Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 289, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ANALYSIS AND APPLICATION OF DIFFUSION EQUATIONS INVOLVING A NEW FRACTIONAL DERIVATIVE WITHOUT SINGULAR KERNEL

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Communicated by Mokhtar Kirane

Abstract. In this article, a family of nonlinear diffusion equations involving multi-term Caputo-Fabrizio time fractional derivative is investigated. Some maximum principles are obtained. We also demonstrate the application of the obtained results by deriving some estimation for solution to reaction-diffusion equations.

1. Introduction

Luchko [8] obtained a maximum principle for the generalized time-fractional diffusion equation on an open bounded domain by applying an extremum principle involving Caputo-Dzherbashyan fractional derivative. Later, the maximum principles for generalized time-fractional diffusion equations (multi-term diffusion equation and the diffusion equation of distributed order) with Caputo and Riemann-Liouville type derivatives were presented by Luchko [9], and Al-Refai and Luchko [1] respectively. Alsaedi et al. [2] proved an inequality for fractional derivatives related to the Leibniz rule and used it to derive maximum principles for time and space fractional heat equations with nonlinear diffusion. For further details on the topic, see [3, 5, 6, 7, 10, 12].

Here, in contrast to the above referenced works, we study a nonlinear diffusion equation with multi-term Caputo-Fabrizio time fractional derivative (without singular kernel) given by

$$\mathcal{P}({}^{CF}D_t u)(x,t) = -L(u)(x,t) + F(x,t,u(x,t)), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded open domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, L is a uniformly elliptic operator given by

$$L(u) = -\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i},$$

$$\mathcal{P}({}^{CF}D_t u) = {}^{CF}D_t^{\alpha} u + \sum_{i=1}^{m} \lambda_i^{CF} D_t^{\alpha_i} u, \quad 0 < \alpha_m < \cdots < \alpha_1 < \alpha < 1,\ \lambda_i \geq 0,$$

2010 *Mathematics Subject Classification.* 26A33, 35B05, 35B50.
Key words and phrases. Caputo-Fabrizio derivative; maximum principle; multi-term time fractional diffusion equations.

Submitted September 21, 2017. Published November 21, 2017.

where $i = 1, 2, \dots, m$, ${}^{CF}D_t^\alpha$ denotes Caputo-Fabrizio fractional derivative of order $0 < \alpha < 1$ defined by

$$ {}^{CF}D_t^\alpha f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s)) f'(s)ds, \quad t \geq 0, $$

$M(\alpha)$ is a normalization constant depending on α. For details about fractional derivative without singular kernel, we refer the reader to [4].

2. Preliminaries

In this section, we present some useful theorems related to our work.

Lemma 2.1. [11, 13] *Let $u \in C^2(\overline{\Omega})$ be a function attaining its maximum at a point x_0 inside $\Omega \subseteq \mathbb{R}^n$ and $A = (a_{ij})_{n\times n}$, $x \in \Omega$ be a positive definite matrix. Then*

$$ \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}\Big|_{x=x_0} \leq 0. $$

Theorem 2.2. *Let $0 < \alpha < 1$. Assume that $f \in C^1([0,T])$ attains its maximum on the interval $[0,T]$ at the point $t_0 \in (0,T]$. Then*

$$ {}^{CF}D_t^\alpha f(t_0) \geq \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \exp(-\frac{\alpha t_0}{1-\alpha})\big[f(t_0) - f(0)\big] \geq 0. $$

Proof. As in [12], we introduce an auxiliary function

$$ y(t) = f(t) - f(t_0), \quad t \in [0,T]. $$

It is easy to deduce that $y \in C^1([0,T])$ and the following properties hold:

(1) $y(t) \leq 0$, for all $t \in [0,T]$;
(2) $y(t_0) = y'(t_0) = 0$;
(3) $y(t) = (t - t_0)x(t)$ with $x \in C([0,T])$ and $x(t) \leq 0$ for all $t \in [0, t_0]$.

