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SPECTRAL PROPERTIES OF A FOURTH-ORDER EIGENVALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. In this article we consider eigenvalue problems for fourth-order ordinary differential equation with spectral parameter in boundary conditions. We study the location of eigenvalues on the real axis, find the multiplicities of eigenvalues, investigate the oscillation properties of eigenfunctions, and the basis properties in the space L_p, $1 < p < \infty$, of the subsystems of eigenfunctions of this problem.

1. Introduction

We consider the eigenvalue problem

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \quad 0 < x < 1, \tag{1.1}$$

$$y''(0) = y''(1) = 0, \tag{1.2}$$

$$Ty(0) - a\lambda y(0) = 0, \tag{1.3}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is positive and absolutely continuous function on $[0, 1]$, a, c are real constants such that $a > 0$, $c < 0$.

Problem (1.1)-(1.4) arises in the dynamical boundary-value problem describing free bending vibrations of a homogeneous rod of constant rigidity, in cross sections of which the longitudinal force acts, both ends of which are fixed elastically and on these ends the servocontrol forces in acting. For more details on the physical meaning of this problem, see [12, 25].

The study of boundary-value problems for ordinary differential operators with spectral parameter in boundary conditions has a long history. In his Memoire [24] Poisson solved the problem of the motion of a body suspended by the end of an inextensible thread. Krylov [19] and Timoshenko [30] considered the problem of the longitudinal vibrations of the rod, which is one of the most interesting exactly solvable models. Additional information on specific physical problems leading to the boundary-value problems for ordinary differential operators with spectral parameter in boundary conditions can be found in the books [12, 19, 25, 29, 30] and in the papers [10, 11, 13, 15, 21, 28, 31]. Spectral problems for ordinary differential

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operators with spectral parameter in the boundary conditions have been considered in various formulations by many authors [1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 26, 27, 28, 31]. In [3, 5, 13, 14, 15, 18, 20, 23, 26, 27, 31] the authors studied the basis property in various function spaces of the root functions systems of the Sturm-Liouville problem with spectral parameter in the boundary conditions. The basis properties of subsystems of root functions in the space L_p, $1 < p < \infty$, of the boundary-value problems for fourth order ordinary differential equations with spectral parameter in one of the boundary conditions are studied in [1, 2, 6, 17].

In the recent paper [4] the basis properties of eigenfunctions of a fourth-order eigenvalue problem with spectral parameter entering in two of the boundary conditions at the point $x = 1$ are studied. In this paper, are found sufficient conditions for the subsystems of eigenfunctions of this problem to form a basis in the space $L_p(0, 1)$, $1 < p < \infty$.

The purpose of the present paper is to study the basis property of the subsystems of eigenfunctions of boundary-value problem (1.1)-(1.4) in $L_p(0, 1)$, $1 < p < \infty$.

This article has the following structure. Some statements necessary in the sequel are given in Section 2. In Section 3 we investigate the main properties of solution of problem (1.1)-(1.3) which play an essential role in the study of the oscillatory properties of eigenfunctions of (1.1)-(1.4). In Section 4 we give an operator interpretation of boundary-value problem (1.1)-(1.4), where we associate with the problem a self-adjoint operator in the Hilbert space $L_2(0, 1) \oplus \mathbb{C}^2$, and provide some spectral properties of the corresponding operator. Here we study the structure of root subspaces, the location of eigenvalues on the real axis and the oscillation properties of eigenfunctions of problem (1.1)-(1.4). We show that the eigenvalues of boundary-value problem (1.1)-(1.4) are nonnegative, simple and they form an infinitely increasing sequence. In Section 5 we obtain sufficient conditions for the subsystems of eigenfunctions of (1.1)-(1.4) to form a basis in the space $L_p(0, 1)$, $1 < p < \infty$. More precisely, we prove that the system of eigenfunctions of this problem after removing two eigenfunctions corresponding to eigenvalues with numbers of different parity forms a basis in the space $L_p(0, 1)$, $1 < p < \infty$, which is an unconditional basis for $p = 2$.

2. Preliminaries

To study the spectral properties of problem (1.1)-(1.4), we will need the following statements.

