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STURM-LIOUVILLE PROBLEMS WITH RETARDED ARGUMENT AND A FINITE NUMBER OF TRANSMISSION CONDITIONS

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Abstract. The main goal of the present paper is to study the asymptotic behaviour of eigenvalues and eigenfunctions of a discontinuous boundary-value problem with retarded argument with a finite number of transmission conditions.

1. Introduction

Spectral properties of boundary-value problems with retarded argument and with discontinuities inside the interval are studied by many authors [1, 4, 5, 7, 10, 17, 18, 19, 23]. Following these studies, in this work, we consider the boundary-value problem for the differential equation

$$y''(x) + q(x)y(x - \Delta(x)) + \mu^2 y(x) = 0 \tag{1.1}$$

on $[0, r_1) \cup (r_1, r_2) \cup \dots \cup (r_m, \pi]$, with boundary conditions

$$d_1 y(0) + d_2 y'(0) = 0, \tag{1.2}$$

$$y'(\pi) + \mu^2 y(\pi) = 0, \tag{1.3}$$

and transmission conditions

$$y(r_i - 0) - \delta_i y(r_i + 0) = 0, \quad i = \overline{1, m}, \tag{1.4}$$

$$y'(r_i - 0) - \delta_i y'(r_i + 0) = 0, \quad i = \overline{1, m} \tag{1.5}$$

where the real-valued function $q(x)$ is continuous in $[0, r_1) \cup (r_1, r_2) \cup \dots \cup (r_m, \pi]$ and has finite limits

$$q(r_i \pm 0) = \lim_{x \to r_i \pm 0} q(x),$$

the real valued function $\Delta(x) \geq 0$ continuous in $[0, r_1) \cup (r_1, r_2) \cup \dots \cup (r_m, \pi]$ and has finite limits

$$\Delta(r_i \pm 0) = \lim_{x \to r_i \pm 0} \Delta(x),$$

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$x-\Delta(x)\geq 0$ if $x\in[0,r_1)$; $x-\Delta(x)\geq r_1$, if $x\in(r_1,r_2)$;..., $x-\Delta(x)\geq r_{m-1}$, if $x\in(r_m,\pi)$; μ is a real positive eigenparameter; $r_i,\delta_i\neq 0$ are arbitrary real numbers such that $0<r_1<r_2<\dots<r_m<\pi$ and $d_1d_2\neq 0$.

The goal of this article is to obtain asymptotic formulas for eigenvalues of eigenfunctions for problem (1.1)–(1.5). To this aim, first, the principal term of asymptotic distribution of eigenvalues and eigenfunctions of (1.1)–(1.5) was obtained up to $O(1/N)$, but, afterwards under some additional conditions we improve these formulas up to $O(1/N^2)$. Thus, when the number of points of discontinuity is more than one, we see how the asymptotic behaviour of eigenvalues and eigenfunctions of a boundary-value problem with retarded argument which contains a spectral parameter in the boundary conditions change. We point out that our results are extension and/or generalization to those in [3, 9, 11, 12, 13, 14, 15, 19, 20, 21]. For example, if the retardation function $\Delta\equiv 0$ in (1.1) and $\delta_i=1$ $(i=\overline{1,m})$; or $\delta_1\neq 1$ and $\delta_i=1$ $(i=\overline{2,m})$; or $\delta_{1,2}\neq 1$ and $\delta_i=1$ $(i=\overline{3,m})$; or $\delta_i\neq 1$ $(i=\overline{1,m})$ results obtained in this paper coincide with the results of [9, 11, 14, 20], respectively.

Differential equations with deviating argument, in particular differential equations with retarded argument, describe processes with aftereffect; they find many applications, particularly in the theory of automatic control, in the theory of self-oscillatory systems, in the study of problems connected with combustion in rocket engines (see [16] and the references therein).

Boundary value problems containing a spectral parameter in the boundary conditions have many interesting applications, especially in mathematical physics (e.g. [22, pp. 146-152]). It must be also noted that recently boundary-value problems with transmission conditions attracted much attention in connection with the inverse acoustic scattering problem (see, e.g., [2, 6, 8] and the references therein).

Let $w_1(x,\lambda)$ be a solution of (1.1) on $[0,h_1]$, satisfying the initial conditions

$$w_1(0,\mu)=d_2 \quad\text{and}\quad w_1'(0,\mu)=-d_1. \tag{1.6}$$

The conditions (1.6) define a unique solution of (1.1) on $[0,h_1]$ [16, p. 12].

