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EXISTENCE OF SOLUTIONS TO NONLINEAR PROBLEMS WITH THREE-POINT BOUNDARY CONDITIONS

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Abstract. Using Leray-Schauder degree theory and the method of upper and lower solutions, we obtain a solution for nonlinear boundary-value problem

$$\begin{gathered}\big(\varphi(u')\big)' = f(t,u,u')\\ l(u,u') = 0,\end{gathered}$$

where $l(u,u') = 0$ denotes the three-point boundary conditions on $[0,T]$, and φ is a homeomorphism such that $\varphi(0) = 0$.

1. Introduction

The purpose of this article is to obtain a solution for the nonlinear problem

$$\begin{gathered}\big(\varphi(u')\big)' = f(t,u,u')\\ l(u,u') = 0,\end{gathered} \tag{1.1}$$

where $l(u,u') = 0$ denotes the boundary conditions $u(T) = u'(0) = u'(T)$ or $u(0) = u(T) = u'(0)$ on the interval $[0,T]$, φ is a singular or classic homeomorphism such that $\varphi(0) = 0$, and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Solvability of two-point boundary value problems can be investigated by various methods: fixed point theorems, topological degree arguments, variational methods, lower and upper functions, etc., see for example, [1, 7, 8, 9, 12, 13] and the reference therein. In particular, the author in [13] proved the existence of solutions for the Dirichlet and mixed problems, assuming f and $\varphi$are continuous and that φ is strictly increasing and satisfies $\varphi(\mathbb{R}) = \mathbb{R}$ and $\varphi^{-1} \in C^1(\mathbb{R})$.

Bereanu and Mawhin [2] proved the existence of solutions for the periodic boundary-value problem

$$\begin{gathered}\big(\varphi(u')\big)' = f(t,u,u')\\ u(0) = u(T), \quad u'(0) = u'(T),\end{gathered}$$

assuming that $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\varphi : \mathbb{R} \to (-a,a)$ $(0 < a \le \infty)$ is an increasing homeomorphism such that $\varphi(0) = 0$. They obtained solutions by using the method of upper and lower solutions and the Leray-Schauder degree. The interest in this class of nonlinear operators $u \mapsto (\varphi(u'))'$ is

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mainly due to the fact that they include the p-Laplacian operator

$$u \mapsto |u'|^{p-2}u',$$

where $p > 1$.

Using the barrier strip argument and topological transversality theorem the authors in [10] obtained the existence of solutions for nonlinear boundary-value problems

$$\begin{gathered} \big(\varphi(u')\big)' = f(t, u, u') \\ u(0) = A, \quad u'(1) = B, \end{gathered}$$

where $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism.

Inspired by these results, the main aim of this paper is to study the existence of solutions for (1.1) using topological methods based upon Leray-Schauder degree. The main contribution of this paper is the extension of some results above cited to a more general type of boundary conditions. Such problems do not seem to have been studied in the literature.

This article is organized as follows. In Section 2, we introduce some notations and preliminaries, which will be crucial in the proofs of our results. Section 3 is devoted to the study of existence of solutions for boundary-value problems

$$\begin{gathered} \big(\varphi(u')\big)' = f(t, u, u') \\ u(T) = u'(0) = u'(T), \end{gathered}$$

where $\varphi : (-a, a) \to \mathbb{R}$ (we call it singular). We call *solution* of this problem any function $u : [0, T] \to \mathbb{R}$ of class C^1 such that $\max_{[0,T]} |u'(t)| < a$, satisfying the boundary conditions and the function $\varphi(u')$ is continuously differentiable and $(\varphi(u'(t)))' = f(t, u(t), u'(t))$ for all $t \in [0, T]$. Combining the method of upper and lower solutions and the fixed point theorem of Schauder, we prove in Section 4 an existence result (Theorem 4.3) for boundary-value problems of the form

$$\begin{gathered} \big(\varphi(u')\big)' = f(t, u, u') \\ u(0) = u(T) = u'(0), \end{gathered}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism. A *solution* of this problem is any function $u : [0, T] \to \mathbb{R}$ of class C^1 satisfying the boundary conditions and satisfying that $\varphi(u')$ is continuously differentiable and $(\varphi(u'(t)))' = f(t, u(t), u'(t))$ for all $t \in [0, T]$.