Lemma 2.1 ([8, Lemma 2.1]). *Let $y(x, \lambda)$ be a nontrivial solution of equation* (1.1) *for $\lambda > 0$. If y, y', y'', Ty are nonnegative and not all equal zero at $x_0 \in (0, 1)$, then they are positive for $x \in (x_0, 1]$. If $y, -y', y'', -Ty$ are nonnegative and not all equal zero at $x_0 \in (0, 1)$, then they are positive for $x \in [0, x_0)$.*

Lemma 2.2 ([8, Lemma 2.2]). *Let $y(x, \lambda)$ be a nontrivial solution of problem* (1.1), (1.2) *for $\lambda > 0$. If $y(x_0, \lambda) = 0$ or $y''(x_0, \lambda) = 0$, then $y'(x, \lambda)Ty(x, \lambda) < 0$ in a some neighborhood of $x_0 \in (0, 1)$; if $y'(x_0, \lambda) = 0$ or $Ty(x_0, \lambda) = 0$, then $y(x, \lambda)y''(x, \lambda) < 0$ in a some neighborhood of $x_0 \in (0, 1)$.*

Consider the boundary condition

$$y(0) \cos \beta + Ty(0) \sin \beta = 0, \tag{2.1}$$

$$y(1)\cos\delta - Ty(1)\sin\delta = 0, \tag{2.2}$$

where $\beta, \delta \in [0, \frac{\pi}{2}]$.

Alongside boundary-value problem (1.1)-(1.4) we shall consider the spectral problem (1.1), (1.2), (2.1), (2.2) and (1.1), (1.2), (2.1), (1.4). The spectral properties of problem (1.1), (1.2), (2.1), (2.2) have been investigated in [8], and of problem (1.1), (1.2), (2.1), (1.4) have been investigated in [2, 6, 17].

Theorem 2.3 ([8, Thms. 5.4 and 5.5]). *The eigenvalues of boundary-value problem* (1.1), (1.2), (2.1), (2.2) *are real, simple and form an infinitely increasing sequence* $\{\lambda_k(\beta,\delta)\}_{k=1}^{\infty}$ *such that* $\lambda_1(\beta,\delta) > 0$ *for* $\beta+\delta < \pi$ *and* $\lambda_1\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 0$. *Moreover, the eigenfunction* $u_k^{(\beta,\,\delta)}(x)$ *corresponding to the eigenvalue* $\lambda_k(\beta,\,\delta)$ *has* $k-1$ *simple zeros in the interval* $(0,1)$.

Theorem 2.4 ([17, Thm. 2.2]). *The eigenvalues of boundary-value problem* (1.1), (1.2), (2.1), (1.4) *are real, simple and form an infinitely increasing sequence* $\{\tilde\lambda_k(\beta)\}_{k=1}^{\infty}$ *such that* $\tilde\lambda_1(\beta) > 0$ *for* $\beta < \pi/2$ *and* $\tilde\lambda_1(\pi/2) = 0$. *Moreover, the eigenfunction* $\tilde u_k^{(\beta)}(x)$ *corresponding to the eigenvalue* $\tilde\lambda_k(\beta)$ *has* $k-1$ *simple zeros in the interval* $(0,1)$.

Remark 2.5. By making the change of variables $x' = 1 - x$ and applying the conclusion of the Theorem 2.4 we have: the eigenvalues of problem (1.1)-(1.3), (2.2), are real, simple and form an infinitely increasing sequence $\{\lambda_k(\delta)\}_{k=1}^{\infty}$ such that $\lambda_1(\delta) > 0$ for $\delta \in [0, \pi/2)$ and $\lambda_1(\pi/2) = 0$; moreover, the eigenfunction $u_k^{(\delta)}(x)$ corresponding to the eigenvalue $\lambda_k(\delta)$ has $k-1$ simple zeros in the interval $(0,1)$.

3. Properties of solution to (1.1)-(1.3)

Theorem 3.1. *For each fixed* $\lambda \in \mathbb{C}$ *there exists a nontrivial solutions* $y(x,\lambda)$ *of problem* (1.1)-(1.3), *which is unique up to a constant coefficient. Moreover, the function* $y(x,\lambda)$ *for each fixed* $x \in [0,1]$ *is an entire function of* λ.

The proof of this theorem is similar to that of [16, Lemma 2] with the use of Lemma 2.1.

