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POSITIVE LYAPUNOV EXPONENT OF DISCRETE ANALYTIC JACOBI OPERATOR

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Abstract. In this article, we study the Lyapunov exponent of discrete analytic Jacobi operator with a family of special mappings on the torus. By applying the theory of subharmonic functions, we prove that the Lyapunov exponent is positive, if the coupling number is large.

1. Introduction

Consider the discrete Jacobi operator on $l^2(\mathbb{Z})$:

$$\begin{aligned}(H_x\phi)(n) &= -a(T^{n+1}(x))\phi(n+1) - \bar{a}(T^n(x))\phi(n-1) + \lambda v(T^n(x))\phi(n) \\ &= E\phi(n), \quad n \in \mathbb{Z},\end{aligned} \tag{1.1}$$

where $a(x), v(x)$ are analytic functions on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, $a(x)$ is not identically zero and $T(x)$ is a mapping from $\mathbb{T}$ to $\mathbb{T}$, satisfying

$$T^n(x) = x + f(n), \tag{1.2}$$

where $f(n)$ is some function from $\mathbb{Z}$ to $\mathbb{T}$. Note that this Jacobi operator can be expressed as

$$\begin{pmatrix}\phi(n+1)\\ \phi(n)\end{pmatrix} = M_n(x,E)\begin{pmatrix}\phi(1)\\ \phi(0)\end{pmatrix},$$

where

$$M_n(x,E) = \prod_{j=n-1}^{0} \frac{1}{a(T^{j+1}(x))}\begin{pmatrix}\lambda v(T^j(x)) - E & -\bar{a}(T^j(x))\\ a(T^{j+1}(x)) & 0\end{pmatrix}$$

is called the transfer matrix of (1.1). Since $a(x)$ is analytic, we know that the number of its zeros is finite. So for almost every $x \in \mathbb{T}$, the matrix $M_n(x,E)$ can be defined. Let

$$L_n(E) = \frac{1}{n}\int_{\mathbb{T}} \log\|M_n(x,E)\|dx.$$

Then the Lyapunov exponent of the Jacobi equation is defined as

$$L(E) = \liminf_{n\to\infty} L_n(E). \tag{1.3}$$

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We remark here that we use $\liminf$ instead of $\lim$, so that definition (1.3) applies to a generic $T(x)$. Moreover, from [10], we have that there exists a constant $C(\lambda)$ such that

$$0 \le L(E) \le C(\lambda).$$

It is well known that non-uniformly hyperbolic dynamic system has many properties, and it is the essential condition for the Anderson Localization, which is a central topic in our field and named by Anderson in [3]. Also, even if $T(x)$ is not ergodic, the positive Lyapunov exponent, defined by the lower limit, is also an important topic in the spectrum theory.

The main result of this paper reads as follows.

Theorem 1.1. *There exists $\lambda_0 = \lambda_0(v, a) > 0$ such that when the coupling number $|\lambda| > \lambda_0$, then for any $T(x) : \mathbb{T} \to \mathbb{T}$ having the form (1.2), we have*

$$L(E) \ge c \log |\lambda| \quad \text{for all } E,$$

where c is a constant depending only on v and a.

This topic comes from the following discrete analytic Schrödinger operator which has been studied by many researchers:

$$(S_x\phi)(n) = \phi(n+1) + \phi(n-1) + \lambda v[x + f(n)]\phi(n) = E\phi(n), \quad n \in \mathbb{Z}.$$

The most focused one is the shift, i.e. $f(n) = n\omega$, and there are many influential works for it, including [BG], [GS] and [AJ]. Such many works shew various results about dynamical system and spectrum theory, including the Anderson localization, Hölder continuity, cantor spectrum and so on. Among these works, the following non-perturbative positive Lyapunov exponent theorem (first proved in [9]) plays a key role: there exist $\lambda_0 = \lambda_0(v)$ and $c = c(v)$ such that $L(E) > c \log |\lambda|$ for any irrational ω and $|\lambda| > \lambda_0$. Clearly, our result is an extension of this result.

Instead of the shift, there are some other articles concerning on the Schrödinger operator with other mappings. Krüger [8] proved that for the polynomial mapping

$$f(n) = a_d n^d + \dots + a_1 n,$$

and for any constant $c > 0$,

$$\operatorname{meas}\{E : L(E) < c\} \to 0$$

as $d \to +\infty$; for any $\epsilon > 0$, there exists $\lambda_0(d, \epsilon)$ such that

$$\operatorname{meas}\{E : L(E) < \log |\lambda|\} < \epsilon$$

for all $|\lambda| > \lambda_0$. He expected that for any $\lambda \neq 0$ and any nonconstant analytic $v(x)$, the Lyapunov exponent is positive for all E when $d \ge 2$. It is obvious that our Theorem 1.1 answers this question for the large coupling number. Moreover, if the potential becomes $v(T^n(x)) = \cos(2\pi n^\rho + x)$, where ρ is not an integer, Bourgain had shown the positive Lyapunov exponent with small λ in [4].