2. Preliminaries

We first introduce some notation. Let $C = C([0, T], \mathbb{R})$ denote the Banach space of continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with the uniform norm $\|\cdot\|_\infty$, and $C^1 = C^1([0, T], \mathbb{R})$ the Banach space of continuously differentiable functions from $[0, T]$ to $\mathbb{R}$ equipped with the usual norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$.

We introduce the following operators: the *Nemytskii operator* $N_f : C^1 \to C$,

$$N_f(u)(t) = f(t, u(t), u'(t)),$$

the *integration operator* $H : C \to C^1$,

$$H(u)(t) = \int_0^t u(s)ds,$$

also the following continuous linear mappings:

$$
\begin{gathered}
K : C \to C^1, \quad K(u)(t) = -\int_t^T u(s)ds, \\
Q : C \to C, \quad Q(u)(t) = \frac{1}{T}\int_0^T u(s)ds, \\
S : C \to C, \quad S(u)(t) = u(T), \\
P : C \to C, \quad P(u)(t) = u(0).
\end{gathered}
$$

The following technical result proved by Bereanu and Mawhin is needed for the construction of the equivalent fixed point problem (see [3]).

Lemma 2.1. *For each $h \in C$, there exists a unique $Q_\varphi = Q_\varphi(h) \in \operatorname{Im}(h)$ (where $\operatorname{Im}(h)$ denotes the range of h) such that*

$$\int_0^T \varphi^{-1}(h(t) - Q_\varphi(h))dt = 0.$$

Moreover, the function $Q_\varphi : C \to \mathbb{R}$ is continuous and sends bounded sets into bounded sets.

3. Boundary value problems with singular φ-Laplacian

In this section we are interested in boundary-value problems of the type

$$
\begin{gathered}
\big(\varphi(u')\big)' = f(t,u,u') \\
u(T) = u'(0) = u'(T),
\end{gathered}
\tag{3.1}
$$

where $\varphi : (-a,a) \to \mathbb{R}$ is a homeomorphism such that $\varphi(0) = 0$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We remember that an *solution* of this problem is any function $u : [0,T] \to \mathbb{R}$ of class C^1 such that $\max_{[0,T]} |u'(t)| < a$, satisfying the boundary conditions and the function $\varphi(u')$ is continuously differentiable and $(\varphi(u'(t)))' = f(t,u(t),u'(t))$ for all $t \in [0,T]$.

Let us consider the operator $M_1 : C^1 \to C^1$,

$$M_1(u) = S(u) + Q(N_f(u)) + K(\varphi^{-1}[H(N_f(u) - Q(N_f(u))) + \varphi(S(u))]).$$

Here φ^{-1} is understood as the operator $\varphi^{-1} : C \to B_a(0) \subset C$ defined for $\varphi^{-1}(v)(t) = \varphi^{-1}(v(t))$. The symbol $B_a(0)$ denoting the open ball of center 0 and radius a in C. It is clear that φ^{-1} is continuous and sends bounded sets into bounded sets. When $\varphi : \mathbb{R} \to \mathbb{R}$, such an operator has been considered in [5].

Lemma 3.1. *A map $u \in C^1$ is a solution of* (3.1) *if and only if u is a fixed point of the operator M_1.*

Proof. For $u \in C^1$, we have the following equivalences:

$$
\begin{aligned}
&(\varphi(u'))' = N_f(u),\ u'(T) = u'(0),\ u'(0) = u(T) \\
&\Leftrightarrow (\varphi(u'))' = N_f(u) - Q(N_f(u)),\ Q(N_f(u)) = 0,\ u'(0) = u(T) \\
&\Leftrightarrow \varphi(u') = H(N_f(u) - Q(N_f(u))) + \varphi(u'(0)),\ Q(N_f(u)) = 0,\ u'(0) = u(T) \\
&\Leftrightarrow u' = \varphi^{-1}[H(N_f(u) - Q(N_f(u))) + \varphi(u'(0))],\ Q(N_f(u)) = 0,\ u'(0) = u(T) \\
&\Leftrightarrow u = u(T) + K(\varphi^{-1}[H(N_f(u) - Q(N_f(u))) + \varphi(u'(0))]), \\
&\qquad Q(N_f(u)) = 0,\ u'(0) = u(T) \\
&\Leftrightarrow u = u(T) + Q(N_f(u)) + K(\varphi^{-1}[H(N_f(u) - Q(N_f(u))) + \varphi(u(T))])
\end{aligned}
$$

