Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 308, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

HÖLDER CONTINUITY FOR (p,q)-LAPLACE EQUATIONS THAT DEGENERATE UNIFORMLY ON PART OF THE DOMAIN

SARVAN T. HUSEYNOV

Communicated by Ludmila S. Pulkina

Abstract. In this article we consider $p(x)$-Laplace equations with two-phase degree $p(x)$, taking two values p and q, when the boundary of the phase interface is a hyperplane. Assuming that in the part of the domain where $q < p$ the equation degenerates uniformly for a small parameter, Hölder continuity of the solution is established.

1. Formulation of results

Consider in the domain $D \subset \mathbb{R}^n$, $n \geq 2$ and the family of the elliptic equations

$$L_\varepsilon u = \operatorname{div}(\omega_\varepsilon(x)|\nabla u|^{p(x)-2}\nabla u) = 0 \tag{1.1}$$

with positive weight $\omega_\varepsilon(x)$ and degree $p(x)$, that will be defined below. Suppose that the domain D is divided by the hyperplane $\Sigma = \{x : x_n = 0\}$ into two parts $D^{(1)} = D \cap \{x : x_n > 0\}$ and $D^{(2)} = D \cap \{x : x_n < 0\}$. Also assume that for $\varepsilon \in (0, 1]$,

$$\omega_\varepsilon(x) = \begin{cases} \varepsilon, & x \in D^{(1)} \\ 1, & x \in D^{(2)}, \end{cases} \tag{1.2}$$

and for $1 < q < p$,

$$p(x) = \begin{cases} q, & x \in D^{(1)} \\ p, & x \in D^{(2)}, \end{cases} \tag{1.3}$$

To define the solution of equation (1.1) we define a class of functions related with the degree $p(x)$:

$$W_{\rm loc}(D) = \{u : u \in W^{1,1}_{\rm loc}(D),\ |\nabla u|^{p(x)} \in L^1_{\rm loc}(D)\},$$

where $W^{1,1}_{\rm loc}(D)$ is a Sobolev space of the locally integrable in D functions together with their first order generalized derivatives.

As a solution of equation (1.1) we take the function $u \in W_{\rm loc}(D)$, satisfying

$$\int_D \omega_\varepsilon(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx = 0 \tag{1.4}$$

for all test functions $\varphi \in C_0^\infty(D)$.

2010 *Mathematics Subject Classification.* 35J92, 35J65, 35J70, 35J62.
Key words and phrases. (p,q)-Laplacian; elliptic equation; Hölder continuity.

Submitted November 3, 2017. Published December 14, 2017.

For the degree $p(\cdot)$, given by equality (1.3), the smooth functions are dense in $W_{\rm loc}(D)$ (see [1]), and as a result in integral identity (1.4) as test functions the finite functions from $W_{\rm loc}(D)$ may be taken.

The p-Laplace type equation with variable nonlinearity degree $p(x)$ and the variational problems with integrant satisfying the non standard coercivity and growth conditions usually arise in the modeling of the composite materials, electroreological fluids (the characteristics of which depend on the electromagnetic filed), in the problems of image processing. In this paper the plane junction of two different phases is considered as a model case. The case is complicated by the presence of the uniform degeneracy over ε in the domain $D^{(1)}$.

In each of the domains $D^{(i)}$, $i=1,2$ regularity of the solution is described by the well developed theory (see [2]). In [3] is proved that for the degree p, given by equality (1.3), any solution of equation (1.1) by each fixed value $\varepsilon \in (0,1]$ in the arbitrary subdomain $D' \Subset D$ belongs to the space $C^{\alpha}(D')$ of the Hölder in D' functions. We are interested in the problem of independence of the degree α on ε.

Consider the family $\{u^{\varepsilon}(x)\}$ of the solutions of the equation $L_{\varepsilon}u^{\varepsilon}=0$, uniformly bounded over ε in L^{∞} on the compact subspaces of D. The main aim of this work is to prove the following statement.

Theorem 1.1. *There exists a constant $\alpha \in (0,1)$, not depending on ε, such that the family $\{u^{\varepsilon}(x)\}$ is compact in $C^{\alpha}(D')$ for arbitrary subdomain $D' \Subset D$.*

Note that in the case $p=q$ a similar result is obtained in [4,5].

Choice of the weight of type (1.2) and the degree p from (1.3) makes the case nonsymmetric with respect to the domains $D^{(1)}$ and $D^{(2)}$, and use of known results does not allow the one to prove the above statement. We will proceed from the modification of the Mozer's technique [6], developed in [7,8], where the domains $D^{(1)}$ and $D^{(2)}$ play different roles in the proof of the Theorem 1.1.