For the Lyapunov exponent of the Jacobi operator, the work [7] is the most famous. They considered the following called extended Harper's model, which is a case of Jacobi operator with the shift:

$$a(x) = \lambda_3 \exp[-2\pi i(x + \frac{\beta}{2})] + \lambda_2 + \lambda_1 \exp[2\pi i(x + \frac{\beta}{2})], \quad 0 \le \lambda_2,\ 0 \le \lambda_1 + \lambda_3,$$
$$v(x) = 2\cos(2\pi x).$$

Then the Lyapunov exponent on the spectrum is zero when $0 \le \lambda_1 + \lambda_3 \le \lambda_2$ and $1 \le \lambda_2$, or $\max\{1, \lambda_2\} \le \lambda_1 + \lambda_3$, and is given by the following formula when $0 \le \lambda_1 + \lambda_3 \le 1$, and $0 \le \lambda_2 \le 1$:

$$L(E) = \begin{cases} \log\big(\frac{1+\sqrt{1-4\lambda_1\lambda_3}}{2\max\{\lambda_1,\lambda_3\}}\big), & \text{if } 0 \le \lambda_2 \le \lambda_1 + \lambda_3 \le 1; \\ \log\big(\frac{1+\sqrt{1-4\lambda_1\lambda_3}}{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}\big), & \text{if } 0 \le \lambda_1 + \lambda_3 \le \lambda_2 \le 1. \end{cases}$$

It is easy to see that $L(E) \approx -\log\hat\lambda$ when $\hat\lambda(:= \max\{\lambda_1, \lambda_2, \lambda_3\})$ is small enough. Thus, this formula satisfies Theorem 1.1 as $1/\hat\lambda \simeq \lambda$, and the zero Lyapunov exponent part also shows that it is necessary to assume that the coupling number λ is large enough.

Define

$$L^a(E) = \liminf_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \| \prod_{j=n-1}^{0} A(T^j(x), E)\| dx,$$

where

$$A(x, E) = \begin{pmatrix} \lambda v(x) - E & -\bar a(x) \\ a(T(x)) & 0 \end{pmatrix}.$$

It is easy to see that $L(E) = L^a(E) - D$, where D is a constant defined by

$$D := \int_{\mathbb{T}} \log|a(x)| dx.$$

Note that $D \ll \log\lambda$ with large λ. Thus, we only need to prove the following result.

Lemma 1.2. *If $|\lambda| > \lambda_0$, then*

$$L^a(E) \ge 2c\log|\lambda| \quad \text{for all } E,$$

where λ_0 and c are the same as in Theorem 1.1.

In Section 3, it is proved that Lemma 1.2 is also valid for the more general matrix

$$A(x, E) := \begin{pmatrix} \lambda v_{11}(x) - E & v_{12}(x) & \dots & v_{1m}(x) \\ v_{21}(x) & v_{22}(x) & \dots & v_{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}(x) & v_{m2}(x) & \dots & v_{mm}(x) \end{pmatrix}, \tag{1.4}$$

where every $v_{ij}(x)$ is an analytic function on $\mathbb{T}$.

This article is organized as follows. In Section 2, we introduce some properties about the subharmonic functions, especially the inequality of subharmonic functions under harmonic measure. The proof of Lemma 1.2 with the general matrix (1.4) is presented in Section 3.

2. Subharmonic Functions

Let $u(z)$ be a real function defined on some domain $\Omega \in \mathbb{C}$.

Definition 2.1 (Subharmonic Function)**.** We call $u(z)$ a subharmonic function on Ω if

(1) $u(z) : \Omega \to [-\infty, +\infty)$;
(2) $u(z)$ is upper semicontinuous from Ω into $[-\infty, +\infty)$;

(3) for any $z_1 \in \Omega$, there exists $r_1 = r_1(z_1) > 0$ such that for any $0 < r < r_1$ holds

$$u(z_1) \leq \frac{1}{2\pi} \int_0^{2\pi} u\left(z_1 + re^{i\theta}\right) d\theta. \tag{2.1}$$

Remark 2.2. We recall the following the Jensen formula

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(z_0 + re^{i\theta}\right) \right| d\theta = \log |f(z_0)| + \sum_{|z-z_0|<r, f(z)=0} \log \frac{r}{|z - z_0|},$$

which makes $u(z) = \log |f(z)|$ be subharmonic in Ω, when $f(z)$ is an analytic function in Ω. Similarly, for fixed $f(n)$ and E, $u_n(z) = \frac{1}{n} \log \|M_n^a(z, E)\|$ is also a subharmonic function.