$$\Leftrightarrow u = S(u) + Q(N_f(u)) + K(\varphi^{-1}[H(N_f(u) - Q(N_f(u))) + \varphi(S(u))]).$$

$\square$

Remark 3.2. Note that $u'(T) = u'(0) \Leftrightarrow Q(N_f(u)) = 0$.

Lemma 3.3. *The operator $M_1 : C^1 \to C^1$ is completely continuous.*

Proof. Let $\Lambda \subset C^1$ be a bounded set. Then, if $u \in \Lambda$, there exists a constant $\rho > 0$ such that

$$\|u\|_1 \leq \rho. \tag{3.2}$$

Next, we show that $\overline{M_1(\Lambda)} \subset C^1$ is a compact set. Let $(v_n)_n$ be a sequence in $M_1(\Lambda)$, and let $(u_n)_n$ be a sequence in Λ such that $v_n = M_1(u_n)$. Using (3.2), we have that there exists a constant $L_1 > 0$ such that, for all $n \in \mathbb{N}$,

$$\|N_f(u_n)\|_\infty \leq L_1,$$

which implies

$$\|H(N_f(u_n) - Q(N_f(u_n)))\|_\infty \leq 2L_1T.$$

Hence the sequence $(H(N_f(u_n) - Q(N_f(u_n))))_n$ is bounded in C. Moreover, for $t, t_1 \in [0, T]$ and for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&|H(N_f(u_n) - Q(N_f(u_n)))(t) - H(N_f(u_n) - Q(N_f(u_n)))(t_1)| \\
&\leq \Big| \int_{t_1}^{t} N_f(u_n)(s)ds \Big| + \Big| \int_{t_1}^{t} Q(N_f(u_n))(s)ds \Big| \\
&\leq L_1|t - t_1| + |t - t_1| \|Q(N_f(u_n))\|_\infty \\
&\leq 2L_1|t - t_1|,
\end{aligned}$$

which implies that $\big(H(N_f(u_n) - Q(N_f(u_n)))\big)_n$ is equicontinuous. Thus, by the Arzelà-Ascoli theorem there is a subsequence of $(H(N_f(u_n) - Q(N_f(u_n))))_n$, which we call $(H(N_f(u_{n_j}) - Q(N_f(u_{n_j}))))_j$, which is convergent in C. Then, passing to a subsequence if necessary, we obtain that the sequence

$$(H(N_f(u_{n_j}) - Q(N_f(u_{n_j}))) + \varphi(S(u_{n_j})))_j$$

is convergent in C. Using the fact that $\varphi^{-1} : C \to B_a(0) \subset C$ is continuous it follows from

$$M_1(u_{n_j})' = \varphi^{-1}[(H(N_f(u_{n_j}) - Q(N_f(u_{n_j}))) + \varphi(S(u_{n_j})))]$$

that the sequence $(M_1(u_{n_j})')_j$ is convergent in C. Therefore, passing if necessary to a subsequence, we have that $(v_{n_j})_j = (M_1(u_{n_j}))_j$ is convergent in C^1. Finally, let $(v_n)_n$ be a sequence in $\overline{M_1(\Lambda)}$. Let $(z_n)_n \subseteq M_1(\Lambda)$ be such that

$$\lim_{n\to\infty} \|z_n - v_n\|_1 = 0.$$

Let $(z_{n_j})_j$ be a subsequence of $(z_n)_n$ such that converge to z. It follows that $z \in \overline{M_1(\Lambda)}$ and $(v_{n_j})_j$ converge to z. This completes the proof. $\square$

