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PERIODIC OSCILLATIONS OF THE RELATIVISTIC PENDULUM WITH FRICTION

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Abstract. We consider the existence and multiplicity of periodic oscillations for the forced pendulum model with relativistic effects by using the Poincaré-Miranda theorem. Some detailed information about the bound for the period of forcing term is obtained. To support our analytical work, we also consider a forced pendulum oscillator with the special force $\gamma_0 \sin(\omega t)$ including a sufficiently small parameter. The result shows us that for all $\omega \in (0, +\infty)$, there exists a $2\pi/\omega$ periodic solution under our settings.

1. Introduction

In this article, we consider the existence and multiplicity of periodic oscillations for the forced pendulum model with relativistic effects

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{c^2}}}\right)' + kx' + a\sin x = p(t), \tag{1.1}$$

where $c > 0$ is the speed of light in the vacuum, $k > 0$ is a possible viscous friction coefficient and p is a continuous and T-periodic forcing term with mean value zero

$$\bar{p} = \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t = 0.$$

This equation has received much attention as a prototype of equation with singular ϕ-Laplacian (see [5] and [1, 3, 6]). An essential difference between the relativistic and the newtonian ($c = +\infty$) case has been explained in [9]. In [9], Torres proved the following theorem.

Theorem 1.1. *Let us assume that $2cT \leq 1$. For any a, k and any continuous T-periodic function $p(t)$ with mean value zero,* (1.1) *has at least one T-periodic solution.*

The proof of the above theorem is an interesting application of the Schauder fixed point theorem. The bound in Theorem 1.1 was improved to $2cT \leq 4\sqrt{3} \approx 6.9282$ in [10] for the general pendulum-type equation was considered, see also [2, 4].

Motivated by [9], in this paper we give detailed information on the bounds for T which depend on the parameters a, k and the forcing term p. Without loss of

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generality, we assume $a \geq 0$; otherwise we only require replacing x with $x + \pi$. Let us define

$$\|p\|_\infty = \sup_{t \in [0,T]} |p(t)|,$$

and the constant

$$c_* = \frac{c\big(kTc_* + 3k\pi + 2(a + \|p\|_\infty)T\big)}{\sqrt{c^2 + \big(kTc_* + 3k\pi + 2(a + \|p\|_\infty)T\big)^2}} < c. \tag{1.2}$$

Our main result reads as follows.

Theorem 1.2. *For any values a, k and for any continuous and T-periodic function $p(t)$ with mean value zero satisfying $2c_*T \leq 2\pi$, (1.1) has at least two distinct T-periodic solutions.*

The proof of Theorem 1.2 is an elementary application of a variation of the Poincaré-Miranda theorem (see [11]) which will be given in the next section. We remark that the two distinct T-periodic solutions in Theorem 1.2 are indeed geometrically different periodic solutions, which generalizes Theorem 1.1. Moreover, when k or $\|p\|_\infty$ tend to infinity, we show that $c_* \to c$ so that $2cT \leq 2\pi$. This case does not improve the previous bound.

To support our analytical work, based on the method of averaging, we also consider the existence of periodic oscillations for a special forced pendulum oscillator with a sufficiently small parameter ε,

$$\Big(\frac{x'}{\sqrt{1 - \frac{x'^2}{c^2}}}\Big)' + \varepsilon^2 kx' + a\sin x = \varepsilon^3 \gamma_0 \sin(\omega t), \tag{1.3}$$

where $\omega^2 = a + \varepsilon^2 \beta_0$ with $\beta_0 > 0$. We summarize our results as follows.

Theorem 1.3. *For any γ_0, k, $\beta_0 > 0$ and $\omega > 0$, (1.3) has at least one $2\pi/\omega$-periodic solution when ε is sufficiently small. Moreover, this periodic solution is stable for $k > 0$ and is unstable for $k < 0$.*

Noticed that $T = 2\pi/\omega \to +\infty$ as $\omega \to 0$. Thus, in this case, (1.3) does not meet the hypotheses of Theorem 1.1. From (1.2) we also see that $c_* \to 0$ when $\varepsilon \to 0$, satisfying the hypotheses of Theorem 1.2.

In Section 2, we introduce a variation of the Poincaré-Miranda theorem in two-dimensional case which is used to prove Theorem 1.2. We prove Theorem 1.2 in Section 3. In the last section, we prove Theorem 1.3 using the method of averaging and perform some numerical simulations.

2. A variation of the Poincaré-Miranda theorem

We first introduce a variation of the Poincaré-Miranda theorem (see [8, 11] for instance) in two-dimensional case, which goes back to Poincaré (1883) and has been used many times in the study of boundary value problems and the existence of periodic solutions. For an example see [7] and the references therein.