Lemma 2.3. *Let $u(z) \in C^2(\Omega)$. Then*

$$\iint_{D(z_0,r_0)} \big(\log \frac{r_0}{|z - z_0|} \big) \Delta u(x, y)\, dx\, dy = \int_0^{2\pi} u\left(z_0 + r_0 e^{i\theta}\right) d\theta - 2\pi u(z_0).$$

Proof. Recall the Green's formula

$$\iint_{\mathscr{A}} (g\Delta f - f\Delta g)\; dx\, dy = \oint_{\partial\mathscr{A}} \left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n} \right) ds.$$

Define $\mathscr{A} = \{z : \rho < |z - z_0| < r_0\}$. Then

$$\iint_{\mathscr{A}} (g\Delta f - f\Delta g)\; dx\, dy = r_0 \int_{\Gamma_{r_0}} (g\partial_r f - f\partial_r g)\, d\theta - \rho \int_{\Gamma_\rho} (g\partial_r f - f\partial_r g)\, d\theta,$$

where $\Gamma_r = \{z : |z - z_0| = r\}$. Taking $f = u(z)$ and $g = \log \frac{r_0}{|z-z_0|}$, we obtain

$$\begin{gathered} \Delta g(z) = 0, \quad \forall z \in \mathscr{A}, \\ g(z) = 0, \quad \forall z \in \Gamma_{r_0}, \\ \partial_r g(z) = -\frac{1}{r}, \quad \forall z \in \Gamma_r. \end{gathered}$$

Thus

$$\begin{aligned} &\iint_{\mathscr{A}} \left(\log \frac{r_0}{|z - z_0|} \right) \Delta u(x, y)\, dx\, dy \\ &= \int_0^{2\pi} u(z_0 + r_0 e^{i\theta}) d\theta - \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta - \rho \log \frac{r_0}{\rho} \int_0^{2\pi} \partial_r u(z_0 + \rho e^{i\theta}) d\theta. \end{aligned}$$

Let $\rho \to 0$, then

$$\begin{gathered} \mathscr{A} \to D(z_0, r_0), \\ \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta \to 2\pi u(z_0), \\ \rho \log \frac{r_0}{\rho} \int_0^{2\pi} \partial_r u(z_0 + \rho e^{i\theta}) d\theta \leq \rho \log \frac{r_0}{\rho} C(u) \to 0. \end{gathered}$$

Now the proof is complete. □

Corollary 2.4. *Let $u(z) \in C^2(\Omega)$, then $u(z)$ is subharmonic if and only if $\Delta u(z) \geq 0$ for any $z \in \Omega$.*

Proof. It is easy to see that if $\Delta u(z) \geq 0$ for any $z \in \Omega$, then

$$\iint_{D(z_0,r_0)} \Big(\log \frac{r_0}{|z-z_0|}\Big)\Delta u(x,y)\,dx\,dy = \int_0^{2\pi} u\left(z_0 + r_0 e^{i\theta}\right)d\theta - 2\pi u(z_0) \geq 0,$$

since $\log \frac{r_0}{|z-z_0|} \geq 0$ for any $|z-z_0| \leq r_0$.

Now we show that if $u(z)$ is subharmonic, Lemma 2.3 implies $\Delta u(z) \geq 0$ for any $z \in \Omega$. Indeed, assume that $\Delta u(z_0) < 0$ for some $z_0 \in \Omega$. Then $\Delta u(z) \leq -\delta$ for any $|z-z_0| < r_0$, with some $\delta_0 > 0$ and $r_0 > 0$. Therefore,

$$\begin{aligned}
\iint_{D(z_0,r_0)} \Big(\log \frac{r_0}{|z-z_0|}\Big)\Delta u(x,y)\,dx\,dy &\leq -\delta_0 \iint_{D(z_0,r_0)} \Big(\log \frac{r_0}{|z-z_0|}\Big)\,dx\,dy \\
&\leq -\delta \iint_{|z-z_0|<\frac{r_0}{2}} \Big(\log \frac{r_0}{|z-z_0|}\Big)\,dx\,dy \\
&\leq -\frac{\pi}{4}\delta_0 r_0^2 \log 2 < 0,
\end{aligned}$$

which contracts to Lemma 2.3 and the definition of the subharmonic function. □

Remark 2.5. It is well known that $u(z)$ is harmonic if and only if $\Delta u(z) = 0$ for any $z \in \Omega$. In the sense of distribution, we can define Δu for the continuous function $u(z)$. Thus, $u(z)$ is subharmonic if and only if $\Delta u(z) \geq 0$ for any $z \in \Omega$.

Definition 2.6. Given a domain Ω and a function $f \in C(\partial\Omega)$, the Dirichlet problem for f on Ω is to find a function $u \in C(\bar{\Omega})$ such that $\Delta u = 0$ on Ω and $u\big|_{\partial\Omega} = f$.

Then the following results deals with the Dirichlet problem on the upper half-plane $\mathbb{H}$.