To apply the Leray-Schauder degree to the equivalent fixed point operator M_1, for $\lambda \in [0, 1]$, we introduce the family of boundary-value problems

$$\begin{gathered}
(\varphi(u'))' = \lambda N_f(u) + (1 - \lambda)Q(N_f(u)) \\
u(T) = u'(0) = u'(T).
\end{gathered} \tag{3.3}$$

Note that (3.3) coincide with (3.1) for $\lambda = 1$. So, for each $\lambda \in [0,1]$, the nonlinear operator associated with (3.3) by Lemma 3.1 is the operator $M(\lambda,\cdot)$, where M is defined on $[0,1]\times C^1$ by

$$M(\lambda,u) = S(u) + Q(N_f(u)) + K(\varphi^{-1}[\lambda H(N_f(u) - Q(N_f(u))) + \varphi(S(u))]). \tag{3.4}$$

Using the same arguments as in the proof of Lemma 3.3 we show that the operator M is completely continuous. Moreover, using the same reasoning as in Lemma 3.1, system (3.3) is equivalent to the problem

$$u = M(\lambda,u).$$

The following lemma gives a priori bounds for the possible fixed points of M.

Lemma 3.4. *Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous. If $(\lambda,u)\in[0,1]\times C^1$ is such that $u = M(\lambda,u)$, then*

$$\|u\|_1 \le a(2+T).$$

Proof. Let $[0,T]\times C^1$ be such that $u = M(\lambda,u)$. Then

$$u = M(\lambda,u) = S(u) + Q(N_f(u)) + K(\varphi^{-1}[\lambda H(N_f(u) - Q(N_f(u))) + \varphi(S(u))]).$$

Differentiating, we obtain

$$u' = [M(\lambda,u)]' = \varphi^{-1}[\lambda H(N_f(u) - Q(N_f(u))) + \varphi(S(u))],$$

so that $\|u'\|_\infty \le a$. Because $u\in C^1$ is such that $u(T) = u'(0)$, we have

$$|u(t)| \le |u(T)| + \int_0^T |u'(s)|ds \le a + aT, \quad t\in[0,T],$$

and hence $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty \le a + aT + a = a(2+T)$. This completes the proof □

3.1. **Existence result.** We can now prove an existence theorem for (3.1). We denote by $\deg_B$ the Brouwer degree and for $\deg_{LS}$ the Leray-Schauder degree, and define the mapping $G:\mathbb{R}^2\to\mathbb{R}^2$ by

$$G(x,y) = (xT + yT^2 - yT - \frac{1}{T}\int_0^T f(t, x+yt, y)dt, y - x - yT).$$

Theorem 3.5. *Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous. Then for all $\rho > a(2+T)$*

$$\deg_{LS}(I - M(1,\cdot), B_\rho(0), 0) = \deg_B(G, B_\rho(0)\cap\mathbb{R}^2, 0).$$

If furthermore

$$\deg_B(G, B_\rho(0)\cap\mathbb{R}^2, 0) \neq 0,$$

then problem (3.1) has at least one solution.

Proof. Let M be the operator given by (3.4) and let $\rho > a(2+T)$. Using Lemma 3.4, we have that, for each $\lambda\in[0,T]$, the Leray-Schauder degree $\deg_{LS}(I - M(\lambda,\cdot), B_\rho(0), 0)$ is well defined, and by the homotopy invariance, one has

$$\deg_{LS}(I - M(0,\cdot), B_\rho(0), 0) = \deg_{LS}(I - M(1,\cdot), B_\rho(0), 0).$$

On the other hand,

$$\deg_{LS}(I - M(0,\cdot), B_\rho(0), 0) = \deg_{LS}(I - (S + QN_f + KS), B_\rho(0), 0).$$

But the range of the mapping

$$u \mapsto S(u) + Q(N_f(u)) + K(S(u))$$

is contained in the subspace of related functions, isomorphic to $\mathbb{R}^2$. Using homotopy invariance and reduction properties of Leray-Schauder degree [6], we obtain

$$\begin{aligned}
&\deg_{LS}(I-(S+QN_f+KS),B_\rho(0),0)\\
&=\deg_B\big(I-(S+QN_f+KS)\big|_{\overline{B_\rho(0)\cap\mathbb{R}^2}},B_\rho(0)\cap\mathbb{R}^2,0\big)\\
&=\deg_B(G,B_\rho(0)\cap\mathbb{R}^2,0)\neq 0.
\end{aligned}$$

Then, $\deg_{LS}(I-M(1,\cdot),B_\rho(0),0)\neq 0$. Hence, there exists $u\in B_\rho(0)$ such that $M_1(u)=u$, which is a solution for (3.1). ☐

Let us give now an application of Theorem 3.5.