After defining the above solution, then we shall define the solution $w_{i+1}(x,\mu)$ of (1.1) on $[r_i,r_{i+1}]$ by means of the solution $w_i(x,\mu)$ using the initial conditions

$$w_{i+1}(r_i,\mu)=\delta_i^{-1}w_i(r_i,\mu)\text{and}\quad w_{i+1}'(r_i,\mu)=\delta_i^{-1}w_i'(r_i,\mu),\ i=\overline{2,m-1} \tag{1.7}$$

The conditions (1.7) define a unique solution of (1.1) on $[r_i,r_{i+1}]$.

Continuing in this manner we may define the solution $w_{m+1}(x,\mu)$ of (1.1) on $[r_m,\pi]$ by means of the solution $w_m(x,\mu)$ using the initial conditions

$$w_{m+1}(r_m,\mu)=\delta_m^{-1}w_m(r_m,\mu)\text{and}\quad w_{m+1}'(r_m,\mu)=\delta_m^{-1}w_m'(r_m,\mu). \tag{1.8}$$

The conditions (1.8) define a unique solution of (1.1) on $[r_m,\pi]$.

Consequently, the function $w(x,\mu)$ is defined on $[0,r_1)\cup(r_1,r_2)\cup\dots\cup(r_m,\pi]$ by the equality

$$w(x,\lambda)=\begin{cases} w_1(x,\mu), & x\in[0,r_1),\\ w_i(x,\mu), & x\in(r_i,r_{i+1}),\ i=\overline{2,m-1},\\ w_{m+1}(x,\mu), & x\in(r_m,\pi]\end{cases}$$

is a solution of (1.1) on $[0,r_1)\cup(r_1,r_2)\cup\dots\cup(r_m,\pi]$; which satisfies one of the boundary conditions and transmission conditions.

Lemma 1.1. *Let $w(x,\mu)$ be a solution of* (1.1). *Then the following integral equations hold:*

$$\begin{aligned} w_1(x,\mu) &= d_2\cos\mu x - \frac{d_1}{\mu}\sin\mu x \\ &\quad - \frac{1}{\mu}\int_0^x q(\tau)\sin\mu(x-\tau)w_1(\tau-\Delta(\tau),\mu)\,d\tau, \end{aligned} \tag{1.9}$$

$$\begin{aligned} w_{i+1}(x,\mu) &= \frac{1}{\delta_i}w_i(r_i,\mu)\cos\mu(x-r_i) + \frac{w_i'(r_i,\lambda)}{\mu\delta_i}\sin\mu(x-r_i) \\ &\quad - \frac{1}{\mu}\int_{r_i}^x q(\tau)\sin s(x-\tau)w_{i+1}(\tau-\Delta(\tau),\mu)\,d\tau, \end{aligned} \tag{1.10}$$

Proof. To prove this lemma, it suffices to substitute $-\mu^2 w_1(\tau,\mu) - w_1''(\tau,\mu)$ and $-\mu^2 w_{i+1}(\tau,\mu) - w_{i+1}''(\tau,\mu)$ by $-q(\tau)w_1(\tau-\Delta(\tau),\mu)$ and $-q(\tau)w_{i+1}(\tau-\Delta(\tau),\mu)$ in the integrals in (1.9), (1.10) respectively, and then integrate by parts twice. □

2. An existence theorem

In this chapter, we show that the characteristic function of the problem (1.1)–(1.5) has an infinite set of roots.

Theorem 2.1. *Problem* (1.1)-(1.5) *can have only simple eigenvalues.*

Proof. Let $\widetilde{\mu}$ be an eigenvalue of (1.1)-(1.5) and

$$\widetilde{y}(x,\widetilde{\mu}) = \begin{cases} \widetilde{y}_1(x,\widetilde{\mu}), & x\in[0,r_1), \\ \dots \\ \widetilde{y}_{m+1}(x,\widetilde{\mu}), & x\in(r_m,\pi] \end{cases}$$

be a corresponding eigenfunction. Then, from (1.2) and (1.6), it follows that the determinant