Remark 3.2. Since any solution $y(x,\lambda)$ of problem (1.1)-(1.3) has a representation $y(x,\lambda) = u(x,\lambda) + iv(x,\lambda)$, where the functions $u(x,\lambda)$ and $v(x,\lambda)$ are real valued, and the coefficients $q(x)$ and a are real, it follows that the functions $u(x,\lambda)$ and $v(x,\lambda)$ are solutions of problem (1.1)-(1.3) for $\lambda \in \mathbb{R}$. If $u(x,\lambda)$ is a nontrivial solution of (1.1)-(1.3), then $u(0,\lambda) \neq 0$ for $\lambda > 0$ and $u(1,\lambda) \neq 0$ for $\lambda \le 0$. Indeed, if $u(0,\lambda) = 0$ for $\lambda > 0$, then it follows from (1.2)-(1.3) that $u(0,\lambda) = u''(0,\lambda) = Tu(0,\lambda) = 0$ for $\lambda > 0$. Hence $u'(0,\lambda) \neq 0$ for $\lambda > 0$ and it follows by continuity that $u'(x,\lambda) \neq 0$ in an open interval $(0,a)$ for some $a \in (0,1)$. We can assume without loss of generality that $u'(x,\lambda) > 0$ for $x \in (0,a)$. Then $u(x,\lambda) > 0$ for $x \in (0,a)$. Since $\lambda > 0$ it follows by (1.1) that $(Tu(x,\lambda))' > 0$ for $x \in (0,a)$, so that $Tu(x,\lambda) > 0$ in $(0,a)$. In view of the equality $Tu(x,\lambda) = u'''(x,\lambda) - q(x)\,u'(x,\lambda)$ we obtain that $u'''(x,\lambda) > q(x)\,u'(x,\lambda) > 0$ for $x \in (0,a)$. Hence $u''(x,\lambda) > 0$ in $(0,a)$. Then the first statement of Lemma 2.1 implies that $u''(1,\lambda) > 0$. But the boundary condition (1.2) implies that $u''(1,\lambda) = 0$, a contradiction. If $u(1,\lambda) = 0$ for $\lambda \le 0$, then the function $u(x,\lambda)$ solves the problem (1.1)-(1.3), (2.2) for $\delta = 0$ which contradicts the condition $\lambda \le 0$ in view of Remark 2.5.

Now let $y(x, \lambda)$ be a solution of problem (1.1)-(1.3), normalized for example by the condition

$$y(0, \lambda) = 1, \tag{3.1}$$

if $\lambda > 0$, and by

$$y(1, \lambda) = 1, \tag{3.2}$$

if $\lambda \leq 0$. If $\lambda > 0$ ($\lambda \leq 0$), then it follows from representation $w(x, \lambda) = u(x, \lambda) + iv(x, \lambda)$ that $u(0, \lambda) = 1$ and $v(0, \lambda) = 0$ ($u(1, \lambda) = 1$ and $v(1, \lambda) = 0$). Hence the above reasoning we see that $v(x, \lambda) \equiv 0$ for $\lambda \in \mathbb{R}$, i.e. $y(x, \lambda) = u(x, \lambda)$ for $\lambda \in \mathbb{R}$. Therefore, the solution $y(x, \lambda)$ of (1.1)-(1.3), (3.1) for $\lambda > 0$ and of (1.1)-(1.3), (3.2) for $\lambda \leq 0$ is a real valued for $\lambda \in \mathbb{R}$.

In the sequel we assume that the function $y(x, \lambda), x \in [0, 1], \lambda \in \mathbb{C}$, is a solution of problem (1.1)-(1.3), (3.1) for $\lambda > 0$ and of problem (1.1)-(1.3), (3.2) for $\lambda \leq 0$. Consider the equation

$$y(x, \lambda) = 0,$$

for $x \in [0, 1]$ and $\lambda \in \mathbb{R}$. The zeros of this equation are functions of λ.

Lemma 3.3. *Let $\lambda \in \mathbb{R}$. Then every zero $x(\lambda) \in (0, 1]$ of the function $y(x, \lambda)$ is simple and is a C^1 function of λ.*

Proof. Let $x_0 \in (0, 1]$ and $\lambda_0 > 0$ such that $y(x_0, \lambda_0) = y'(x_0, \lambda_0) = 0$. If $x_0 \in (0, 1)$ and $y''(x_0, \lambda_0)\ Ty(x_0, \lambda_0) \geq 0$, then the first statement of Lemma 2.1 implies that $y''(1, \lambda_0) > 0$. This is in contradiction with the condition $y''(1, \lambda_0) = 0$. If $x_0 \in (0, 1)$ and $y''(x_0, \lambda_0)Ty(x_0, \lambda_0) < 0$, then the second part of the same lemma yield a contradiction with the boundary condition $y''(0, \lambda_0) = 0$. If $x_0 = 1$, then by (1.2) we have $y(1, \lambda_0) = y'(1, \lambda_0) = y''(1, \lambda_0) = 0$. Let $b \in (0, 1)$ be the fixed point such that $y''(x, \lambda_0) \neq 0$ for $x \in (b, 1)$. We can assume without loss of generality that $y''(x, \lambda_0) > 0$ for $x \in (b, 1)$. Then $y'(x, \lambda_0) < 0$, $y(x, \lambda_0) > 0$ for $x \in (b, 1)$ and $Ty(1, \lambda_0) = y'''(1, \lambda_0) < 0$. Since $\lambda_0 > 0$ it follows by (1.1) that $(Ty(x, \lambda_0))' > 0$ for $x \in (b, 1)$, so that $Ty(x, \lambda_0) < 0$ for $x \in (b, 1)$. Hence the second statement of Lemma 2.1 implies that $y''(0, \lambda_0) > 0$ which contradicts the condition $y''(0, \lambda_0) = 0$.