Example 3.6. We consider the boundary-value problem

$$\begin{gathered}
(\varphi(u'))'=e^{u'}+e\\
u(T)=u'(0)=u'(T),
\end{gathered}\tag{3.5}$$

where $\varphi(s)=s/\sqrt{1-s^2}$.

It is not difficult to verify that $\varphi:(-1,1)\to\mathbb{R}$ is a homeomorphism and $f(t,x,y)=e^y+e$ is a continuous function. If we choose $\rho>2+T$, then the equation

$$\begin{aligned}
G(x,y)&=\Big(xT+yT^2-yT-\frac{1}{T}\int_0^T f(t,x+yt,y)dt,y-x-yT\Big)=(0,0)\\
&=\Big(xT+yT^2-yT-\frac{1}{T}\int_0^T(e^y-e)dt,y-x-yT\Big)=(0,0)\\
&=\Big(xT+yT^2-yT-e^y+e,y-x-yT\Big)=(0,0)
\end{aligned}$$

does not have solutions on $\partial B_\rho(0)\cap\mathbb{R}^2$. Then we have that the Brouwer degree $\deg_B(G,B_\rho(0)\cap\mathbb{R}^2,(0,0))$ is well defined and, by the properties of that degree, we have that

$$\deg_B(G,B_\rho(0)\cap\mathbb{R}^2,(0,0))=\sum_{(x,y)\in G^{-1}(0,0)}\operatorname{sgn} J_G(x,y)=\operatorname{sgn}(-e)=-1,$$

where $(0,0)$ is a regular value of G and $J_G(x,y)=\det G'(x,y)$ is the Jacobian of G at (x,y). So, using Theorem 3.5, we obtain that the boundary-value problem (3.5) has at least one solution.

Remark 3.7. Using the family of boundary-value problems

$$\begin{gathered}
(\varphi(u'))'=\lambda N_f(u)+(1-\lambda)Q(N_f(u))\\
u(0)=u'(0)=u'(T)
\end{gathered}\tag{3.6}$$

which gives the completely continuous homotopy $\widetilde{M}$ defined on $[0,1]\times C^1$ by

$$\widetilde{M}(\lambda,u)=P(u)+Q(N_f(u))+H(\varphi^{-1}[\lambda H(N_f(u)-Q(N_f(u)))+\varphi(P(u))]),$$

and similar a priori bounds as in the Lemma 3.4, it is not difficult to see that (3.6) has a solution for $\lambda=1$.

4. Boundary value problems with φ-Laplacian

In this section we study the existence of solutions for the boundary-value problem

$$\begin{gathered}(\varphi(u'))' = f(t,u,u')\\ u(0) = u(T) = u'(0),\end{gathered} \tag{4.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $\varphi(0) = 0$ and $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous.

Let us consider the operator $M_1 : C^1 \to C^1$,

$$M_1(u)\varphi^{-1}(-Q_\varphi(H(N_f(u)))) + H(\varphi^{-1}[H(N_f(u)) - Q_\varphi(H(N_f(u)))]).$$

As in the previous section, here φ^{-1} with an abuse of notation is understood as the operator $\varphi^{-1} : C \to C$ defined for $\varphi^{-1}(v)(t) = \varphi^{-1}(v(t))$. It is clear that φ^{-1} is continuous and sends bounded sets into bounded sets.

To transform problem (4.1) to a fixed point problem we use Lemma 2.1.