$$W[\widetilde{y}_1(0,\widetilde{\mu}), w_1(0,\widetilde{\mu})] = \begin{vmatrix} \widetilde{y}_1(0,\widetilde{\mu}) & d_2 \\ \widetilde{y}_1'(0,\widetilde{\mu}) & -d_1 \end{vmatrix} = 0,$$

and the functions $\widetilde{y}_1(x,\widetilde{\mu})$ and $w_1(x,\widetilde{\mu})$ are linearly dependent on $[0,r_1]$. We can also prove that the functions $\widetilde{y}_{i+1}(x,\widetilde{\mu})$ and $w_{i+1}(x,\widetilde{\mu})$ are linearly dependent on $[r_i,r_{i+1}]$, $i=\overline{2,m-1}$ and $\widetilde{y}_{m+1}(x,\widetilde{\mu})$ and $w_{m+1}(x,\widetilde{\mu})$ are linearly dependent on $[r_m,\pi]$. Hence

$$\widetilde{y}_i(x,\widetilde{\mu}) = K_i w_i(x,\widetilde{\mu}) \quad (i=\overline{1,m+1}) \tag{2.1}$$

for some $K_i\neq 0$. We must show that $K_i=K_{i+1}$. From the equalities (1.4) and (2.1), we have

$$\begin{aligned} \widetilde{y}(r_i-0,\widetilde{\mu}) - \delta_i\widetilde{y}(r_i+0,\widetilde{\mu}) &= \widetilde{y}_i(r_i,\widetilde{\mu}) - \delta_i\widetilde{y_{i+1}}(r_i,\widetilde{\mu}) \\ &= K_i w_i(r_i,\widetilde{\mu}) - \delta_i K_{i+1}w_{i+1}(r_i,\widetilde{\mu}) \\ &= K_i\delta_i w_{i+1}(h_i,\widetilde{\mu}) - K_{i+1}\delta_i w_{i+1}(h_i,\widetilde{\mu}) \\ &= \delta_i(K_i-K_{i+1})w_{i+1}(h_i,\widetilde{\mu}) = 0. \end{aligned}$$

Since $\delta_i(K_i-K_{i+1})\neq 0$ it follows that

$$w_{i+1}(r_i,\widetilde{\mu}) = 0. \tag{2.2}$$

By the same procedure from equality (1.5) we can derive that

$$w_{i+1}'(r_i,\widetilde{\mu}) = 0. \tag{2.3}$$

From the fact that $w_i(x,\widetilde{\mu})$ is a solution of the differential (1.1) on $[r_i, r_{i+1}]$ and satisfies the initial conditions (2.2) and (2.3) it follows that $w_{i+1}(x,\widetilde{\mu}) = 0$ identically on $[r_i, \pi]$.

By using this method, we may also find

$$w_{m+1}(r_i,\widetilde{\mu}) = w'_{m+1}(r_i,\widetilde{\mu}) = 0.$$

From the latter discussions of $w_{m+1}(x,\widetilde{\mu})$ it follows that $w_m(x,\widetilde{\mu}) = 0$, $w_i(x,\widetilde{\mu}) = 0$, $w_1(x,\widetilde{\mu}) = 0$ identically on (r_{m-1}, r_m), (r_{i-1}, r_i) and $[0, r_1)$. But this contradicts (1.6), thus completing the proof. □

The function $w(x,\mu)$ is defined in introduction is a nontrivial solution of (1.1) satisfying conditions (1.2) and (1.4)-(1.5). Putting $w(x,\mu)$ into (1.3), we get the characteristic equation

$$H(\mu) \equiv w'(\pi,\mu) + \mu^2 w(\pi,\mu) = 0. \tag{2.4}$$

By Theorem 2.1 the set of eigenvalues of boundary-value problem (1.1)-(1.5) coincides with the set of real roots of (2.7). Let

$$q_1 = \int_0^{r_1} |q(\tau)|\,d\tau, \quad q_i = \int_{r_{i-1}}^{ri} |q(\tau)|\,d\tau, \quad q_{m+1} = \int_{r_m}^{\pi} |q(\tau)|\,d\tau, i = \overline{2,m}$$

Lemma 2.2. *(1) Let $\mu \ge 2q_1$. Then for the solution $w_1(x,\mu)$ of* (2.1)*, the following inequality holds:*

$$|w_1(x,\mu)| \le \frac{1}{q_1}\sqrt{4q_1^2 d_2^2 + d_1^2}, \quad x \in [0, r_1]. \tag{2.5}$$