In consequence, we obtain

$$ \begin{aligned} {}^{CF}D_t^\alpha y(t) &= \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s)) y'(s)ds \\ &= \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s)) f'(s)ds \\ &= {}^{CF}D_t^\alpha f(t). \end{aligned} $$

Note that $y(t) \leq 0$ for all $t \in [0, t_0]$ and $y(t_0) = 0$. Then, integrating by parts, we obtain

$$ \begin{aligned} {}^{CF}D_t^\alpha y(t_0) &= \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^{t_0} \exp(-\frac{\alpha}{1-\alpha}(t_0-s)) y'(s)ds \\ &= \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \exp(-\frac{\alpha}{1-\alpha}(t_0-s)) y(s)\Big|_0^{t_0} \\ &\quad - \frac{(2-\alpha)M(\alpha)\alpha}{2(1-\alpha)(1-\alpha)} \int_0^{t_0} \exp(-\frac{\alpha}{1-\alpha}(t_0-s)) y(s)ds \\ &\geq -\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \exp(-\frac{\alpha t_0}{1-\alpha}[f(0) - f(t_0)]) \end{aligned} $$

$$= \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \exp(-\frac{\alpha t_0}{1-\alpha}[f(t_0) - f(0)]) \geq 0.$$

□

3. Some maximum principles

In the section, we derive some maximum principles for the following parabolic type fractional differential operator without singular kernel

$$\begin{aligned} Q_\alpha(u) &= P({}^{CF}D_t u)(x,t) + L(u) - h(x,t)u \\ &= P({}^{CF}D_t u)(x,t) - \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} \\ &\quad + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} - h(x,t)u, \end{aligned} \tag{3.1}$$

where $h(x,t) \leq 0$ ($(x,t) \in \overline{\Omega} \times [0,T]$) is a bounded function. Let us begin with a weak maximum principle.

Theorem 3.1. *If $u \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T))$ satisfies $Q_\alpha(u) \leq 0$ (Q is defined by (3.1)), then the following inequality holds:*

$$u(x,t) \leq \max\Big\{ \max_{x \in \overline{\Omega}} u(x,0), \max_{(x,t) \in \partial\Omega \times (0,T]} u(x,t),\ 0 \Big\}.$$

Proof. Assume that the function $u(x,t)$ attains its positive maximum $u(x_0,t_0)$ at a point $(x_0,t_0) \in \Omega \times (0,T]$. By Lemma 2.1, we obtain

$$Lu(x,t)|_{(x_0,t_0)} = -\sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j}|_{(x_0,t_0)} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i}|_{(x_0,t_0)} \geq 0. \tag{3.2}$$

By Theorem 2.2, we have

$$\begin{aligned} &\mathcal{P}({}^{CF}D_t u)(x_0,t_0) \\ &= {}^{CF}D_t^\alpha u(x_0,t_0) + \sum_{i=1}^{m} \lambda_i {}^{CF}D_t^{\alpha_i} u(x_0,t_0) \\ &= \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \exp(-\frac{\alpha t_0}{1-\alpha}) \big[u(x_0,t_0) - u(x_0,0)\big] \\ &\quad + \sum_{i=1}^{m} \lambda_i \frac{(2-\alpha_i)M(\alpha_i)}{2(1-\alpha_i)} \exp(-\frac{\alpha_i t_0}{1-\alpha_i}) \big[u(x_0,t_0) - u(x_0,0)\big] > 0. \end{aligned} \tag{3.3}$$

Applying the condition $h(x,t) \leq 0$, one can easily deduce from inequalities (3.2) and (3.3) that

$$(Q_\alpha u)(x_0,t_0) > P({}^{CF}D_t)u(x_0,t_0) - h(x_0,t_0)u(x_0,t_0) > 0,$$

which contradicts the condition $(Q_\alpha u)(x,t) \leq 0$ for all $(x,t) \in \Omega \times (0,T]$. □

Analogously, we can prove the following minimum principle.