Lemma 4.1. *A map $u \in C^1$ is a solution of* (4.1) *if and only if u is a fixed point of the operator M_1.*

Proof. If $u \in C^1$ is solution of (4.1), then

$$(\varphi(u'(t)))' = N_f(u)(t) = f(t,u(t),u'(t)), \quad u(0) = u(T), \quad u(0) = u'(0)$$

for all $t \in [0,T]$. Applying H to both members and using the fact that $u(0) = u'(0)$, we deduce that

$$\varphi(u'(t)) = \varphi(u(0)) + H(N_f(u))(t).$$

Composing with the function φ^{-1}, we obtain

$$u'(t) = \varphi^{-1}[H(N_f(u))(t) + c],$$

where $c = \varphi(u(0))$. Integrating from 0 to $t \in [0,T]$, we have

$$u(t) = u(0) + H(\varphi^{-1}[H(N_f(u)) + c])(t).$$

Because $u(0) = u(T)$, we have

$$\int_0^T \varphi^{-1}[H(N_f(u))(t) + c]dt = 0.$$

Using Lemma 2.1, it follows that $c = -Q_\varphi(H(N_f(u)))$. Hence,

$$u = \varphi^{-1}(-Q_\varphi(H(N_f(u)))) + H(\varphi^{-1}[H(N_f(u)) - Q_\varphi(H(N_f(u)))]).$$

Now suppose that $u \in C^1$ be such that $u = M_1(u)$. It follows that

$$u(t) = \varphi^{-1}(-Q_\varphi(H(N_f(u)))) + H(\varphi^{-1}[H(N_f(u)) - Q_\varphi(H(N_f(u)))])(t) \tag{4.2}$$

for all $t \in [0,T]$. Since

$$\int_0^T \varphi^{-1}[H(N_f(u))(t) - Q_\varphi(H(N_f(u)))]dt = 0,$$

we have $u(0) = u(T)$. Differentiating (4.2), we obtain

$$\begin{aligned}u'(t) &= \varphi^{-1}[H(N_f(u)) - Q_\varphi(H(N_f(u)))](t)\\ &= \varphi^{-1}[H(N_f(u))(t) - Q_\varphi(H(N_f(u)))].\end{aligned}$$

Applying φ to both of its members, and differentiating we have

$$(\varphi(u'(t)))' = N_f(u)(t), \quad u(0) = u(T), \quad u(0) = u'(0)$$

for all $t \in [0,T]$. This completes the proof. $\square$

Using an argument similar to the one introduced in the proof of [5, Lemma 4.2], it is not difficult to see that $M_1 : C^1 \to C^1$ is well defined and completely continuous.

4.1. **Upper and lower solutions.** The functions considered as lower and upper solutions for the initial problem (4.1) are defined as follows.

Definition 4.2. A lower solution α (resp. upper solution β) of (4.1) is a function $\alpha \in C^1$ such that $\varphi(\alpha') \in C^1$, $\alpha'(0) \geq \alpha(0) = \alpha(T)$ (resp. $\beta \in C^1$, $\varphi(\beta') \in C^1$, $\beta'(0) \leq \beta(0) = \beta(T)$) and

$$(\varphi(\alpha'(t)))' \geq f(t,\alpha(t),\alpha'(t)) \quad (\text{resp. } (\varphi(\beta'(t)))' \leq f(t,\beta(t),\beta'(t)) \tag{4.3}$$

for all $t \in [0,T]$.

We can now prove some existence results for (4.1). These results are inspired on works by Bereanu and Mawhin [2] and Carrasco and Minhós [4].

Theorem 4.3. *Suppose that* (4.1) *has a lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0,T]$. If there exists a continuous function $g(t,x)$ on $[0,T] \times \mathbb{R}$ such that*

$$|f(t,x,y)| \leq |g(t,x)|, \quad \textit{for all } (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \tag{4.4}$$

then (4.1) *has a solution u such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0,T]$.*

Proof. Let α, β be, respectively, lower and upper solutions of (4.1). Let $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(t,x) = \begin{cases} \beta(t), & x > \beta(t) \\ x, & \alpha(t) \leq x \leq \beta(t) \\ \alpha(t), & x < \alpha(t), \end{cases}$$

and define $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $F(t,x,y) = f(t,\gamma(t,x),y) + \frac{x-\gamma(t,x)}{1+|x-\gamma(t,x)|}$. We consider the modified problem

$$\begin{gathered} (\varphi(u'))' = F(t,u,u') \\ u(0) = u(T) = u'(0). \end{gathered} \tag{4.5}$$

For clearness, the proof will follow several steps.