Statement of Theorem 1.1 remains true also for the solutions of the equation

$$\operatorname{div}\left(\omega_{\varepsilon}(x)|\nabla u|^{p(x)-2}a\nabla u\right)=0$$

with measurable uniformly positive defined matrix a. Wherein Hölder degree of the solutions will be additionally depend on the ellipticity coefficients of this matrix.

2. Auxiliary statements

Here and below u denotes the solution of equation (1.1), $B_R \subset D$ are balls with centers in Σ, $B_R^{(i)} = B_R \cap D^{(i)}$ are semiballs ($i=1,2$), $|E|$ is n-dimensional Lebesgue measure of the measurable set $E \subset \mathbb{R}^n$, and

$$\fint_E f\,dx = \frac{1}{|E|}\int_E f\,dx.$$

Below we use for $i=1,2$ Sobolev's embedding theorem in the semiballs:

$$\begin{gathered}\Big(\fint_{B_R^{(i)}} |\varphi|^{kq}dx\Big)^{1/k} \le CR^q \fint_{B_R^{(i)}} |\nabla\varphi|^q\,dx,\\ q\ge 1, \quad k=\frac{n}{n-1}, \quad \varphi \in C_0^{\infty}(B_R),\end{gathered} \tag{2.1}$$

where $C=C(n,q)$.

Everywhere below $M = \sup_{B_{R_0}} |u(x)|$, where $B_{R_0} \subset D$, $R_0 \leq 1/4$, and for $R \leq R_0/6$ is taken

$$M_6 = \sup_{B_{6R}} u, \quad m_6 = \inf_{B_{6R}} u, \quad v(x) = \ln \frac{M_6 - m_6 + 2R}{M_6 - u(x) + R}. \tag{2.2}$$

It is easy to see that

$$\frac{R}{4(M+1)} \leq v(x) \leq \frac{2(M+1)}{R} \quad x \in B_{6R}. \tag{2.3}$$

The odd continuation of the function f from $D^{(2)}$ in to $D^{(1)}$ with respect to the hyperplane Σ is denoted as $\tilde{f}$.

Lemma 2.1. *For any $R \leq \rho < r \leq 3R$ the inequality*

$$\sup_{B_\rho} v \leq C(n,p,q,M)\Big(\frac{r}{r-\rho}\Big)^a \Big(\fint_{B_r} v^p\,dx\Big)^{1/p} \tag{2.4}$$

holds with constant $a(n,p) > 0$.

Proof. Choosing in (1.4) the test function as $\varphi(x) = v^{\gamma+q-p}(x)(M_6 - u(x) + R)^{1-p}\eta^p(x)$, where

$$\gamma \geq 1 + p - q, \tag{2.5}$$

$\eta \in C_0^\infty(B_{4R})$, $0 \leq \eta(x) \leq 1$, we find that

$$\begin{aligned}
&(\gamma+q-p)\int_{B_{4R}} \omega_\varepsilon(x)|\nabla u|^{p(x)}(M_6-u+R)^{-p}v^{\gamma+q-p-1}\eta^p\,dx \\
&+ (p-1)\int_{B_{4R}} \omega_\varepsilon(x)|\nabla u|^{p(x)}(M_6-u+R)^{-p}v^{\gamma+q-p}\eta^p\,dx \\
&\leq p\int_{B_{4R}} \omega_\varepsilon(x)|\nabla u|^{p(x)-1}(M_6-u+R)^{1-p}v^{\gamma+q-p}\eta^{p-1}|\nabla\eta|\,dx.
\end{aligned}$$

Omitting the second term in the left hand side and applying to the integrant in the right hand side the Young inequality with corresponding ε we arrive to the estimate

$$\begin{aligned}
&\int_{B_{4R}} \omega_\varepsilon(x)|\nabla u|^{p(x)}(M_6-u+R)^{-p}v^{\gamma+q-p-1}\eta^p\,dx \\
&\leq C(p)\int_{B_{4R}} \omega_\varepsilon(x)(M_6-u+R)^{p(x)-p}v^{\gamma+q-p+p(x)-1}|\nabla\eta|^{p(x)}\,dx.
\end{aligned} \tag{2.6}$$

We narrow the integration domain in the left-hand side of (2.6) to the semiball $B_{4R}^{(2)}$. Then considering (1.2) and (1.3) we can write

$$\begin{aligned}
&\int_{B_{4R}^{(2)}} |\nabla u|^p(M_6-u+R)^{-p}v^{\gamma+q-p-1}\eta^p\,dx \\
&\leq C(p)\Big(\int_{B_{4R}^{(1)}} (M_6-u+R)^{q-p}v^{\gamma+2q-p-1}|\nabla\eta|^q\,dx \\
&\quad + \int_{B_{4R}^{(2)}} v^{\gamma+q-1}|\nabla\eta|^p\,dx\Big).
\end{aligned} \tag{2.7}$$