Theorem 3.2. *If $u \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T))$ satisfies $Q_\alpha(u) \geq 0$, then the following inequality holds:*

$$u(x,t) \geq \min\big\{ \min_{x \in \overline{\Omega}} u(x,0), \min_{(x,t) \in \partial\Omega \times (0,T]} u(x,t),\ 0 \big\}.$$

4. Applications of maximum and minimum principles

Here we present some new results for multi-dimensional time-fractional diffusion equation by using the maximum and minimum principles derived in the previous section. Consider the linear problem

$$
\begin{gathered}
P({}^{CF}D_t u)(x,t) + L(u) - h(x,t)u = f(x,t), \quad (x,t) \in \Omega \times (0,T], \\
u(x,0) = g(x), \quad x \in \Omega, \\
u(x,t) = \mu(x,t), \quad (x,t) \in \partial\Omega \times (0,T).
\end{gathered} \tag{4.1}
$$

A direct application of Theorems 3.1 and 3.2 leads to the following two comparison results for the problem (4.1).

Theorem 4.1. *Assume that $f(x,t) \le 0$, $g(x) \le 0$ and $\mu(x,t) \le 0$. If $u(x,t) \in C^{2,1}(\Omega \times (0,T))$ is a solution of the problem (4.1), then $u(x,t) \le 0$, $(x,t) \in \overline{\Omega} \times [0,T]$.*

Theorem 4.2. *Assume that $f(x,t) \ge 0$, $g(x) \ge 0$ and $\mu(x,t) \ge 0$. If $u(x,t) \in C^{2,1}(\Omega \times (0,T))$ is a solution of the problem (4.1), then $u(x,t) \ge 0$, $(x,t) \in \overline{\Omega} \times [0,T]$.*

Theorem 4.3. *There exists at most one solution for the problem (4.1).*

Proof. Let us suppose that (4.1) has two solutions u_1 and u_2. Letting $u = u_1 - u_2$, we obtain

$$
\begin{gathered}
P({}^{CF}D_t u)(x,t) + L(u) - h(x,t)u = 0, \quad (x,t) \in \Omega \times (0,T), \\
u(x,0) = 0, \quad x \in \Omega, \\
u(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T).
\end{gathered}
$$

By Theorems 4.1 and 4.2, it follows that $u = 0$, that is, $u_1 = u_2$. □

Theorem 4.4. *The solution u of (4.1) depends continuously on the given initial value $g(x)$ and the boundary value $\mu(x,t)$.*

Proof. Let u_i denote the solution of (4.1) with the corresponding data $g_i(x)$ and $\mu_i(x,t)$, $i = 1,2$. Fix $\eta = \frac{\varepsilon}{2}$, $\varepsilon > 0$ and suppose that $\max_{x\in\Omega} |g_1(x) - g_2(x)| \le \eta$ and $\max_{(x,t)\in\partial\Omega\times(0,T)} |\mu_1(x,t) - \mu_2(x,t)| \le \eta$. Taking $u = u_1 - u_2$, we obtain

$$
\begin{gathered}
P({}^{CF}D_t u)(x,t) + L(u) = 0, \quad (x,t) \in \Omega \times (0,T), \\
u(x,0) = g_1(x) - g_2(x), \quad x \in \Omega, \\
u(x,t) = \mu_1(x,t) - \mu_2(x,t) \quad (x,t) \in \partial\Omega \times (0,T).
\end{gathered}
$$

Since $h(x,t) \le 0$, it follows by Theorems 3.1 and 3.2 that

$$
\begin{gathered}
u(x,t) \le \max\big\{ \max_{x\in\overline{\Omega}} (g_1 - g_2), \max_{(x,t)\in\partial\Omega\times(0,T]} (\mu_1 - \mu_2), \ 0 \big\} < \varepsilon, \\
u(x,t) \ge \min\big\{ \min_{x\in\overline{\Omega}} (g_1 - g_2), \min_{(x,t)\in\partial\Omega\times(0,T]} (\mu_1 - \mu_2), \ 0 \big\} > -\varepsilon.
\end{gathered}
$$