(2) Let $\mu \ge \max\{2q_1, 2q_2, \dots, 2q_{m+1}\}$. Then for the solution $w_{i+1}(x,\mu)$ $(i = \overline{1,m})$ of (2.2)*, the following inequality holds:*

$$|w_{i+1}(x,\mu)| \le \frac{4^i}{q_1 \prod_{j=1}^{i} |\delta_j|}\sqrt{4q_1^2 d_2^2 + d_1^2}, \quad x \in [r_1, r_2]. \tag{2.6}$$

The proof of the above lemma is similar to that of [19, Lemma 2].

Theorem 2.3. *Problem* (1.1)-(1.5) *has an infinite set of positive eigenvalues.*

Proof. We readily see that

$$\begin{aligned} \frac{\partial}{\partial x} w_{i+1}(x,\mu) &= -\frac{\mu}{\delta_i} w_i(r_i,\mu)\sin\mu(x - r_i) + \frac{\frac{\partial}{\partial x} w_{i+1}(r_i,\mu)}{\delta_i}\cos\mu(x - r_i) \\ &\quad - \int_{r_i}^{x} q(\tau)\cos\mu(x-\tau) w_{i+1}(\tau - \Delta(\tau),\mu)\,d\tau. \end{aligned} \tag{2.7}$$

Let μ be sufficiently big. With the helps of (1.8), (1.9)), (2.6), (2.7), (2.4) and (2.5), Equation (2.7) can be reduced to the form

$$\mu\cos\mu\pi + O(1) = 0. \tag{2.8}$$

Obviously, for big μ, (2.8) has an infinite set of roots. Thus, the proof of theorem is complete. □

3. Asymptotic formulas for eigenvalues and eigenfunctions

Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that μ is sufficiently big. From (1.9) and (2.5), we obtain

$$w_1(x,\mu) = O(1) \quad \text{on} \quad [0, r_1]. \tag{3.1}$$

Equations (1.10) and (2.6), lead to

$$w_{i+1}(x,\mu) = O(1), \quad (i = \overline{1, m-1}) \quad \text{on } [r_i, r_{i+1}]. \tag{3.2}$$

$$w_{m+1}(x,\mu) = O(1) \quad \text{on } [r_m, \pi]. \tag{3.3}$$

The existence and continuity of the derivatives $\frac{\partial}{\partial\mu} w_1(x,\mu)$ for $0 \le x \le r_1, |\mu| < \infty$, $\frac{\partial}{\partial\mu} w_{i+1}(x,\mu)$ for $r_i \le x \le r_{i+1}$ $(i = \overline{1, m-1}), |\mu| < \infty$ and $\frac{\partial}{\partial\mu} w_{m+1}(x,\mu)$ for $r_m \le x \le \pi$, $|\mu| < \infty$ follows from [16, Theorem 1.4.1].

Lemma 3.1. *The following statements hold:*

$$\frac{\partial}{\partial\mu} w_1(x,\mu) = O(1), \quad x \in [0, r_1], \tag{3.4}$$

$$\frac{\partial}{\partial\mu} w_{i+1}(x,\mu) = O(1), \quad (i = \overline{1, m-1})\ x \in [r_i, r_{i+1}], \tag{3.5}$$

$$\frac{\partial}{\partial\mu} w_{m+1}(x,\mu) = O(1), \quad x \in [r_m, \pi]. \tag{3.6}$$

Proof. By differentiating (1.9) with respect to μ, we get, by (3.1)-(3.3)

$$\begin{aligned}\frac{\partial}{\partial\mu} w_{m+1}(x,\mu) = &-\frac{1}{\mu}\int_{r_m}^{x} q(\tau)\sin\mu(x-\tau)\frac{\partial}{\partial\mu} w_{m+1}(\tau-\Delta(\tau),\mu) \\ &+ R(x,\mu), \quad (|R(x,\mu)| \le \qquad R_0).\end{aligned} \tag{3.7}$$