Now let $x_0 \in (0, 1]$ and $\lambda_0 \leq 0$ such that $y(x_0, \lambda_0) = y'(x_0, \lambda_0) = 0$. Then λ_0 is a nonpositive eigenvalue of the problem defined on $[0, x_0]$ and determined by equation (1.1) with the boundary conditions $y''(0) = 0$, (1.3) and $y(x_0) = y'(x_0) = 0$. By Remark 2.5 all the eigenvalues of this problem are positive, contradiction.

The smoothness of the function $x(\lambda)$ follows from the implicit function theorem. The proof of this lemma is complete. □

From the continuity of the zeros of $y(x, \lambda)$ as functions of λ, together with Remark 3.2 (see (3.1), (3.2)), it follows an important corollary.

Corollary 3.4. *As $\lambda > 0$ ($\lambda \leq 0$) varies the function $y(x, \lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0, 1]$ through its endpoint $x = 1$ ($x = 0$).*

We consider the function

$$F(x, \lambda) = \frac{y(x, \lambda)}{Ty(x, \lambda)}.$$

It follows by Lemmas 2.2, 3.3 and Remark 2.5 that the function $H(x, \lambda)$ is a finite order meromorphic function of λ for all finite λ and fixed $x \in (0, 1]$.

Let $M_k = (\lambda_{k-1}(0), \lambda_k(0))$, $k \in \mathbb{N}$, where $\lambda_0(0) = -\infty$.

Obviously, the eigenvalues $\lambda_k(0)$ and $\lambda_k(\pi/2)$ $k \in \mathbb{N}$, of problem (1.1)-(1.3), (2.2) for $\delta = 0$ and $\delta = \pi/2$ are zeros of the entire functions $y(1, \lambda)$ and $Ty(1, \lambda)$, respectively. We note that the function

$$G(\lambda) = \frac{1}{F(1, \lambda)} = \frac{Ty(1, \lambda)}{y(1, \lambda)}$$

is defined for

$$\lambda \in M \equiv \left(\cup_{k=1}^{\infty} M_k\right) \cup (\mathbb{C} \backslash \mathbb{R}),$$

and is a meromorphic function of finite order, $\lambda_k(\pi/2)$ and $\lambda_k(0)$, $k \in \mathbb{N}$, are the zeros and poles of this function, respectively.

Lemma 3.5. *For each $\lambda \in M$ the following relation holds*

$$\frac{dG(\lambda)}{d\lambda} = \frac{1}{y^2(1, \lambda)} \Big\{ \int_0^1 y^2(x, \lambda) dx + a y^2(0, \lambda) \Big\}. \tag{3.3}$$

Proof. By (1.1) we have

$$(Ty(x, \mu))' \, y(x, \lambda) - (Ty(x, \lambda))' \, y(x, \mu) = (\mu - \lambda) \, y(x, \mu) y(x, \lambda).$$

Integrating this relation from 0 to 1, using the formula for the integration by parts and taking into account boundary conditions (1.2) and (1.3) we obtain

$$\begin{aligned} & y(1, \lambda) \, Ty(1, \mu) - y(1, \mu) \, Ty(1, \lambda) \\ & = (\mu - \lambda) \Big\{ \int_0^1 y(x, \mu) y(x, \lambda) \, dx + a y(0, \mu) y(0, \lambda) \Big\} \end{aligned} \tag{3.4}$$

In view of (3.4) for $\lambda, \mu \in M_k$, $k \in \mathbb{N}$, we have

$$\frac{Ty(1, \mu)}{y(1, \mu)} - \frac{Ty(1, \lambda)}{y(1, \lambda)} = (\mu - \lambda) \, \frac{\int_0^1 y(x, \mu) y(x, \lambda) dx + a y(0, \mu) y(0, \lambda)}{y(1, \mu) y(1, \lambda)} \tag{3.5}$$

Dividing both sides of relation (3.5) by $\mu - \lambda$ ($\mu \neq \lambda$) and passing to the limit as $\mu \to \lambda$ we obtain (3.3). The proof of this lemma is complete. ☐

It follows from (3.3) that

$$\frac{\partial F(1, \lambda)}{\partial \lambda} = - \frac{\int_0^1 y^2(x, \lambda) dx + a y^2(0, \lambda)}{(Ty(1, \lambda))^2} < 0. \tag{3.6}$$

Lemma 3.6. *It holds*

$$\lim_{\lambda \to -\infty} G(\lambda) = -\infty. \tag{3.7}$$

The proof of this Lemma is similar to that of [4, Lemma 3.4]; we omit it.

Remark 3.7. It follows from the relation $Ty(1, \lambda_1(\pi/2)) = 0$ that $G(0) = 0$.