Lemma 2.7. *Suppose that $f \in C(\mathbb{R} \cup \{\infty\})$. Then there exists a unique function $u = u_f \in C(\overline{\mathbb{H} \cup \{\infty\}})$ such that u is harmonic on $\mathbb{H}$ and $u\big|_{\partial\mathbb{H}} = f$.*

Before giving the proof, we show the following lemma, which was first proved in [1].

Lemma 2.8 (Ahlfors)**.** *Suppose the function $u(z)$ is subharmonic and bounded above on a region Ω such that $\bar{\Omega} \neq \mathbb{C}$. Let F be a finite subset of $\partial\Omega$ and suppose*

$$\limsup_{z \to \zeta} u(z) \leq 0 \tag{2.2}$$

for all $\zeta \in \partial \backslash F$. Then $u(z) \leq 0$ on Ω.

Proof of Lemma 2.7. Assume that f is real valued and $f(\infty) = 0$. For $\epsilon > 0$, let us take disjoint open intervals $I_j = (t_j, t_{j+1})$ and real constants c_j, $j = 1, \dots, n$, such that the step function

$$f_\epsilon(t) = \sum_{j=1}^n c_j \chi_{I_j}$$

satisfies

$$\|f - f_\epsilon\|_{L^\infty(\mathbb{R})} < \epsilon. \tag{2.3}$$

Set

$$u_\epsilon(z) = \sum_{j=1}^n c_j \mu(z, I_j, \mathbb{H}),$$

where

$$\mu(z, I_j, \mathbb{H}) = \frac{1}{\pi} \arg\Big(\frac{z - t_{j+1}}{z - t_j}\Big) = \frac{1}{\pi} \Im[\log(z - t_{j+1}) - \log(z - t_j)].$$

If $t \in \mathbb{R} \backslash \partial I_j$, then

$$\begin{gathered}\sum_{j=1}^{n} \mu(z, I_j, \mathbb{H}) \to 1 \quad \text{as } z \to \bigcup_{j=1}^{n} I_j, \\ \sum_{j=1}^{n} \mu(z, I_j, \mathbb{H}) \to 0 \quad \text{as } z \to \mathbb{R} \backslash \bigcup_{j=1}^{n} \bar{I}_j, \end{gathered} \tag{2.4}$$

which imply $\lim_{\mathbb{H} \ni z \to t} u_\epsilon(z) = f_\epsilon(t)$. Therefore, by (2.3) and (2.2),

$$\sup_{\mathbb{H}} |u_{\epsilon_1}(z) - u_{\epsilon_2}(z)| < \epsilon_1 + \epsilon_2.$$

Consequently the limit

$$u(z) := \lim_{\epsilon \to 0} u_\epsilon(z)$$

exists, and the limit $u(z)$ is harmonic on $\mathbb{H}$ and satisfies

$$\sup_{\mathbb{H}} |u(z) - u_\epsilon(z)| \leq 2\epsilon.$$

We claim that

$$\limsup_{z \to t} |u_\epsilon(z) - f(t)| \leq \epsilon \tag{2.5}$$

for all $t \in \mathbb{R}$. It is clear that (2.5) holds when $\zeta \notin \bigcup_{j=1}^{n} \partial I_j$. To verify (2.5) at the endpoint $t_{j+1} \in \partial I_j \cap I_{j+1}$, by (2.4) and Lemma 2.8, we have

$$\sup_{\mathbb{H}} \Big| c_j \mu(z, I_j, \mathbb{H}) + c_{j+1} \mu(z, I_{j+1}, \mathbb{H}) - \Big(\frac{c_j + c_{j+1}}{2}\Big) \mu(z, I_j \cup I_{j+1}, \mathbb{H}) \Big| \leq \big| \frac{c_j - c_{j+1}}{2} \big|,$$

where

$$\lim_{z \to t_{j+1}} \big(\frac{c_j + c_{j+1}}{2}\big) \mu(z, I_j \cup I_{j+1}, \mathbb{H}) = \frac{c_j + c_{j+1}}{2}.$$

Hence all limit values of $u_\epsilon(z)$ at t_{j+1} lie in the closed interval with endpoints c_j and c_{j+1}, and then (2.3) yields (2.5) for the endpoint t_{j+1}.