According to (2.3) and (1.3), in $B_{4R}^{(1)}$ the following inequalities are valid

$$(M_6-u+R)^{q-p} \leq R^{q-p}, \quad v^q \leq C(p,q,M)R^{q-p}v^p.$$

Additionally

$$|\nabla v| = \frac{|\nabla u|}{M_6 - u + R}, \tag{2.8}$$

and from (2.7) considering given above relations we obtain the estimate

$$\begin{aligned} &\int_{B_{4R}^{(2)}} |\nabla v|^p v^{\gamma+q-p-1}\eta^p\,dx \\ &\le C(p,q,M)\Big(R^{q-p}\int_{B_{4R}^{(1)}} v^{\gamma+q-1}|\nabla\eta|^q\,dx + \int_{B_{4R}^{(2)}} v^{\gamma+q-1}|\nabla\eta|^p\,dx\Big). \end{aligned} \tag{2.9}$$

By the Soboloev's embedding theorem (2.1),

$$\begin{aligned} &\Big(\fint_{B_{4R}^{(2)}} (v^{\gamma+q-1}\eta)^k\,dx\Big)^{1/k} \\ &\le C(n,p,q,M)(\gamma+q-1)^p\Big(R^q\fint_{B_{4R}^{(1)}} v^{\gamma+q-1}|\nabla\eta|^q\,dx + R^p\fint_{B_{4R}^{(2)}} v^{\gamma+q-1}|\nabla\eta|^p\,dx\Big) \end{aligned}$$

in the semiball $B_{4R}^{(2)}$.

Choosing here radial-symmetric with respect to the center of the ball B_R, cutoff function $\eta = 1$ in B_ρ, $|\nabla\eta| \le Cr(R(r-\rho))^{-1}$, we have

$$\Big(\fint_{B_\rho^{(2)}} v^{(\gamma+q-1)k}\,dx\Big)^{1/k} \le C(n,p,q,M)(\gamma+q-1)^p\big(\frac{r}{r-\rho}\big)^p\fint_{B_r} v^{\gamma+q-1}\,dx. \tag{2.10}$$

Now we prove a similar estimate in the semiball $B_{4R}^{(1)}$. Let

$$G_R = B_{4R}^{(1)} \cap \{x : v(x) > \tilde v(x)\}. \tag{2.11}$$

In (1.4) use the test function

$$\varphi(x) = \begin{cases} (v^\gamma(x) - \tilde v^\gamma(x))(M_6 - u(x) + R)^{1-q}\eta^p(x) & \text{in } G_R \\ 0 & B_{4R}\setminus G_R, \end{cases}$$

where $\gamma > 1$ satisfies to condition (2.5), the radial-symmetric function $\eta(x)$ has the same properties as above. Considering (1.2), and (1.3) we have

$$\begin{aligned} &\gamma\int_{G_R} |\nabla u|^q v^{\gamma-1}(M_6 - u + R)^{-q}\eta^p\,dx \\ &\le \gamma\int_{G_R} |\nabla u|^{q-1}|\nabla\tilde v|\tilde v^{\gamma-1}(M_6 - u + R)^{1-q}\eta^p\,dx \\ &\quad + p\int_{G_R} |\nabla u|^{q-1}(v^\gamma + \tilde v^\gamma)(M_6 - u + R)^{1-q}|\nabla\eta|\eta^{p-1}\,dx. \end{aligned} \tag{2.12}$$

Since $\tilde v(x) < v(x)$ by $x \in G_R$ and $0 \le \eta \le 1$, applying the Young inequality to the integrant in the right hand side of (2.12), we obtain

$$\begin{aligned} &|\nabla u|^{q-1}|\nabla\tilde v|\tilde v^{\gamma-1}(M_6 - u + R)^{1-q}\eta^p \\ &\le \delta|\nabla u|^q v^{\gamma-1}(M_6 - u + R)^{-q}\eta^p + C(\delta,q)|\nabla\tilde v|^q\tilde v^{\gamma-1}\eta^p, \end{aligned}$$

$$\begin{aligned} &|\nabla u|^{q-1}(v^\gamma + \tilde v^\gamma)(M_6 - u + R)^{1-q}|\nabla\eta|\eta^{q-1} \\ &\le \delta|\nabla u|^q v^{\gamma-1}(M_6 - u + R)^{-q}\eta^p + C(\delta,q)v^{\gamma+q-1}|\nabla\eta|^q. \end{aligned}$$