Step 1 We show that if u is a solution of (4.5), then $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0,T]$ and hence u is a solution of (4.1). Let u be a solution of the modified problem (4.5) and suppose by contradiction that there is some $t_0 \in [0,T]$ such that

$$\max_{[0,T]}(\alpha(t) - u(t)) = \alpha(t_0) - u(t_0) > 0. \tag{4.6}$$

If $t_0 \in (0,T)$, there are sequences (t_k) in $[t_0 - \epsilon, t_0)$ and (t'_k) in $(t_0, t_0 + \epsilon]$ converging to t_0 such that $\alpha'(t_k) - u'(t_k) \geq 0$ and $\alpha'(t'_k) - u'(t'_k) \leq 0$. Therefore $\alpha'(t_0) = u'(t_0)$. Using the fact that φ is an increasing homeomorphism, this implies $(\varphi(\alpha'(t_0)))' \leq (\varphi(u'(t_0)))'$. By (4.3) we get the contradiction

$$\begin{aligned} (\varphi(\alpha'(t_0)))' &\leq (\varphi(u'(t_0)))' = F(t_0,u(t_0),u'(t_0)) \\ &\leq f(t_0,\alpha(t_0),\alpha'(t_0))) + \frac{u(t_0) - \alpha(t_0)}{1 + |u(t_0) - \alpha(t_0)|} \end{aligned}$$

$$< f(t_0, \alpha(t_0), \alpha'(t_0))) \le (\varphi(\alpha'(t_0)))'.$$

So $\alpha(t) \le u(t)$ for all $t \in (0,T)$. If the maximum is attained at $t_0 = 0$ then

$$\max_{[0,T]}(\alpha(t) - u(t)) = \alpha(0) - u(0) > 0.$$

Using the fact that u is solution of (4.5) and $\alpha'(0) \le u'(0)$, we obtain the following contradiction

$$\alpha(0) \le \alpha'(0) \le u'(0) = u(0) < \alpha(0).$$

If

$$\max_{[0,T]}(\alpha(t) - u(t)) = \alpha(T) - u(T) > 0,$$

then, since $\alpha(0) = \alpha(T)$ and $u(0) = u(T)$ we obtain again a contradiction. In consequence we have that $\alpha(t) \le u(t)$ for all $t \in [0,T]$. In a similar way we can prove that $u(t) \le \beta(t)$ for all $t \in [0,T]$.

Step 2 We show that problem (4.5) has at least one solution. Let $u \in C^1$ and define the operator $M_F : C^1 \to C^1$ by

$$M_F(u) = \varphi^{-1}(-Q_\varphi(H(N_F(u)))) + H(\varphi^{-1}[H(N_F(u)) - Q_\varphi(H(N_F(u)))]),$$

with $F(t,u,u') = f(t, \gamma(t,u), u') + \frac{u-\gamma(t,u)}{1+|u-\gamma(t,u)|}$, by Lemma 4.1 it is enough to prove that M_F has a fixed point. Under the hypothesis of the theorem, the operator M_F is bounded. Indeed, if $v = M_F(u)$ then

$$\varphi(v') = [H(N_F(u)) - Q_\varphi(H(N_F(u)))], \tag{4.7}$$

where

$$\begin{aligned} |H(N_F(u))(t)| &\le \int_0^T |f(s, \gamma(s,u(s)), u'(s)) + \frac{u(s) - \gamma(s,u(s))}{1 + |u(s) - \gamma(s,u(s))|}|ds \\ &\le \int_0^T |f(s, \gamma(s,u(s)), u'(s))|ds + T \\ &\le \int_0^T |g(s, \gamma(s,u(s)))|ds + T \\ &\le \sigma T + T, \end{aligned}$$

with $\sigma := \sup_{s\in[0,T]} |g(s, \gamma(s,u(s)))|$. Using (4.7), we have that