Let $t \in \mathbb{R}$. By (2.5)

$$\limsup_{z \to t} |u(z) - f(t)| \leq \sup_{\mathbb{H}} |u(z) - u_\epsilon(z)| + \limsup_{z \to t} |u_\epsilon(z) - f(t)| \leq 3\epsilon.$$

The same estimate holds if $t = \infty$. Therefore u extends to be continuous on $\overline{\mathbb{H}}$ and $u\big|_{\partial \mathbb{H}} = f$. The uniqueness of u follows immediately from the maximum principle. $\square$

For $a < b$, elementary calculus gives

$$\mu(x + iy, (a, b), \mathbb{H}) = \int_a^b \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}.$$

If $E \subset \mathbb{R}$ is measurable, the harmonic measure of E at $z \in \mathbb{H}$ is defined as

$$\mu(z, E, \mathbb{H}) = \int_E \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}.$$

For $z = x + iy \in \mathbb{H}$, the density

$$P_z(t) = \frac{1}{\pi} \frac{y}{(x - t)^2 + y^2}$$

is called the Poisson kernel for $\mathbb{H}$. If $f \in C(\mathbb{R} \cup \{\infty\})$, the proof of Lemma 2.7 shows that

$$u_f(z) = \int_{\mathbb{R}} f(t) P_z(t) dt,$$

and for this reason u_f is also called the Poisson integral of f.

Now we consider the Dirichlet problem on a Jordan domain Ω, which will be solved by the following Carathéodory lemma.

Lemma 2.9 (Carathéodory)**.** *Let ψ be a conformal mapping from the unit disc $\mathbb{D}$ onto a Jordan domain Ω. Then ψ has continuous extension to $\overline{\mathbb{D}}$, and the extension is a one-to-one map from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$.*

Let E be a measurable set on $\partial\mathbb{D}$, then define the harmonic measure of E at $z \in \mathbb{D}$ to be

$$\mu(z, E, \mathbb{D}) := \mu(\psi(z), \psi(E), \mathbb{H}), \tag{2.6}$$

where ψ is any conformal map of $\mathbb{D}$ onto $\mathbb{H}$. By Lemma 2.8 the definition (2.6) does not depend on the choice of ψ. It follows by the change of variables $\psi(z) = i\frac{1+z}{1-z}$ that

$$\mu(z, E, \mathbb{D}) = \int_E \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

By Lemma 2.9, if f is continuous on $\partial\mathbb{D}$, the solution of the Dirichlet problem for f on $\mathbb{D}$ is

$$\begin{aligned} u(z) = u_f(z) &= \int_{\partial\mathbb{D}} f(\zeta) d\mu_\zeta(z, \partial\mathbb{D}, \mathbb{D}) \\ &= \int_0^{2\pi} f(e^{it}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} d\theta, \quad \forall z = re^{it} \in \mathbb{D}, \end{aligned}$$

where the kernel

$$P_{\mathbb{D}}(\theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}, \quad z = re^{i\theta}$$

is the Poisson kernel for the disc and the function $u = u_f$ is called the Poisson integral of f on $\mathbb{D}$.

Let ψ be a conformal mapping from the unit disc onto the Jordan domain Ω, f be a continuous function on $\Gamma = \partial\Omega$. Then $f \circ \psi$ is also continuous on $\partial\mathbb{D}$, and

$$u(z) := u_f(z) = \int_0^{2\pi} f \circ \psi(e^{i\theta}) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi}, \quad w = \psi^{-1}(z),$$

is harmonic on Ω, and by Lemma 2.9,

$$\lim_{\ni z \to \zeta} u(z) = f(\zeta), \quad \forall \zeta \in \Gamma.$$

In general, the solution of the Dirichlet problem on a Jordan domain Ω can be written as

$$u(z) = \int_{\partial\Omega} f(\zeta) d\mu_\zeta(z, \partial\Omega, \Omega),$$

where $\mu(z, E, \Omega)$ is the harmonic measure of $E \subset \Gamma = \partial\Omega$ defined by

$$\mu(z, E, \Omega) = \mu(w, \psi^{-1}(E), \mathbb{D}) = \int_{\psi^{-1}(E)} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi}. \tag{2.7}$$

Note that again by Lemma 2.8 this harmonic measure does not depend on the choice of ψ. It also means that if u is a harmonic on Ω, for any Jordan domain $\Omega' \subset \Omega$ satisfying $\overline{\Omega'} \subset \Omega$, such that for any $z \in \Omega'$

$$u(z) = \int_{\partial\Omega'} u(\zeta) d\mu_\zeta(z, \partial\Omega', \Omega').$$

Obviously, it is an extension of mean-value property of harmonic function.

Moreover, this harmonic measure theory can also solve the following inequality of boundary-value problem

$$\begin{gathered} -\Delta u(z) \leq 0, \quad \forall z \in \Omega, \\ u(\zeta) = f(\zeta), \quad \forall \zeta \in \partial\Omega. \end{gathered} \tag{2.8}$$

Before giving the solution, we would better introduce some equivalent characterizations of the subharmonic function.