From this and (2.12) (see also (2.8)) after a corresponding choice of δ we find that

$$\begin{aligned}&\int_{G_R} |\nabla v|^q v^{\gamma-1}\eta^p\,dx\\ &\le C(p,q)\Big(\int_{G_R} |\nabla \tilde v|^q \tilde v^{\gamma-1}\eta^p\,dx + \int_{G_R} v^{\gamma+q-1}|\nabla\eta|^q\,dx\Big).\end{aligned}\tag{2.13}$$

Expanding the integrals in the right hand side of (2.13) to larger set $B^{(1)}_{4R}$, we rewrite (2.13) in the form

$$\begin{aligned}&\int_{G_R} |\nabla v|^q v^{\gamma-1}\eta^p\,dx\\ &\le C(p,q)\Big(\int_{B^{(1)}_{4R}} |\nabla \tilde v|^q \tilde v^{\gamma-1}\eta^p\,dx + \int_{B^{(1)}_{4R}} v^{\gamma+q-1}|\nabla\eta|^q\,dx\Big).\end{aligned}\tag{2.14}$$

We can not estimate the gradient $v(x)$ over the set $B^{(1)}_{4R}\setminus G_R$. But this is not important. Consider in $D^{(1)}$ the auxiliary function

$$w(x) = \max\,(v(x),\tilde v(x)).\tag{2.15}$$

Since $w(x)=v(x)$ for $x\in G_R$ and $w(x)=\tilde v(x)$ for $x\in B^{(1)}_{4R}\setminus G_R$, we have

$$\int_{B^{(1)}_{4R}} |\nabla w|^q w^{\gamma-1}\eta^p\,dx \le \int_{G_R} |\nabla v|^q v^{\gamma-1}\eta^p\,dx + \int_{B^{(1)}_{4R}} |\nabla \tilde v|^q \tilde v^{\gamma-1}\eta^p\,dx,$$

and considering (2.14),

$$\begin{aligned}&\int_{B^{(1)}_{4R}} |\nabla w|^q w^{\gamma-1}\eta^p\,dx\\ &\le C(p,q)\Big(\int_{B^{(1)}_{4R}} |\nabla \tilde v|^q \tilde v^{\gamma-1}\eta^p\,dx + \int_{B^{(1)}_{4R}} v^{\gamma+q-1}|\nabla\eta|^q\,dx\Big).\end{aligned}\tag{2.16}$$

Now let us modify the first integrant in the right-hand side of (2.16). Since $p>q$, according to Young's theorem

$$|\nabla \tilde v|^q \tilde v^{\gamma-1} < R^{p-q}|\nabla \tilde v|^p \tilde v^{\gamma+q-p-1} + R^{-q}\tilde v^{\gamma+q-1}.$$

So from (2.16) we obtain

$$\begin{aligned}&\int_{B^{(1)}_{4R}} |\nabla w|^q w^{\gamma-1}\eta^p\,dx\\ &\le C(p,q)\Big(R^{p-q}\int_{B^{(1)}_{4R}} |\nabla \tilde v|^p \tilde v^{\gamma+q-p-1}\eta^p\,dx\\ &\quad + \int_{B^{(1)}_{4R}} v^{\gamma+q-1}|\nabla\eta|^q\,dx + R^{-q}\int_{B^{(1)}_{4R}} \tilde v^{\gamma+q-1}\eta^p\,dx\Big).\end{aligned}\tag{2.17}$$

Since $\tilde v(x)$ is an odd continuation of $v(x)$ from $D^{(2)}$ to $D^{(1)}$ and the cutoff function $\eta(x)$ is radial symmetric, it follows that

$$\int_{B^{(1)}_{4R}} \tilde v^{\gamma+q-1}\eta^p\,dx = \int_{B^{(2)}_{4R}} v^{\gamma+q-1}\eta^p\,dx \le \int_{B_{4R}} v^{\gamma+q-1}\eta^p\,dx,$$

and considering (2.9),

$$\int_{B^{(1)}_{4R}} |\nabla \tilde v|^p \tilde v^{\gamma+q-p-1}\eta^p\,dx = \int_{B^{(2)}_{4R}} |\nabla v|^p v^{\gamma+q-p-1}\eta^p\,dx$$

$$\le C(p,q,M)\Big(R^{q-p}\int_{B_{4R}^{(1)}} v^{\gamma+q-1}|\nabla\eta|^q\,dx + \int_{B_{4R}^{(2)}} v^{\gamma+q-1}|\nabla\eta|^p\,dx\Big).$$