Let $D_\mu = \max_{[r_m,\pi]} |\frac{\partial}{\partial\mu} w_{m+1}(x,\mu)|$. Then the existence of D_μ follows from continuity of derivation for $x \in [r_m, \pi]$. From (3.7)

$$D_\mu \le \frac{1}{\mu} q_{m+1} D_\mu + R_0.$$

Now let $\mu \ge 2q_{m+1}$. Then $D_\mu \le 2R_0$ and the validity of the asymptotic formula (3.6) follows. Formulas (3.4) and (3.5) may be proved analogically. □

Theorem 3.2. *Let N be a natural number. For each sufficiently big N there is exactly one eigenvalue of the problem* (1.1)-(1.5) *near N^2.*

Proof. We consider the expression which is denoted by $O(1)$ in (2.8). If formulas (3.1)-(3.6) are taken into consideration, it can be shown by differentiation with respect to μ that for big μ this expression has bounded derivative. We shall show that, for big N, only one root (2.8) lies near to each N. We consider the function $\phi(\mu) = \mu\cos\mu\pi + O(1)$. Its derivative, which has the form $\frac{\partial}{\partial\mu}\phi(\mu) = \cos\mu\pi - \mu\pi\sin\mu\pi + O(1)$, does not vanish for μ close to$N$for sufficiently big N. Thus our assertion follows by Rolle's Theorem. □

Let N be sufficiently big. In what follows we shall denote by μ_n^2 the eigenvalue of the problem (1.1)-(1.5) situated near N^2. We set $\mu_N = N + \frac{1}{2} + \delta_N$. Then from (2.8) it follows that $\delta_N = O(\frac{1}{N})$. Consequently

$$\mu_N = N + \frac{1}{2} + O\Big(\frac{1}{N}\Big), \tag{3.8}$$

Formula (3.8) make it possible to obtain asymptotic expressions for eigenfunction of the problem (1.1)-(1.5). From (1.9), (3.1), we get

$$w_1(x,\mu) = d_2 \cos \mu x + O\big(\frac{1}{\mu}\big). \tag{3.9}$$

From expressions of (1.10), (3.5), (3.9), we easily see that

$$w_{i+1}(x,\mu) = \frac{d_2}{\prod_{j=1}^{i} \delta_j} \cos \mu x + O\big(\frac{1}{\mu}\big), \quad (i=\overline{1,m}). \tag{3.10}$$

By substituting (3.8) in (3.9) and (3.10), we find that

$$\begin{gathered}
U_{1N} = w_1(x,\mu_N) = d_2 \cos\big((N+\frac{1}{2})x\big) + O\big(\frac{1}{N}\big), \\
U_{(i+1)N} = w_{i+1}(x,\mu_N) = \frac{d_2}{\prod_{j=1}^{i} \delta_j} \cos\big((N+\frac{1}{2})x\big) + O\big(\frac{1}{N}\big),\ (i=\overline{1,m}).
\end{gathered}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

(a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,r_1)\cup(r_1,r_2)\cup\dots\cup(r_m,\pi]$ and have finite limits $q'(r_i\pm 0)=\lim_{x\to r_i\pm 0} q'(x)$, and $\Delta''(r_i\pm 0)=\lim_{x\to r_i\pm 0}\Delta''(x)$ $(i=\overline{1,m})$.

(b) $\Delta'(x)\le 1$ in $[0,r_1)\cup(r_1,r_2)\cup\dots\cup(r_m,\pi]$, $\Delta(0)=0$, $\lim_{x\to h_1+0}\Delta(x)=0$ and $\lim_{x\to r_i+0}\Delta(x)=0$ $(i=\overline{1,m})$.

It is easy to see that, using (b)

$$x-\Delta(x)\ge 0,\ x\in[0,r_1), \tag{3.11}$$

$$x-\Delta(x)\ge r_i,\ x\in(r_i,r_{i+1}) \quad (i=\overline{1,m-1}), \tag{3.12}$$

$$x-\Delta(x)\ge r_m, \quad x\in(r_m,\pi] \tag{3.13}$$

are obtained. By (3.9)-(3.13), we have

$$w_1(\tau-\Delta(\tau),\mu) = d_2\cos\mu(\tau-\Delta(\tau)) + O\big(\frac{1}{\mu}\big), \tag{3.14}$$

$$w_{i+1}(\tau-\Delta(\tau),\mu) = \frac{d_2}{\prod_{j=1}^{i}\delta_j}\cos\mu(\tau-\Delta(\tau)) + O\big(\frac{1}{\mu}\big) \tag{3.15}$$

on $[0,r_1)$, (r_i,r_{i+1}) $(i=\overline{1,m-1})$ and $(r_m,\pi]$ respectively.