Lemma 2.10. *Let $u : \Omega \to [-\infty, +\infty)$ be an upper semicontinuous function. Then the following statements are equivalent:*

(a) *The function u is subharmonic on Ω.*

(b) *Whenever $\bar{D}(z_0, r_0) = \{z : |z - z_0| \leq r_0\} \subset \Omega$, then for any $r \leq r_0$*

$$u(z_0 + re^{it}) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2r_0 r \cos(\theta - t) + r^2} u(z_0 + r_0 e^{i\theta}) d\theta.$$

(c) *If Ω' is a relatively compact subdomain of Ω, and h is a harmonic function on Ω' satisfying*

$$\limsup_{z \to \zeta} (u - h)(z) \leq 0$$

for all $\zeta \in \partial\Omega'$, then $u \leq h$ on Ω'.

Proof. (a)$\Rightarrow$(c): Given Ω' and h as in (c), the function $u - h$ is subharmonic on Ω', so the result follows by the maximum principle of subharmonic functions.

(c)$\Rightarrow$(b): Suppose that $\bar{D} := \bar{D}(z_0, r_0) \subset \Omega'$. For $n \geq 1$, define $\psi_n : \partial D \to \mathbb{R}$ by

$$\psi_n(z_0 + r_0 e^{i\theta}) = \sup_{0 \leq \theta' < 2\pi} \Big(u(z_0 + r_0 e^{i\theta'}) - n\|\theta - \theta'\| \Big), \quad \theta \in [0, 2\pi),$$

where

$$\|\theta - \theta'\| = \min_{k \in \mathbb{Z}} |\theta - \theta' + 2k\pi|.$$

Then for each n, we have

$$|\psi_n(z_0 + r_0 e^{i\theta}) - \psi_n(z_0 + r_0 e^{i\theta'})| \leq n\|\theta - \theta'\|,$$

thus ψ_n is continuous on ∂D. Clearly, $\psi_1 \geq \psi_2 \geq \cdots \geq u$, and so in particular $\lim_{n \to +\infty} \psi_n \geq u$. On the other hand,

$$\psi_n(z_0 + r_0 e^{i\theta}) \leq \max\Big(\sup_{\|\theta' - \theta\| < \rho} u(z_0 + r_0 e^{i\theta'}), \sup_{\partial D} u - n\rho \Big), \quad \forall \rho > 0.$$

Thus

$$\lim_{n \to +\infty} \psi_n(z_0 + r_0 e^{i\theta}) \leq \sup_{\|\theta' - \theta\| < \rho} u(z_0 + r_0 e^{i\theta'}), \quad \forall \rho > 0.$$

As u is upper semicontinuous, letting $\rho \to 0$, we have $\lim_{n\to+\infty} \psi_n(z) \le u(z)$ for any $z \in \partial D$. Thus, these continuous functions $\psi_n : \partial D \to \mathbb{R}$ converges to u on ∂D with $\psi_n \ge u$ for any n. Define the Poisson integrals $P_D\psi_n : D \to \mathbb{R}$ by

$$P_D\psi_n(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r_0^2 - r}{r^2 - 2rr_0\cos(\theta - t) + r^2}\psi_n(z_0 + r_0e^{it})dt, \quad \forall r < r_0,$$

which is harmonic on D obviously. Also $\lim_{z\to\zeta} P_D\psi_n(z) = \psi_n(\zeta)$ for all $\zeta \in \partial D$, and hence

$$\limsup_{z\to\zeta}(u - P_D\psi_n)(z) \le u(\zeta) - \psi_n(\zeta) \le 0.$$

It follows from (c) that $u \le P_D\psi_n$ on D. Letting $n \to +\infty$ and using the monotone convergence theorem, the desired inequality is obtained.

(b)⇒(a) is obvious. □

By Corollary 2.4, $u(z)$ is a solution of (2.8), if and only if $u(z)$ is a subharmonic function on Ω and for any $\zeta \in \partial\Omega$, $u(\zeta) = f(\zeta)$. Let ψ be a conformal mapping from $\mathbb{D}$ onto this Jordan domain Ω, then the necessary and sufficient condition that $u(z)$ is a solution becomes $u \circ \psi$ is a subharmonic function on $\mathbb{D}$ and satisfies $u \circ \psi(e^{i\theta}) = f \circ \psi(e^{i\theta})$. By Lemma 2.10, we have the fact that

$$u\circ\psi(re^{it}) \le \frac{1}{2\pi}\int_0^{2\pi} f\circ\psi(e^{i\theta})\frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}d\theta = \int_{\partial\mathbb{D}} f\circ\psi(\zeta)d\mu_\zeta(z, \partial\mathbb{D}, \mathbb{D}).$$

By (2.7), we have

$$u(z) \le \int_{\partial\Omega} f(\zeta)d\mu_\zeta(z, \partial\Omega, \Omega), \quad \forall z \in \Omega.$$

Therefore, the following result holds.