Therefore from (2.17) we arrive at the estimate

$$\begin{aligned}
&\int_{B_{4R}^{(1)}} |\nabla w|^q w^{\gamma-1}\eta^p\,dx\\
&\le C(p,q,M)\Big(\int_{B_{4R}^{(1)}} v^{\gamma+q-1}|\nabla\eta|^q\,dx + R^{p-q}\int_{B_{4R}^{(2)}} v^{\gamma+q-1}|\nabla\eta|^p\,dx\\
&\quad + R^{-q}\int_{B_{4R}} v^{\gamma+q-1}\eta^p\,dx\Big).
\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}
&\int_{B_{4R}^{(1)}} |\nabla(w^{(\gamma+q-1)/q}\eta^{p/q})|^q\,dx\\
&\le C(p,q,M)(\gamma+q-1)^q\Big(\int_{B_{4R}^{(1)}} v^{\gamma+q-1}|\nabla\eta|^q\,dx\\
&+ R^{p-q}\int_{B_{4R}^{(2)}} v^{\gamma+q-1}|\nabla\eta|^p\,dx + R^{-q}\int_{B_{4R}} v^{\gamma+q-1}\eta^p\,dx + \int_{B_{4R}^{(1)}} w^{\gamma+q-1}|\nabla\eta|^q\,dx\Big).
\end{aligned}$$

From definition (2.15) of the function w and the radial symmetricity of the cutoff function η, we obtain

$$\int_{B_{4R}^{(1)}} w^{\gamma+q-1}|\nabla\eta|^q\,dx \le \int_{B_{4R}} v^{\gamma+q-1}|\nabla\eta|^q\,dx.$$

From (2.5) and (1.3), we have $(\gamma+q-1)^q \le (\gamma+q-1)^p$. Therefore

$$\begin{aligned}
&\int_{B_{4R}^{(1)}} |\nabla(w^{(\gamma+q-1)/q}\eta^{p/q})|^q\,dx\\
&\le C(p,q,M)(\gamma+q-1)^p\Big(\int_{B_{4R}^{(1)}} v^{\gamma+q-1}|\nabla\eta|^q\,dx\\
&\quad + R^{p-q}\int_{B_{4R}^{(2)}} v^{\gamma+q-1}|\nabla\eta|^p\,dx + R^{-q}\int_{B_{4R}} v^{\gamma+q-1}\eta^p\,dx + \int_{B_{4R}} v^{\gamma+q-1}|\nabla\eta|^q\,dx\Big).
\end{aligned}$$

From this following to the Sobolev's embedding theorem (2.1) in the semiball $B_{4R}^{(1)}$ and from the choice of the cutoff function η we find that

$$\Big(\fint_{B_\rho^{(1)}} w^{(\gamma+q-1)k}\,dx\Big)^{1/k} \le C(n,p,q,M)(\gamma+q-1)^p\big(\frac{r}{r-\rho}\big)^p\fint_{B_r} v^{\gamma+q-1}\,dx.$$

Or, since $w \ge v$ on $B_\rho^{(1)}$, it follows that

$$\Big(\fint_{B_\rho^{(1)}} v^{(\gamma+q-1)k}\,dx\Big)^{1/k} \le C(n,p,q,M)(\gamma+q-1)^p\big(\frac{r}{r-\rho}\big)^p\fint_{B_r} v^{\gamma+q-1}\,dx. \quad (2.18)$$

Summing (2.10) and (2.18) one can get

$$\Big(\fint_{B_\rho} v^{(\gamma+q-1)k}\,dx\Big)^{1/k} \le C(n,p,q,M)(\gamma+q-1)^p\big(\frac{r}{r-\rho}\big)^p\fint_{B_r} v^{\gamma+q-1}\,dx. \quad (2.19)$$

Let us iterate this inequality. Let $j = 0, 1, \dots$. Denote $r_j = \rho + 2^{-j}(r-\rho)$, $\chi_j = pk^j$ and take $r = r_j$, $\rho = r_{j+1}$, $\gamma = \chi_j + 1 - q$ in (2.19). Note that by such choice of γ the above assumption (2.5) becomes true. As a result for

$$\Phi_j = \Big(\fint_{B_{r_j}} v^{\chi_j}\,dx\Big)^{1/\chi_j}$$

we get the recurrence relation

$$\Phi_{j+1} \le C^{1/\chi_j}(n,p,q,M)\left(2^j\,(1+\chi_j)\right)^{p/\chi_j}\Big(\frac{r}{r-\rho}\Big)^{p/\chi_j}\Phi_j,$$

from which follows (see [6]) the estimate (2.4). The proof is complete. □

The proof of the next statement uses the scheme given in [9].