Under conditions (a) and (b) the following two formulas

$$\begin{gathered}
\int_0^x q(\tau)\cos\mu(2\tau-\Delta(\tau))\,d\tau = O(1\mu), \\
\int_0^x q(\tau)\sin\mu(2\tau-\Delta(\tau))\,d\tau = O(1/\mu)
\end{gathered} \tag{3.16}$$

can be proved by the same technique in [16, Lemma 3.3.3].

Using (3.14), (3.15) and (3.16), after long operations we have

$$\begin{aligned}
&-\frac{d_1+d_2}{\prod_{j=1}^{m}\delta_j}\sin\mu\pi + \frac{\mu d_2}{\prod_{j=1}^{m}\delta_j}\cos\mu\pi - \frac{d_2\sin\mu\pi}{2\prod_{j=1}^{m}\delta_j}\int_0^\pi q(\tau)\cos\mu\Delta(\tau)\,d\tau \\
&+\frac{d_2\cos\mu\pi}{2\prod_{j=1}^{m}\delta_j}\int_0^\pi q(\tau)\sin\mu\Delta(\tau)\,d\tau + O\big(\frac{1}{\mu}\big) = 0.
\end{aligned}$$

Again, if we take $\mu_N = N + \frac{1}{2} + \delta_N$, for sufficiently big N, we obtain

$$\delta_N = \frac{1}{(N+\frac{1}{2})\pi}\Big(\frac{d_1}{d_2} - 1 - \frac{1}{2}\int_0^\pi q(\tau)\cos\big((N+\frac{1}{2})\Delta(\tau)\big)\,d\tau\Big) + O(1/N^2)$$

and finally

$$\mu_N = N + \frac{1}{2} + \frac{1}{(N+\frac{1}{2})\pi}(\frac{d_1}{d_2} - 1 - \frac{1}{2}\int_0^\pi q(\tau)\cos\big((N+\frac{1}{2})\Delta(\tau)\big)\,d\tau) + O(1/N^2). \tag{3.17}$$

Thus, we have proven the following theorem.

Theorem 3.3. *If conditions (a) and (b) are satisfied then, the eigenvalues μ_N of the problem* (1.1)-(1.5) *have the* (3.17) *asymptotic formula for $N \to \infty$.*

Now, we may obtain sharper asymptotic formulas for the eigenfunctions. From (1.9)), (3.14), (3.16) and replacing μ by μ_N we have

$$\begin{aligned} u_{1N}(x) = d_2\Big\{&\frac{\sin((N+\frac{1}{2})x)}{N\pi}\Big[\Big(\frac{d_1}{d_2} + \frac{1}{2}\int_0^x q(\tau)\cos((N+\frac{1}{2})\Delta(\tau))\,d\tau\Big)\pi \\ &+ \Big(\frac{d_1}{d_2} - 1 - \frac{1}{2}\int_0^\pi q(\tau)\cos\Big((N+\frac{1}{2})\Delta(\tau)\Big)\,d\tau\Big)x\Big] \\ &+ \cos\big((N+\frac{1}{2})x\big)\Big[1 + \frac{1}{2N}\int_0^x q(\tau)\sin\big((N+\frac{1}{2})\Delta(\tau)\big)\,d\tau\Big]\Big\} + O(1/N^2). \end{aligned}$$

From (1.10), (3.15) and (3.16), and replacing μ by μ_N we have

$$\begin{aligned} u_{(i+1)N}(x) = \frac{d_2}{\prod_{j=1}^i \delta_j}\Big\{&\cos\big((N+\frac{1}{2})x\big)\Big[1 + \frac{1}{2N}\int_0^x q(\tau)\sin\big((N+\frac{1}{2})\Delta(\tau)\big)\,d\tau\Big] \\ &+ \frac{\sin\big((N+\frac{1}{2})x\big)}{N\pi}\Big[\Big(\frac{d_1}{d_2} - 1 - \frac{1}{2}\int_0^\pi q(\tau)\cos\big((N+\frac{1}{2})\Delta(\tau)\big)\,d\tau\Big)x \\ &- \Big(\frac{d_1}{d_2} + \frac{1}{2}\int_0^x q(\tau)\cos\big((N+\frac{1}{2})\Delta(\tau)\big)\,d\tau\Big)\pi\Big]\Big\} + O(1/N^2). \end{aligned}$$

References

[1] F. A. Akgün, A. Bayramov, M. Bayramoglu; *Discontinuous boundary value problems with retarded argument and eigenparameter dependent boundary conditions*, Mediterr. J. Math., 10 (1) (2013), 277-288.