$$|\varphi(v'(t))| \le 2(\sigma T + T) := \delta \quad t \in [0,T], \tag{4.8}$$

and hence

$$\|v'\|_\infty \le \omega, \tag{4.9}$$

where $\omega = \max\{|\varphi^{-1}(\delta)|,\ |\varphi^{-1}(-\delta)|\}$. Because $v \in C^1$ is such that $v(0) = v'(0)$, we have

$$|v(t)| \le |v(0)| + \int_0^T |v'(s)|ds \le \omega + T\omega \quad t \in [0,T],$$

and hence

$$\|v\|_1 = \|v\|_\infty + \|v'\|_\infty \le \omega + T\omega + \omega = \omega(2+T).$$

As the operator M_F is completely continuous and bounded, we can use Schauder's fixed point theorem to deduce the existence of at least one fixed point in $\overline{B_{\omega(2+T)}(0)}$. The proof is complete. □

Corollary 4.4. *Let $f(t,x,y) = f(t,x)$ be a continuous function. If (4.1) has a lower solution α and a upper solution β such that $\alpha(t) \le \beta(t)$ for all $t \in [0,T]$, then problem (4.1) has a solution such that $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in [0,T]$.*

Example 4.5. We consider the boundary-value problem

$$\begin{gathered} \left(|u'|^{p-2}u'\right)' = \frac{\sin(u+1) - 1 + 4ue^{u^2 t}}{1+t^2}, \\ u(0) = u(T) = u'(0), \end{gathered} \tag{4.10}$$

where $p \in (1,\infty)$. As $f(t,x)$ is a continuous function, and the functions $\alpha(t) = -1$ and $\beta(t) = 1$ are lower and upper solutions of (4.10), respectively, then, by Corollary (4.4), we obtain that (4.10) has at least one solution u such that $-1 \le u(t) \le 1$ for all $t \in [0,T]$.

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References

[1] C. Bereanu, J. Mawhin; *Boundary value problems for some nonlinear systems with singular φ-laplacian*, J. Fixed Point Theory Appl. 4 (2008), 57-75.

[2] C. Bereanu, J. Mawhin; *Periodic solutions of nonlinear perturbations of φ-laplacians with possibly bounded φ*, Nonlinear Anal. 68 (2008), 1668-1681.

[3] C. Bereanu, J. Mawhin; *Existence and multiplicity results for some nonlinear problems with singular φ-laplacian*, J. Differential Equations. 243 (2007), 536-557.

[4] H. Carrasco, F. Minhós; *Unbounded solutions for functional problems on the half-line*, Abstract and Applied Analysis. doi:10.1155/2016/8987374.

[5] D. P. D. Santos; *Existence of solutions for some nonlinear problems with boundary value conditions*, Abstract and Applied Analysis. doi:10.1155/2016/5283263.

[6] K. Deimling; *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[7] M. García-Huidobro, R. Manásevich, F. Zanolin; *A Fredholm-like result for strongly nonlinear second order ODE's*, J. Differential Equations. 114 (1994), 132-167.

[8] J. Graef, L. Kong, F. Minhos, J. Fialho; *On the lower and upper solutions method for higher order functional boundary-value problems*, Applicable Analysis and Discrete Mathematic. 5 (2011), 133-146.

[9] Y. X. Huang, G. Metzen; *The existence of solutions to a class of semilinear equations*, Differential Integral Equations. 8 (1995), 429-452.

[10] R. Ma, L. Zhang, R. Liu; *Existence results for nonlinear problems with φ-Laplacian*, Electronic Journal of Qualitative Therory of Differential Equations. 22 (2015), 1-7.

[11] R. Manásevich, J. Mawhin; *Periodic solutions for nonlinear systems with p-laplacian-like operators*, Differential Equations. 145 (1997), 367-393.

[12] J. Mawhin; *Some boundary value problems for Hartman-type perturbations of the ordinary vector p-Laplacian*, Nonlinear Anal. 40 (2000), 497-503.

[13] D. O'Regan; *Some general principles and results for $(\varphi(y'))' = qf(t,y,y')$, $0 < t < 1$*, SIAM J. Math. Anal. 24 (1993), 648-668.

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