Lemma 2.2. *If for any $R \le \rho < r \le 3R$, inequality (2.4) is valid then the following inequity holds*

$$\sup_{B_R} v \le C(n,p,q,M)\fint_{B_{2R}} v\,dx. \tag{2.20}$$

Proof. Without loss of generality we assume that

$$\fint_{B_{2R}} v\,dx = 1. \tag{2.21}$$

Denote by $B(t)$ the concentric with B_R ball of radius $3Rt$ and let

$$J(t) = \Big(\fint_{B(t)} v^p\,dx\Big)^{1/p}.$$

Taking $r = 3Rt$, $\rho = 3R\tau$, rewrite (2.4) in the form

$$\sup_{B(\tau)} v(x) \le C(n,p,q,M)(t-\tau)^{-a}J(t), \quad 1/3 \le \tau < t \le 1. \tag{2.22}$$

In particular $\sup_{B_R} v \le C(n,p,q,M)J(1/2)$, and for the proof of the lemma it is sufficient to set the estimate $J(1/2) \le C(n,p,q,M)$. Since, considering (2.21)

$$J(\tau) \le C(n,p,q,M)\big(\sup_{B(\tau)} v\big)^{\delta}, \quad \delta = 1 - p^{-1},$$

then according to (2.22),

$$J(\tau) \le C^{\delta}\,(t-\tau)^{-a\delta}\,J^{\delta}(t), \quad 1/3 \le \tau < t \le 1,$$
$$\ln J(\tau) \le \delta \ln C + a\delta \ln\frac{1}{t-\tau} + \delta \ln J(t).$$

Take here $\tau = t^b$, where $b > 1$. It easy to see that

$$\int_{(1/2)^{1/b}}^{1} \frac{\ln J(t^b)}{t}\,dt \le C(n,p,q,b,M) + \delta\int_{1/2}^{1}\frac{\ln J(t)}{t}\,dt.$$

Making substitution of the variables $\xi = t^b$ one can get

$$(1/b - \delta)\int_{1/2}^{1}\frac{\ln J(\xi)}{\xi}\,d\xi \le C(n,p,q,b,M).$$

Let us choose here the constant $b > 1$ satisfying the inequality $1/b - \delta > 0$. As $J(\xi) \ge C(n,p)J(1/2)$ by $\xi \in [1/2, 1]$. Then

$$\ln\left(C(n,p)J(1/2)\right) \le \frac{C(n,p,q,b,M)}{(1/b-\delta)\ln 2}$$

that leads us to the seeking estimate for $J(1/2)$. This completes the proof. □

Inequality (2.20) will be applied in some modified form. Denote by Q_r, $r \ge 3R$ the balls with centers in $D^{(2)}$, obtained by the parallel replacement of the ball B_r with the center in x_0 along the normal to Σ in the distance R. Suppose that $Q_r^{(i)} = D^{(i)} \cap Q_r$, $i = 1, 2$. Let additionally $w(x) = \max(v(x), \tilde{v}(x))$ by $x \in B_{4R}^{(1)}$ and $w(x) = v(x)$ by $x \in B_{4R}^{(2)}$. Expanding the integral in the right hand side of (2.20) up to larger set and replacing in the part $D^{(1)}$ the function $v(x)$ by $w(x)$, we obtain

$$\sup_{B_R} v(x) \le C(n,p,M)\Big(\fint_{Q_{3R}} w\,dx + \fint_{Q_{3R}^{(2)}} v\,dx\Big). \tag{2.23}$$

3. Hölder continuity of the solutions

From the results in [2] it is known that the solutions of equation (1.1) are Hölder property inside of $D^{(1)}$ and $D^{(2)}$. It remains to prove the Hölder property of the solution on $\Sigma \cap D$, since the seeking holder property inside of D may be obtained by elementary "union" of the Hölder property on $\Sigma \cap D$ and in $D^{(1)}$, $D^{(2)}$.

Let M means exact upper bound of the module of the solution in the ball $B_{R_0} \subset D$ of radius $R_0 \le 1/2$ and $\operatorname{osc}\{u, B_r\} = \sup_{B_r} u(x) - \inf_{B_r} u(x)$, where B_r are balls with centers in $x_0 \in \Sigma \cap D$. Hölder continuity of the solutions in the point x_0 follows from the following "scattering lemma":

$$\operatorname{osc}\{u, B_R\} \le (1-\delta)\operatorname{osc}\{u, B_{6R}\} + R, \quad \delta = \delta(n,p,q,M) > 0, \quad R \le R_0/6. \tag{3.1}$$

From this lemma (see [10]) it follows the estimate

$$\operatorname{osc}\{u, B_r\} \le Cr^{\alpha}(R_0^{-\alpha}\operatorname{osc}\{u, B_{R_0}\} + 1), \quad r \le R_0$$

with positive constants $C = C(\delta)$ and $\alpha = \alpha(\delta)$. In particular

$$|u(x) - u(x_0)| \le C|x - x_0|^{\alpha}(R_0^{-\alpha}\operatorname{osc}\{u, B_{R_0}\} + 1), \quad |x - x_0| \le R_0,$$

that sets Hölder property of the solutions in the point x_0.