Remark 3.8. By Remarks 2.5 and 3.7, and Lemmas 3.5 and 3.6, we have

$$0 = \lambda_1 \left(\frac{\pi}{2}\right) < \lambda_1(0) < \lambda_2 \left(\frac{\pi}{2}\right) < \lambda_2(0) < \dots. \tag{3.8}$$

Lemma 3.9. *Let $x_0 \in (0, 1]$ and $\lambda_0 > 0$ such that $y(x_0, \lambda_0) = 0$. Then*

$$\frac{\partial F(x_0, \lambda_0)}{\partial x} < 0.$$

Proof. Let $x_0 \in (0,1]$ and $\lambda_0 > 0$ such that $y(x_0,\lambda_0) = 0$. If $x_0 \in (0,1)$, then it follows from Lemma 2.2 that $Ty(x_0,\lambda_0) \neq 0$. The same relation $Ty(x_0,\lambda_0) \neq 0$ follows from (3.8) in the case $x_0 = 1$. Hence we obtain

$$\frac{\partial F(x_0,\lambda_)}{\partial x} = \frac{y'(x_0,\lambda_0)Ty(x_0,\lambda_0) - y(x_0,\lambda_0)(Ty)'(x_0,\lambda_0)}{(Ty\,(x_0,\lambda_0))^2} = \frac{y'(x_0,\lambda_0)}{Ty\,(x_0,\lambda_0)} < 0.$$

The proof of Lemma 3.9 is complete. □

Lemma 3.10. *Let $0 < \mu < \nu$ and $y(x,\mu)$ has m zeros in the interval $(0,1)$. Then $y(x,\nu)$ has at least m zeros in $(0,1)$.*

The proof of the above lemma is similar to that of [4, Lemma 3.6], using formula (3.6) and Lemma 3.9.

Let $m(\lambda)$ be the number of zeros of the function $y(x,\lambda)$ in the interval $(0,1)$. Then it follows from Remark 2.5 that

$$m(\lambda_k(0)) = k-1,\, k \in \mathbb{N}. \tag{3.9}$$

As an immediate consequence of Lemmas 3.3, 3.10, Corollary 3.4 and relations (3.8), (3.9), we obtain the following result.

Lemma 3.11. *If $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)] \cap (0,+\infty)$, then $m(\lambda) = k-1$.*

4. Oscillatory properties of eigenfunctions of (1.1)-(1.4)

Problem (1.1)-(1.4) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ with inner product

$$(\hat{u},\hat{v}) = (\{y,m,n\},\{v,s,t\}) = \int_0^1 y(x)\overline{v(x)}\,dx + |a|^{-1}m\bar{s} + |c|^{-1}n\bar{t}, \tag{4.1}$$

where

$$L\hat{y} = L\{y,m,n\} = \{(Ty(x))',\, Ty(0),\, Ty(1)\}$$

with the domain

$$\begin{gathered} D(L) = \Big\{\{y(x),\, m,\, n\} : y \in W_2^4(0,1), (Ty(x))' \in L_2(0,1), \\ y''(0) = y''(1) = 0,\ m = ay(0),\ n = cy(1)\Big\} \end{gathered}$$

dense everywhere in H [27]. It is obvious that the operator L is well defined in H and the eigenvalue problem (1.1)-(1.4) becomes

$$L\hat{y} = \lambda\hat{y}, \quad \hat{y} \in D(L), \tag{4.2}$$

i.e., the eigenvalues $\lambda_k, k \in \mathbb{N}$, of problem (1.1)-(1.4) and those of the operator L coincide; moreover, between the eigenfunctions, there is a one-to-one correspondence

$$y_k(x) \leftrightarrow \{y_k(x), m_k,\, n_k\}, m_k = ay_k(0),\ n_k = cy_k(1).$$

Problem (1.1)-(1.4) is strongly regular in the sense of [27]; in particular, this problem has discrete spectrum.

Theorem 4.1. *L is a self-adjoint operator in H. The system of eigenvectors $\{y_k(x), m_k, n_k\}$ of the operator L forms a unconditional basis (Riesz basis after normalization) in the space H.*

The proof of this theorem is similar to that of [4, Theorem 5.1]; we omit it.

Remark 4.2. It follows from Theorem 4.1 that the eigenvalues of problem (1.1)-(1.4) are real. Moreover, by (4.2) we have

$$(L\hat{y}(\lambda), \hat{y}(\lambda)) = \lambda(\hat{y}(\lambda), \hat{y}(\lambda)),$$

where $\hat{y}(\lambda) = \{y(x, \lambda), ay(0, \lambda), cy(1, \lambda)\}$, which implies by (4.1) that

$$\begin{aligned}&\int_0^1 \{y''^2(x, \lambda) + q(x)y'^2(x, \lambda)\}\, dx \\ &= \lambda\Big\{\int_0^1 y^2(x, \lambda)dx + ay^2(0, \lambda) - cy^2(1, \lambda)\Big\}.\end{aligned} \tag{4.3}$$

Hence all eigenvalues of problem (1.1)-(1.4) are nonnegative.