[2] T. Aktosun, D. Gintides, V. G. Papanicolaou; *The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation*, Inverse Probl., 27 (115004) (2011), 1-17.

[3] K. Aydemir, O. S. Mukhtarov; *Asymptotic distribution of eigenvalues and eigenfunctions for a multi-point discontinuous Sturm-Liouville problem*, Electron. J. Differential Equations (EJDE), 2016 (131) (2016), 1-14.

[4] M. Bayramoglu, A. Bayramov, E. Şen; *A regularized trace formula for a discontinuous Sturm-Liouville operator with delayed argument*, Electron. J. Differential Equations (EJDE), 2017 (104) (2017). 1-12.

[5] A. Bayramov, S. Çalışkan, S. Uslu; *Computation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument*, Appl. Math. Comput., 191 (2007), 592-600.

[6] S. A. Buterin, C.-F. Yang; *On an inverse transmission problem from complex eigenvalues*, Results Math., 71 (3-4) (2017), 859-866.

[7] F. A. Cetinkaya, K. R. Mamedov; *A boundary value problem with retarded argument and discontinuous coefficient in the differential equation*, Azerb. J. Math., 7 (1) (2017), 130-140.

[8] D. Colton, Y-J. Leung, S. Meng; *Distribution of complex transmission eigenvalues for spherically symmetric stratified media*, Inverse Probl., 31 (035006) (2015), 1-19.
[9] C. T. Fulton; *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edinburgh, A 77 (1977), 293-308.
[10] F. Hira; *A trace formula for the Sturm-Liouville type equation with retarded argument*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 66 (1) (2017), 124-132.
[11] M. Kadakal, O. S. Mukhtarov; *Sturm-Liouville problems with discontinuities at two points*, Computers and Mathematics with Applications 54 (2007), 1367-1379.
[12] B. M. Levitan; Expansion in characteristic functions of differential equations of the second order, GITTL, Moscow, 1950. (Russian)
[13] B. M. Levitan, I. S. Sargsyan; Sturm-Liouville and Dirac operators, Springer-Verlag, New-York, 1991.
[14] A. S. Makin; *Asymptotics of the spectrum of the Sturm-Liouville operator with regular boundary conditions*, Differential Equations, 44 (5) (2008), 645–658.
[15] O. S. Mukhtarov, M. Kadakal, F. S. Muhtarov; *On discontinuous Sturm-Liouville problems with transmission conditions*, J. Math. Kyoto Univ., 44 (4) (2004), 779-798.
[16] S. B. Norkin; Differential equations of the second order with retarded argument, Translations of Mathematical Monographs, AMS, Providence, RI, 1972.
[17] E. Şen, M. Acikgoz, S. Araci; *Spectral problem for Sturm-Liouville operator with retarded argument which contains a spectral parameter in the boundary condition*, Ukrainian Mathematical Journal, 68 (8) (2017), 1263-1277.
[18] E. Şen, A. Bayramov; *Spectral analysis of boundary value problems with retarded argument*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 66 (2) (2017), 175-194.
[19] E. Şen, A. Bayramov; *Calculation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument which contains a spectral parameter in the boundary condition*, Mathematical and Computer Modelling, 54 (11-12) (2011), 3090-3097.
[20] E. Şen, O. S. Mukhtarov; *Spectral properties of discontinuous Sturm–Liouville problems with a finite number of transmission conditions*, Mediterr. J. Math., 13 (1) (2016), 153-170.
[21] E. C. Titchmarsh; Eigenfunctions expansion associated with second order differential equation 1, 2nd ed., Oxford University Press, London, 1962.
[22] A. N. Tikhonov, A. A. Samarskii; Equations of Mathematical Physics, Pergamon press, London, 1963.
[23] C.-F. Yang; *Trace and inverse problem of a discontinuous Sturm-Liouville operator with retarded argument*, J. Math. Anal. Appl., 395 (2012), 30-41.

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