Using denotation (2.2), consider two sets:

$$F = \{x \in Q_{3R} : u(x) \le (M_6 + m_6)/2\}, \tag{3.2}$$

$$G = \{x \in Q_{3R} : M_6 + m_6 - u(x) \le (M_6 + m_6)/2\}. \tag{3.3}$$

One of the following inequalities is always true:

$$|F| \ge \frac{1}{2}|Q_{3R}| \tag{3.4}$$

or

$$|G| \ge \frac{1}{2}|Q_{3R}|. \tag{3.5}$$

If we show that from the condition (3.4) for $u(x)$ follows

$$\sup_{B_R} u(x) \le M_6 - \delta\operatorname{osc}\{u, B_{6R}\} + R, \quad \delta > 0, \tag{3.6}$$

then this result applied to the function $M_6+m_6-u(x)$ guarantees under condition (3.5) the estimate

$$\sup_{B_R}(M_6+m_6-u(x)) \le M_6-\delta\operatorname{osc}\{u,B_{6R}\}+R$$

and in both cases we arrive to (3.1).

The following embedding fact will be used below.

$$\int_{B_r}|\varphi|\,dx \le C(n,\nu)r\int_{B_r}|\nabla\varphi|\,dx, \tag{3.7}$$

for $\varphi\in C^\infty\left(\bar{B}_r\right)$, $\varphi|_E=0$, $|E|\ge\nu|B_r|$, $\nu>0$.

Note that this embedding theorem holds also in the case of truncated balls $B_r\cap D^{(2)}$ with centers in $D^{(2)}$.

Proof of theorem 1.1. For the sake of simplicity assuming the fulfilment of condition (3.4), consider the function $v(x)$, introduced in (2.2). Our aim is obtaining the estimate

$$\sup_{B_R} v(x)\le c_0(n,p,q,M). \tag{3.8}$$

From this explicitly follows the scattering property (3.6) ($\delta=e^{-c_0}$), effecting Hölder property of the solution in the point x_0. To result (3.8) it needs to estimate the integrals in the right hand side of (2.23). Those estimations are based on the following inequalities

$$\int_{Q^{(2)}_{3R}}|\nabla v|\,dx\le C(n,p,q,M)R^{n-1}, \tag{3.9}$$

$$\int_{Q_{3R}}|\nabla w|\,dx\le C(n,p,q,M)R^{n-1} \tag{3.10}$$

that will we set now.

Choosing in (1.4) the test function

$$\varphi(x)=(M_6-u(x)+R)^{1-p}\eta^p(x),$$

where $\eta\in C_0^\infty(Q_{4R})$ is a radial-symmetric with respect to the center of the ball Q_{4R} cutoff function, satisfying the condition $0\le\eta\le1$, $\eta=1$ in Q_{3R} and $|\nabla\eta|\le CR^{-1}$, we obtain

$$\begin{aligned}&\int_{Q_{4R}}\omega_\varepsilon(x)|\nabla u|^{p(x)}(M_6-u+R)^{-p}\eta^p\,dx\\ &\le C(p)\int_{Q_{4R}}\omega_\varepsilon(x)|\nabla u|^{p(x)-1}(M_6-u+R)^{1-p}\eta^{p-1}|\nabla\eta|\,dx.\end{aligned}$$

Appling Young's inequality to the integrant in the right-hand side gives

$$\begin{aligned}&\int_{Q_{4R}}\omega_\varepsilon(x)|\nabla u|^{p(x)}(M_6-u+R)^{-p}\eta^p\,dx\\ &\le C(p)\int_{Q_{4R}}\omega_\varepsilon(x)(M_6-u+R)^{p(x)-p}|\nabla\eta|^{p(x)}\,dx.\end{aligned}$$

and

$$\int_{Q^{(2)}_{4R}}|\nabla u|^p(M_6-u+R)^{-p}\eta^p\,dx\le C(p)\int_{Q_{4R}}(M_6-u+R)^{p(x)-p}|\nabla\eta|^{p(x)}\,dx.$$

Repeating now the considerations of lemma 2.1, used in the setting of estimate (2.9) from relation (2.7), it can be obtained

$$\int_{Q^{(2)}_{4R}} |\nabla v|^p \eta^p \, dx \le C(p,q,M)\Big(R^{q-p} \int_{Q^{(1)}_{4R}} |\nabla \eta|^q \, dx + \int_{Q^{(2)}_{4R}} |\nabla \eta|^p dx\Big).$$