We note that the eigenvalues (with regard to multiplicities) of the problem (1.1)-(1.4) are the roots of the equation

$$Ty(1, \lambda) - c\lambda y(1, \lambda) = 0. \tag{4.4}$$

Remark 4.3. Let λ be an eigenvalue of problem (1.1)-(1.4). Hence by Remark 4.2 we have $\lambda \geq 0$. If $\lambda = 0$, then it follows from (4.3) that $y(x, \lambda) \equiv \text{const} \neq 0$, which implies by (3.2) that $y(0, 0) = 1$. If $\lambda > 0$, then $y(1, \lambda) \neq 0$ by virtue of (3.8).

In turn, by Remark 4.3 each root (with regard of multiplicities) of equation (4.4) is also a root of the equation

$$G(\lambda) = c\lambda\,. \tag{4.5}$$

Lemma 4.4. *The eigenvalues of boundary-value problem* (1.1)-(1.4) *are simple and form a countable set without finite limit point.*

Proof. The entire function occurring on the left-hand side of (4.4) does not vanish for nonreal λ in view of Remark 4.2. Hence it is distinct from the identically zero function and its zeros form an at most countable set without finite limit point.

Now we claim that (4.5) has only simple roots. In fact, if λ is a multiple root of equation (4.5), then $G(\lambda) = c\lambda$ and $G'(\lambda) = c$. Hence by Remarks 4.2 and 4.3 it follows from (3.3) that

$$\int_0^1 y^2(x, \lambda)dx + ay^2(0, \lambda) - cy^2(1, \lambda) = 0,$$

which is impossible in view of conditions $a > 0$ and $c < 0$. The proof is complete. □

Theorem 4.5. *Boundary-value problem* (1.1)-(1.4) *has a sequence of eigenvalues*

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_k \to +\infty.$$

The corresponding eigenfunctions $y_k(x)$, $k \in \mathbb{N}$, *have* $k - 1$ *simple zeros in* $(0, 1)$.

Proof. By relations (3.3), (3.7) and (3.8), we have

$$\lim_{\lambda\to\lambda_k(0)-0} G(\lambda) = +\infty, \quad \lim_{\lambda\to\lambda_{k-1}(0)+0} G(\lambda) = -\infty, \quad k \in \mathbb{N}.$$

Hence the function $G(\lambda)$ takes each value in $(-\infty, +\infty)$ at a unique point in the interval $(\lambda_{k-1}(0), \lambda_k(0))$, $k \in \mathbb{N}$. For the function $H(\lambda) = c\,\lambda$ we have $H'(\lambda) = c$. Since $c < 0$ it follows that this function is strictly decreasing in the interval $(-\infty, +\infty)$.

It follows from the preceding considerations that in the interval $(\lambda_{k-1}(0), \lambda_k(0))$, $k \in \mathbb{N}$, there exists a unique $\lambda = \lambda_k$ such that

$$G(\lambda) = H(\lambda)$$

i.e., condition (1.4) is satisfied. Therefore, λ_k is an eigenvalue of boundary-value problem (1.1)-(1.4) and $y_k(x) = y(x, \lambda_k)$ is the corresponding eigenfunction.

By Remark 3.7 it follows from the preceding considerations that $\lambda_1 = \lambda_1\left(\frac{\pi}{2}\right) = 0$ and $\lambda_k > \lambda_1\left(\frac{\pi}{2}\right)$ for $k \geq 2$. Consequently, by Remark 2.5, we have $\lambda_k > 0$ for $k > 2$. Hence it follows by Lemma 3.11 and Remark 4.3 that $m(\lambda_k) = k - 1$. The proof is complete. $\square$

It follows from [17, (3.1)–(3.4)] that

$$\sqrt[4]{\lambda_k(0,0)} = k\pi + O\big(\frac{1}{k}\big), \tag{4.6}$$

$$u_k^{(0,0)}(x) = \sin k\pi x + O\big(\frac{1}{k}\big), \tag{4.7}$$

$$\sqrt[4]{\lambda_k(0)} = (k-1)\pi + O\big(\frac{1}{k}\big), \tag{4.8}$$

$$u_k^{(0)}(x) = \sin(k-1)\pi x + O\big(\frac{1}{k}\big), \tag{4.9}$$

where relations (4.7) and (4.9) hold uniformly for $x \in [0, 1]$.