Corollary 2.11. *Let $u : \Omega \to [-\infty, +\infty)$ be an upper semicontinuous function. Then $u(z)$ is a subharmonic function on Ω, if and only if for any Jordan subdomain Ω' satisfying $\overline{\Omega'} \subset$ and any $z \in \Omega'$, it has*

$$u(z) \le \int_{\partial\Omega'} u(\zeta)d\mu_\zeta(z, \partial\Omega', \Omega'),$$

where $\mu(z, \partial\Omega', \Omega')$ is the harmonic measure of $\partial\Omega'$ at $z \in \Omega'$.

3. Proof of Lemma 1.2

Let v be a 1-periodic nonconstant real analytic function on $\mathbb{R}$. Then there exists $\rho_v > 0$ such that

$$v(x) = \sum_{k\in\mathbb{Z}} \hat{v}(k)e^{2\pi ikx} \quad \text{with } |\hat{v}(k)| \sim e^{-\rho_v|k|}.$$

Moreover, there exists a holomorphic extension

$$v(z) = \sum_{k\in\mathbb{Z}} \hat{v}(k)e^{2\pi ikz}$$

to the strip $|\Im z| < \frac{\rho_v}{5}$, satisfying

$$|v(z)| \le \sum_{k\in\mathbb{Z}} |\hat{v}(k)|e^{2\pi|k||\Im z|} < \sum_{k\in\mathbb{Z}} e^{-\rho_v|k|}e^{\rho_v|k|\frac{\pi}{5}} < C_v.$$

Before giving the proof of Lemma 1.2 with the general matrix (1.4), we introduce the following lemma from [5], which will be applied soon.

Lemma 3.1. *For all $0 < \delta < \rho$, there is an ϵ such that*
$$\inf_{E_1} \sup_{\frac{\delta}{2}<y<\delta} \inf_{x\in[0,1]} |v(x+iy) - E_1| > \epsilon.$$

Proof of Lemma 1.2. Without loss of generality, let $\lambda > 0$. Assume that
$$v_{ij}(x) = \sum_{k\in\mathbb{Z}} \hat{v}_{ij}(k) e^{2\pi ikx} \quad \text{with } |\hat{v}_{ij}(k)| \sim e^{-\rho_{ij}|k|},\ 1 \le i,\ j \le m,$$
$$C_{ij} = \sup_{|\Im z| \le \frac{\rho_{ij}}{5}} |v_{ij}(z)|, \quad 1 \le i,\ j \le m.$$
Let
$$C = \max_{i,j} C_{ij} \text{ and } \ \rho = \min_{i,j} \rho_{ij}.$$
Thus, we can assume that $|E| < mC\lambda$ and then $M_n^a(z,E)$ is analytic on $|\Im z| < \frac{\rho}{5}$ with fixed $f(n)$, E and $\|M_n^a(z,E)\| \le (2mC\lambda)^n$. Define
$$u_n(z) := \frac{1}{n} \log \|M_n^a(z,E)\|,$$
which is a subharmonic function on $|\Im z| < \frac{\rho}{5}$, upper bounded by $\log[2mC\lambda]$.

Fix $0 < \delta \ll \rho$ and ϵ satisfying Lemma 3.1. Define
$$\lambda_0 = 100mC\epsilon^{-100}$$
and let $\lambda > \lambda_0$. Then, for fixed E, there is $\frac{\delta}{2} < y_0 < \delta$ such that
$$\inf_{x\in[0,1]} \left| v_{11}(x+iy_0) - \frac{E}{\lambda} \right| > \epsilon,$$
which implies
$$\inf_{x\in\mathbb{R}} |\lambda v_{11}(x+iy_0) - E| > \lambda\epsilon > 100mC\epsilon^{-99} > 100mC,$$
since $v(x)$ is periodic.

Let
$$M_{n-1}^a(iy_0,E) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} w_1^{n-1} \\ w_2^{n-1} \\ \vdots \\ w_m^{n-1} \end{pmatrix}.$$
Then
$$\begin{aligned} \begin{pmatrix} w_1^n \\ w_2^n \\ \vdots \\ w_m^n \end{pmatrix} &= \begin{pmatrix} \lambda v_{11}(iy_0+f(n)) - E & \dots & v_{1m}(iy_0+f(n)) \\ v_{21}(iy_0+f(n)) & \dots & v_{2m}(iy_0+f(n)) \\ \vdots & \ddots & \vdots \\ v_{m1}(iy_0+f(n)) & \dots & v_{mm}(iy_0+f(n)) \end{pmatrix} \begin{pmatrix} w_1^{n-1} \\ w_2^{n-1} \\ \vdots \\ w_m^{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \big(\lambda v_{11}[iy_0+f(n)] - E\big) w_1^{n-1} + \sum_{j=2}^m v_{1j}[iy_0+f(n)] w_j^{n-1} \\ \sum_{j=1}^m v_{2j}[iy_0+f(n)] w_j^{n-1} \\ \vdots \\ \sum_{j=1}^m v_{mj}[iy_0+f(n)] w_j^{n-1} \end{pmatrix}. \end{aligned}$$
Here we use induction to show that
$$|w_1^n| \ge |w_j^n|,\ j = 2,\dots,m, \quad \text{and} \quad |w_1^n| \ge (\lambda\epsilon - mC)^n,\ n \ge 1. \tag{3.1}$$