From this, considering the choice of the cutoff function, follows the estimate

$$\int_{Q^{(2)}_{3R}} |\nabla v|^p \, dx \le C(n,p,q,M) R^{n-p}, \tag{3.11}$$

that leads to (3.9). In particular since $p > q$, following Young's inequality

$$|\nabla \tilde{v}|^q < R^{p-q} |\nabla \tilde{v}|^p + R^{-q}$$

in the domain $Q^{(1)}_{3R}$ (3.11) and radial simmetricity of η we find that

$$\int_{Q^{(1)}_{4R}} |\nabla \tilde{v}|^q \eta^p \, dx \le R^{p-q} \int_{Q^{(1)}_{4R}} |\nabla \tilde{v}|^p \tilde{\eta}^p \, dx + R^{n-q} \le C(n,p,q,M) R^{n-q}. \tag{3.12}$$

To proof estimate (3.10) we use more complicated test function. Note that $u(x) > \tilde{u}(x)$ on the set $G_R \subset B^{(1)}_{4R}$ (see(2.11)). Let us chose in integral identity (1.4)

$$\varphi(x) = \begin{cases} \big((M_6 - u(x) + R)^{1-q} - (M_6 - \tilde{u}(x) + R)^{1-q}\big)\, \eta^p(x) & \text{in } G_R \\ 0 & \text{in } Q_{4R} \setminus G_R, \end{cases}$$

where η has the same sense as above. Then

$$\begin{aligned} &\int_{G_R} |\nabla u|^q (M_6 - u + R)^{-q} \eta^p \, dx \\ &\le \int_{G_R} |\nabla u|^{q-1} |\nabla \tilde{u}| (M_6 - \tilde{u} + R)^{-q} \eta^p \, dx \\ &\quad + \frac{p}{q-1} \int_{G_R} |\nabla u|^{q-1} (M_6 - u + R)^{-q} |\nabla \eta| \eta^{p-1} \, dx. \end{aligned}$$

Applying Young's inequality to the integrant in the right-hade side, and using the definition G_R, one gets

$$\begin{aligned} &\int_{G_R} |\nabla u|^q (M_6 - u + R)^{-q} \eta^p \, dx \\ &\le C(p,q)\Big(\int_{G_R} |\nabla \tilde{u}|^q (M_6 - \tilde{u} + R)^{-q} \eta^p \, dx + \int_{G_R} |\nabla \eta|^q \, dx\Big). \end{aligned}$$

From this, by relation (2.8) and the choice the cutoff function we obtain

$$\int_{G_R} |\nabla v|^q \eta^p \, dx \le C(n,p,q)\Big(\int_{G_R} |\nabla \tilde{v}|^q \eta^p \, dx + R^{n-q}\Big).$$

Thus,

$$\begin{aligned} \int_{Q^{(1)}_{4R}} |\nabla w|^q \eta^p \, dx &= \int_{Q^{(1)}_{4R} \setminus G_R} |\nabla \tilde{v}|^q \eta^p \, dx + \int_{G_R} |\nabla v|^q \eta^p \, dx \\ &\le C(n,p,q)\Big(\int_{B^{(1)}_{4R}} |\nabla \tilde{v}|^q \eta^p \, dx + R^{n-q}\Big). \end{aligned}$$

Considering (3.12),

$$\int_{Q_{3R}^{(1)}} |\nabla w|^q \, dx \le C(n, p, q, M) R^{n-q},$$

which together with (3.9) gives (3.10).

Now let us estimate the integrals in the right-hand side of (2.23). We use the assumption (3.4) for the first one and note that $|F \cap Q_{3R}^{(2)}| \ge \text{const.}|Q_{3R}|$ (see (3.2)). Since $v(x) \le \ln 2$ on F and $w(x) = v(x)$ in $Q_{3R}^{(2)}$, then for $E = \{x \in \ Q_{3R} : w(x) \le \ln 2\}$ we have the estimate $|E| \ge \text{const.}|Q_{3R}|$. Therefore by inequality (3.7) in the ball Q_{3R},

$$\int_{Q_{3R} \setminus E} |w - \ln 2| \, dx \le C(n) R \int_{Q_{3R}} |\nabla w| \, dx$$

and according to (3.10),

$$\int_{Q_{3R}} w \, dx \le C(n, p, q, M) R^n.$$

The second integral in (2.23) may be estimated similarly. Really, $|E \cap Q_{3R}^{(2)}| \ge \text{const.}|Q_{3R}|$ and again by the inequality (3.7) in the truncated ball $Q_{3R}^{(2)}$,

$$\int_{Q_{3R}^{(2)} \setminus E} |v - \ln 2| \, dx \le C(n) R \int_{Q_{3R}^{(2)}} |\nabla v| \, dx.$$

The inequality (3.9) leads us to the estimate

$$\int_{Q_{3R}^{(2)}} v \, dx \le C(n, p, q, M) R^n,$$

that proves (3.1). □

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Sarvan T. Huseynov
Baku State University, Baku, Azerbaijan
E-mail address: sarvanhuseynov@rambler.ru