Theorem 4.6. *The following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k-2)\pi + O(1/k), \tag{4.10}$$

$$y_k(x) = \sin(k-2)\pi x + O(1/k) \tag{4.11}$$

where relation (4.11) *holds uniformly for* $x \in [0, 1]$.

The proof of the above theorem is similar to that of [17, Theorem 3.1] using Theorem 4.5. We omit the proof here.

5. Basis property in $L_p(0,1)$, $1 < p < \infty$, of the eigenfunctions of (1.1)-(1.4)

Let

$$\delta_k = (\hat{y}_k, \hat{y}_k)\,. \tag{5.1}$$

Then by conditions $a > 0$, $c < 0$ and (4.1) it follows from (5.1) that

$$\delta_k = \|y_k\|_{L_2}^2 + a^{-1}m_k^2 - c^{-1}n_k^2 > 0. \tag{5.2}$$

Hence, the system of eigenvectors $\{\hat{v}_k\}_{k=1}^{\infty}$, $\hat{v}_k = \delta_k^{-1/2}\hat{y}_k$, of operator L forms an orthonormal basis (i.e. Riesz basis) in H.

Let r and l $(r \neq l)$ be arbitrary fixed natural numbers and

$$\tilde{\Delta}_{r,\,l} = \begin{vmatrix} a\delta_r^{-1/2}y_r(0) & a\delta_1^{-1/2}y_1(0) \\ c\delta_r^{-1/2}y_r(1) & c\delta_1^{-1/2}y_1(1) \end{vmatrix} = ac\delta_r^{-1}\delta_1^{-1}\begin{vmatrix} y_r(0) & y_1(0) \\ y_r(1) & y_1(1) \end{vmatrix}, \tag{5.3}$$

$$\Delta_{r,\,l} = \begin{vmatrix} y_r(0) & y_1(0) \\ y_r(1) & y_1(1) \end{vmatrix}. \tag{5.4}$$

By (5.2) it follows from (5.3) and (5.4) that

$$\tilde{\Delta}_{r,\,l} \neq 0 \;\Leftrightarrow\; \Delta_{r,\,l} \neq 0. \tag{5.5}$$

Theorem 5.1. *Let r and l $(r \neq l)$ be arbitrary fixed natural numbers. If $\Delta_{r,\,l} \neq 0$, then the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,\,l}^{\infty}$ of problem* (1.1)-(1.4) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$; if $\Delta_{r,\,l} = 0$, then this system is incomplete and nonminimal in the space $L_2(0,1)$.*

The proof of the above theorem in the case $p = 2$ is similar to that of [5, Theorem 4.1] using Theorem 4.1 and relation (5.5). In the case $p \in (1, +\infty)\backslash\{2\}$ is similar to that of [17, Theorem 5.1] using asymptotic formulas (4.6)-(4.11).

Using Theorem 5.1, we can obtain sufficient conditions for the subsystem of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,l}^{\infty}$ of problem (1.1)-(1.4) to form a basis in $L_p(0,1), 1 < p < \infty$.

Corollary 5.2. *Let r and l $(r \neq l)$ be arbitrary fixed natural numbers having different parity. Then the system of eigenfunctions $\{y_k(x)\}_{k=1,k\neq r,l}^{\infty}$ of problem* (1.1)-(1.4) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$.*

Proof. By (3.1) from (5.4) it follows that

$$\Delta_{r,\,l} = y_l(1) - y_r(1). \tag{5.6}$$

In view of (3.1) and Theorem 4.5 we have

$$\operatorname{sgn} y_k(1) = (-1)^{k+1}, \quad k \in \mathbb{N}. \tag{5.7}$$

Taking this equality into account, from (5.6) we obtain

$$\Delta_{r,\,l} = (-1)^{l+1}\{(-1)^{r+l}|y_l(1)| - |y_r(1)|\}. \tag{5.8}$$

Now the statement of this corollary follows from Theorem 5.1 in view of relation (5.8). The proof is complete. □

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References

[1] Z. S. Aliyev; Basis properties of a fourth order differential operator with spectral parameter in the boundary condi-tion, *Cent. Eur. J. Math.*, **8**(2) (2010), 378-388.

[2] Z. S. Aliyev; Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition, *Differential Equations*, **47**(6), (2011), 766-777.

[3] Z. S. Aliev; On basis properties of root functions of a boundary value problem containing a spectral parameter in the boundary conditions, *Doklady Math.*, **87**(2) (2013), 137-139.

[4] Z. S. Aliev, S. B. Guliyeva; Properties of natural frequencies and harmonic bending vibrations of a rod at one end of which is concentrated inertial load, *J. Differential Equations*, **263**(9) (2017), 5830-5845.

[5] Z. S. Aliev, A. A. Dunyamalieva; Defect basis property of a system of root functions of a Sturm-Liouville problem with spectral parameter in the boundary conditions, *Diff. Equ.* **51**(10) (2015), 1249-1266.