□

Next, we consider the nonlinear problem

$$
\begin{gathered}
P({}^{CF}D_t u)(x,t) + L(u) = F(x,t,u), \quad (x,t) \in \Omega \times (0,T], \\
u(x,0) = g(x), \quad x \in \Omega, \\
u(x,t) = \mu(x,t), \quad (x,t) \in \partial\Omega \times (0,T).
\end{gathered} \tag{4.2}
$$

Theorem 4.5. *Let $F(x,t,u)$ be a smooth function. If $\frac{\partial F}{\partial u} \le 0$, then* (4.2) *has at most one solution.*

Proof. Suppose that nonlinear problem (4.2) has two solutions u_1 and u_2. Letting $u = u_1 - u_2$, we obtain

$$\begin{gathered} P({}^{CF}D_t u)(x,t) + L(u) = F(x,t,u_1) - F(x,t,u_2), \quad (x,t) \in \Omega \times (0,T), \\ u(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \end{gathered}$$

Since F is a smooth function, there exists $\xi = (1-\lambda)u_1 + \lambda u_2 (\lambda \in (0,1))$ such that

$$F(x,t,u_1) - F(x,t,u_2) = \frac{\partial F}{\partial u}(\xi) u.$$

Using the condition $\frac{\partial F}{\partial u} < 0$ together with Theorems 4.1 and 4.2, we obtain $u = 0$. This implies that $u_1 = u_2$. □

Theorem 4.6. *Let $F(x,t,u)$ be a smooth function such that $\frac{\partial F}{\partial u} \le 0$. Then the solution u of* (4.2) *depends continuously on the given initial and boundary data $g(x)$ and $\mu(x,t)$ respectively.*

Proof. Let u_i be the solutions of (4.2) with the corresponding data $g_i(x)$ and $\mu_i(x,t)$, $i = 1,2$. Fixing $\eta = \frac{\varepsilon}{2}$, $\varepsilon > 0$, assume that $\max_{x\in\Omega} |g_1(x) - g_2(x)| \le \eta$ and $\max_{(x,t)\in\partial\Omega\times(0,T)} |\mu_1(x,t) - \mu_2(x,t)| \le \eta$. Define $u = u_1 - u_2$ so that u satisfies

$$\begin{gathered} P({}^{CF}D_t u)(x,t) + L(u) = F(x,t,u_1) - F(x,t,u_2), \quad (x,t) \in \Omega \times (0,T), \\ u(x,0) = g_1(x) - g_2(x), \quad x \in \Omega, \\ u(x,t) = \mu_1(x,t) - \mu_2(x,t), \quad (x,t) \in \partial\Omega \times (0,T). \end{gathered}$$

As in the proof of Theorem 4.5, we have

$$F(x,t,u_1) - F(x,t,u_2) = \frac{\partial F}{\partial u}(\xi) u,$$

where ξ is between u_1 and u_2. Since $\frac{\partial F}{\partial u} \le 0$, by Theorems 3.1 and 3.2 it follows that

$$\begin{gathered} u(x,t) \le \max\big\{ \max_{x\in\overline{\Omega}} (g_1 - g_2), \max_{(x,t)\in\partial\Omega\times(0,T]} (\mu_1 - \mu_2),\ 0 \big\} < \varepsilon, \\ u(x,t) \ge \min\big\{ \min_{x\in\overline{\Omega}} (g_1 - g_2), \min_{(x,t)\in\partial\Omega\times(0,T]} (\mu_1 - \mu_2),\ 0 \big\} > -\varepsilon. \end{gathered}$$

□

Acknowledgments. The authors thank the editor and the reviewers for their constructive remarks that led to the improvement of the original manuscript. This work is partially supported by National Natural Science Foundation of China (No. 11501342) and the Natural Science Foundation for Young Scientists of Shanxi Province, China (No. 201701D221007).

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