 2$ where $\delta_2 = \delta_1 + \Delta$.

Proof. The sequence $(\lambda_k)_{k=-\infty}^{\infty}$ satisfies the assumptions of Lemma 4.4, and therefore the zeros of the entire function ϕ defined in (4.2) coincide with the given sequence (λ_k). With $\alpha = \coth(ga) > 1$ it follows that $\phi \in \Phi_\alpha$. In view of [17, Theorems 3.5 and 3.8] there are functions $\tilde{\phi}$ and $\tilde{X} \in \Phi_\alpha$ such that

$$
\tilde{\phi}(\lambda)\tilde{X}(-\lambda) - \tilde{\phi}(-\lambda)\tilde{X}(\lambda) = 2i\alpha\lambda\Delta, \quad \lambda \in \mathbb{C}, \tag{5.8}
$$

such that the λ_k with $k \in N_1$ are zeros of $\tilde{\phi}$, the λ_k with $k \in N_2$ are zeros of $\tilde{X}$, and such that all zeros of $\tilde{\phi}$ lie in the closed upper half-plane. From the proof of Theorem 3.5 we know that the sequence (ξ_k) of the zeros of $\tilde{\phi}$ can be indexed in such a way that they satisfy asymptotic condition of the form (4.13), and from the definition of Φ_α it follows that $\xi_{-k} = -\overline{\xi_k}$ for not pure imaginary ξ_k with a suitable indexing. Hence (ξ_k) satisfies conditions (3) and (1) of Theorem 4.6. From the definition of Φ_α we know that 0 can be at most a simple zero of $\tilde{\phi}$ and that $\tilde{\phi}$ has no nonzero real zeros. Therefore (ξ_k) also satisfies condition (2) of Theorem 4.6, and hence Theorem 4.6 shows that there are a real function $q \in L_2(0,a)$, $\beta \in \mathbb{R}$ and $\delta_1 \in \mathbb{R}$ such that the spectrum of (5.1)–(5.3) for $j = 1$ coincides with (ξ_k). With $\delta_2 = \delta_1 + \Delta$, Proposition 5.1 shows that the characteristic function $\phi(\cdot, \delta_2)$ of (5.1)–(5.3) with $j = 2$ satisfies

$$
\tilde{\phi}(\lambda)\phi(-\lambda,\delta_2) - \tilde{\phi}(-\lambda)\phi(\lambda,\delta_2) = 2ia\lambda(\delta_2 - \delta_1) = 2i\alpha\lambda\Delta.
$$

In view of [17, Theorem 3.2], for any given $\tilde{\phi} \in \Phi_\alpha$, this equation has a unique solution in Φ_α, and therefore $\tilde{X} = \phi(\cdot, \delta_2)$. □

Theorem 5.4. *Let $\kappa \in \mathbb{N}$, $N_2 = \pm 1, \pm 2, \dots, \pm\kappa$, $N_1 = \{\pm(\kappa+1), \pm(\kappa+2), \dots\}$. Let $(\lambda_k)_{k-\infty,k\neq 0}^{\infty}$ be a sequence of complex numbers such that*

(1) $\operatorname{Im}\lambda_k > 0$ *for all* $k \in \mathbb{Z} \setminus \{0\}$,
(2) $\lambda_{-k} = -\overline{\lambda_k}$ *for all not pure imaginary* λ_{-k},
(3) $\lambda_k \neq \lambda_j$ *for all* $k \in N_1$ *and* $j \in N_2$.

Let the sequence (λ_k) satisfy (4.7) where g and h are real constants, $(\beta_k) \in l_2$.

Then for any given $\Delta \neq 0$ there exist a real potential $q \in L_2(0,a)$, real numbers $\alpha \in (0,1)$, β and δ_1 such that $(\lambda_k)_{k\in N_1}$ are eigenvalues of problem (5.1)–(5.3) with $j = 1$, and $(\lambda_k)_{k\in N_2}$ are eigenvalues of problem (5.1)–(5.3) with $j = 2$ where $\delta_2 = \delta_1 + \Delta$.

Proof. The sequence $(\lambda_k)_{k=-\infty,k\neq 0}^{\infty}$ satisfies the assumptions of Lemma 4.5, and therefore the zeros of the entire function ϕ defined in (4.8) coincide with the given sequence (λ_k). With $\alpha = \tanh(ga) \in (0,1)$ it follows that $\phi \in \Phi_\alpha$. If $\Delta < 0$, we can proceed as in the proof of Theorem 5.3.

If $\Delta > 0$, it follows from [17, Theorems 3.5 and 3.8] that there are functions $\tilde{\phi}$ and $\tilde{X} \in \Phi_\alpha$ such that (5.8) holds, such that the λ_k with $k \in N_1$ are zeros of $\tilde{\phi}$, the λ_k with $k \in N_2$ are zeros of $\tilde{X}$, and such that all zeros of $\tilde{X}$ lie in the closed upper half-plane. Proceeding as in the proof of Theorem 5.3 with $\tilde{\phi}$ and $\tilde{X}$ interchanged completes this proof. □

Remark 5.5. (1) For $\rho \in W_2^2(0,l)$ it was shown in Section 2 that the string problem (2.4)–(2.6) can be transformed into a Robin-Regge eigenvalue problem (3.1)–(3.3). Hence the eigenvalues of (2.4)–(2.6) lie in the open upper half-plane by Proposition 2.2 and satisfy the asymptotics (3.10) or (3.12) if $\rho^{-\frac{1}{2}}(s(a))\nu$ is less than 1 or larger than 1, respectively.

(2) If in the proofs of Theorems 5.3 and 5.4 both $\tilde{\phi}$ and $\tilde{X}$ have no zeros in the open lower half-plane, then the potentials q and the numbers α, β, δ_1 and δ_2 in Theorems 5.3 and 5.4 are unique, see [17, Corollary 3.7].

Acknowledgments. M. Möller was supported by a grant from the National Research Foundation of South Africa (Grant No. 80956).

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