[6] Z. S. Aliyev, N. B. Kerimov; Spectral properties of the differential operators of the fourth-order with eigenvalue parameter dependent boundary condition, *Inter. J. Math. Math. Sci.*, **2012**, Article ID 456517 (2012), p. 28.

[7] K. Aydemir, O. Sh. Mukhtarov; Asymptotic distribution of eigenvalues and eigenfunctions for a multi-point discontinuous Sturm-Liouville problem, *Elec. J. Diff. Equ.*, **2016**(131) (2016), 1-14.

[8] D. O. Banks, G. J. Kurowski; A Prüfer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations*, **24** (1977), 57-74.

[9] J. Ben Amara; Sturm theory for the equation of vibrating beam, *J. Math. Anal. Appl.*, **349** (2009), 1-9.
[10] J Ben Amara, A. A. Vladimirov; On a fourth-order problem with spectral and physical parameters in the boundary condition, *Izvestiya: Mathematics*, **68**(4) (2004), 645-658.
[11] H. Boerner; Das Eigenwertproblem der selbstadjungierten linearen Differentialgleichung vierter Ordnung, *Math. Z.* 34(1) (1932), 293-319.
[12] B. B. Bolotin; Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems, I, *Engineering Industry, Moscow, 1978* .
[13] C. T. Fulton; Two-point boundary value problems with eigenvalue parameter in the boundary conditions, *Proc. Roy. Soc. Edinburgh, Sect.* ***A*** **77**(3-4) (1977), 293-308.
[14] N. Yu. Kapustin, E. I. Moiseev; On the basis property in the space L_p of systems of eigenfunctions corresponding to two problems with spectral parameter in the boundary condition, *Diff. Equ.*, **36**(10) (2000), 1357-1360.
[15] N. Yu. Kapustin; On a spectral problem arising in a mathematical model of torsional vibrations of a rod with pulleys at the ends, *Diff. Equ.*, **41**(10) (2005), 1490-1492.
[16] N. B. Kerimov, Z. S. Aliev; On oscillation properties of the eigenfunctions of a fourth order differential operator, *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.*, **25**(4) (2005), 63-76.
[17] N. B. Kerimov, Z. S. Aliev; On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in a boundary condition, *Diff. Equ.*, **43**(7), (2007), 905-915.
[18] N. B. Kerimov, R.G. Poladov; Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions, *Doklady Math.*, **85**(1) (2012), 8-13.
[19] A. N. Krylov; *Some differential equations of mathematical physics having applications to technical problems*, Academy of Sciences of the USSR, Moscow, 1932 [in Russian].
[20] E. I. Moiseev, N. Yu. Kapustin; On singularities of the root space of a spectral problem with spectral parameter in a boundary condition, *Doklady Math.* **385**(1) (2002), 20-24.
[21] M. Möller, B. Zinsou; Self-adjoint fourth order differential operators with eigenvalue parameter dependent boundary conditions. *Quaest. Math.*, **34**(3) (2011), 393-406.
[22] M. Möller, B. Zinsou; Sixth order differential operators with eigenvalue dependent boundary conditions, *Appl. Anal. Disc. Math.*, **7**(2) (2013), 378-389.
[23] O. Sh. Mukhtarov, K. Aydemir; Eigenfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point, *Acta Math.Sci.* **35**(3) (2015), 639-649.
[24] S. D. Poisson; Mémoire sur la Manière d'exprimer les Fonctions par des Séries de quantités périodiques, et sur l'Usage de cette Transformation dans la Resolution de differens Problèmes. *École Polytechnique* **18** (1820), 417-489.
[25] M. Roseau; *Vibrations in mechanical systems. Analytical methods and applications*, Springer-Verlag, Berlin, 1987.
[26] E. M. Russakovskii; Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions, *Funct. Anal. Appl.*, **9**(4) (1975), 358-359.
[27] A. A. Shkalikov; Boundary value problems for ordinary differential equations with a parameter in the boundary conditions, *J. Sov. Math.*, **33**(6), 1311-1342 (1986).
[28] W. Sternberg; Die entwicklung unstetiger funktionen nach den eigenfunktionen eines eindimensionalen randwert-problems, *Math. Z.*, 3(1) (1919), 191-208.
[29] A. N. Tikhonov, A. A. Samarskii; *Equations of mathematical physics,* Nauka, Moscow, 1972 [in Russian].
[30] S. P. Timoshenko; *Strength and oscillations of elementary constructions*, Nauka, Moscow, 1975 [in Russian].
[31] J. Walter; Regular eigenvalue problems with eigenvalue parameter in the boundary condition, *Math. Z.* **133**(4) (1973), 301-312.

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