As $w_1^0 = 1$, $w_j^0 = 0$, $j = 2, \dots, m$, it yields

$$|w_1^1| = \lambda\epsilon > 100mC, \quad |w_j^1| < C, \quad j = 2, \dots, m.$$

Let

$$|w_1^t| \geq |w_j^t|, \ j = 1, \dots, m, \quad |w_1^t| > (\lambda\epsilon - mC)|w_1^{t-1}| > (\lambda\epsilon - mC)^t. \tag{3.2}$$

By (3.2), we have

$$|w_1^{t+1}| \geq (\lambda\epsilon - mC)\, w_1^t > (\lambda\epsilon - mC)^{t+1},$$
$$|w_j^{t+1}| \leq mC|w_1^t| < 99mC|w_1^t| \leq (\lambda\epsilon - mC)|w_1^t| \leq |w_1^{t+1}|, \ j = 2, \dots, m,$$

which also satisfy (3.1). By the induction, the expression (3.1) holds for any $n \geq 1$. Thus

$$\|M_n^a(iy_0, E)\| > (\lambda\epsilon - mC)^n \quad \text{and} \quad u_n(iy_0) > \log(\lambda\epsilon - mC).$$

Let $\mathbb{H} = \{z : \Im z > 0\}$ be the upper half-plane and $\mathbb{H}_s$ be the strip $\{z = x + iy : 0 < y < \frac{\rho}{5}\}$. Denote $\mu(z, E, \mathbb{H})$ by the harmonic measure of E at $z \in \mathbb{H}$ and $\mu_s(iy_0, E_s, \mathbb{H}_s)$ by the harmonic measure of E_s at $iy_0 \in \mathbb{H}_s$, where $E \subset \partial\mathbb{H} = \mathbb{R}$ and $E_s \subset \partial\mathbb{H}_s = \mathbb{R} \bigcup [y = \frac{\rho}{5}]$. Note that $\psi(z) = \exp\left(\frac{5\pi}{\rho} z\right)$ is a conformal map from $\mathbb{H}_s$ onto $\mathbb{H}$. Due to (2.7), we obtain

$$\mu_s(iy_0, E_s, \mathbb{H}_s) \equiv \mu(\psi(iy_0), \psi(E_s), \mathbb{H}),$$
$$\mu(z = x + iy, E, \mathbb{H}) = \int_E \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}.$$

Thus

$$\mu_s[y = \frac{\rho}{5}] = \frac{5\pi y_0}{\pi\rho} < \frac{5\delta}{\rho}, \quad \frac{d\mu_s(x)}{dx}\Big|_{y=0} < \frac{d\mu(x)}{dx} = \frac{y_0}{x^2 + y_0^2}.$$

By Corollary 2.11, we have

$$\begin{aligned}
\log(\lambda\epsilon - mC) < u_n(iy_0) &\leq \int_{[y=0] \bigcup [y=\frac{\rho}{5}]} u_n(z)\mu_s(dz) \\
&= \int_{y=0} u_n(x)\mu_s(dx) + \int_{y=\frac{\rho}{5}} u_n(x + iy)\mu_s(dx) \\
&\leq \int_{\mathbb{R}} u_n(x)\frac{y_0}{x^2 + y_0^2} dx + \frac{5\delta}{\rho}\big[\sup_{y=\frac{\rho}{5}} u_n(x + iy)\big] \\
&\leq \int_{\mathbb{R}} u_n(x)\frac{y_0}{x^2 + y_0^2} dx + \frac{\bar{C}\delta}{\rho}\log\lambda.
\end{aligned}$$

Thus

$$\begin{aligned}
L_n(E) = \int_0^1 u_n(\theta)d\theta &\geq \frac{y_0}{2}\int_0^1 u_n(\theta)\Big(\sum_{k\in\mathbb{Z}} \frac{y_0}{y_0^2 + (\theta + k)^2}\Big)d\theta \\
&\geq \frac{y_0}{2}\Big(\log(\lambda\epsilon - mC) - \frac{\hat{C}\delta}{\rho}\log\lambda\Big) \\
&\geq \frac{\delta}{4}\Big(\big(1 - \frac{\hat{C}\delta}{\rho}\big)\log\lambda + \log\epsilon\Big).
\end{aligned}$$

By the setting of λ_0 and $\delta \ll \rho$, for any n it holds that

$$L_n(E) > \frac{\delta}{4}\Big(\frac{1}{2}\log\lambda - \frac{1}{100}\log\lambda_0\Big) > c\log\lambda$$

with some small constant c depending on all v_{ij}. The proof is complete. □

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