Consider the closed rectangle

$$D = \{(x, y) \in \mathbb{R}^2 : \alpha_1 \leq x \leq \alpha_2, \beta_1 \leq y \leq \beta_2\},$$

where α_i, β_i $(i = 1, 2)$ are constants such that $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$. The boundary of the rectangle consists of four faces as follows:

$$\begin{aligned} V_-^1 &= \{(x,y) \in \mathbb{R}^2 : x = \alpha_1, \beta_1 \le y \le \beta_2\}, \\ V_+^1 &= \{(x,y) \in \mathbb{R}^2 : x = \alpha_2, \beta_1 \le y \le \beta_2\}, \\ V_-^2 &= \{(x,y) \in \mathbb{R}^2 : y = \beta_1, \alpha_1 \le x \le \alpha_2\}, \\ V_+^2 &= \{(x,y) \in \mathbb{R}^2 : y = \beta_2, \alpha_1 \le x \le \alpha_2\}. \end{aligned}$$

We say that a continuous map $F = (F_1, F_2) : D \to \mathbb{R}^2$ satisfies the *bend-twist condition* on D provided that

$$F_1(V_-^1)F_1(V_+^1) \le 0, \quad F_2(V_-^2)F_2(V_+^2) \le 0$$

or

$$F_2(V_-^1)F_2(V_+^1) \le 0, \quad F_1(V_-^2)F_1(V_+^2) \le 0,$$

where $F_j(V_-^i)F_j(V_+^i) \le 0$ means that $F_j(V_-^i) \le 0$ and $F_j(V_+^i) \ge 0$, or $F_j(V_-^i) \ge 0$ and $F_j(V_+^i) \le 0$; $F_j(V_\pm^i) < 0$ (resp. $F_j(V_\pm^i) > 0$) means that $F_j(x,y) \le 0$ (resp. $F_j(x,y) \ge 0$) for all $(x,y) \in V_\pm^i$ and there exists at least $(x_0,y_0) \in V_\pm^i$ such that $F_j(x_0,y_0) < 0$ (resp. $F_j(x_0,y_0) > 0$); and $F_j(V_\pm^i) = 0$ means that $F_j(x,y) = 0$ for all $(x,y) \in V_\pm^i$, $i,j = 1,2$.

Theorem 2.1 (See [11, Theorem 2.1])**.** *Assume the continuous map $F : D \to \mathbb{R}^2$ satisfies the bend-twist condition, then there exists at least one point $(x_0,y_0) \in D$ such that $F(x_0,y_0) = 0$.*

3. Proof of Theorem 1.2

Equation (1.1) is equivalent to the plane system

$$x' = \frac{c(y-kx)}{\sqrt{c^2 + (y-kx)^2}}, \tag{3.1}$$

$$y' = -a \sin x + p(t). \tag{3.2}$$

Let α, β be positive constants and

$$\|p\|_\infty = \sup_{t \in [0,T]} |p(t)|, \quad \lambda = kT, \quad \mu = \beta + k\alpha + (a + \|p\|_\infty)T.$$

Define $\phi : (-\infty, +\infty) \to (-c, c)$ by

$$\phi(u) = \frac{cu}{\sqrt{c^2 + u^2}}.$$

It is easy to verify that ϕ is an increasing homeomorphism such that $\phi(-u) = -\phi(u)$.

Lemma 3.1. *Assume that $p(t)$ is a continuous T-periodic function. Then for any values a, k and any initial value $(x_0,y_0) \in \big\{(x,y)\big|\, |x| \le \alpha, |y| \le \beta \text{ and } \alpha, \beta > 0\big\}$, the solution $(x(t;x_0,y_0), y(t;x_0,y_0))$ of (3.1)-(3.2) with the initial value (x_0,y_0) satisfies*

$$|x'(t)| \le c_*(\alpha,\beta) < c, \quad \forall t \in [0,T],$$

where c_*, depending on α, β, is a solution of $u = \phi(\lambda u + \mu)$.

Proof. First we note that $|x'(t)| \le c_1 := c$ for all $t \in [0,T]$. Hence $|x(t)| \le \alpha + c_1 T$ for all $t \in [0,T]$, and by (3.2) we see that $|y(t)| \le \beta + (a + \|p\|_\infty)T$ for all $t \in [0,T]$. Therefore,

$$|y(t) - kx(t)| < k\alpha + \beta + (a + c_1 k + \|p\|_\infty)T = \lambda c_1 + \mu, \quad \forall t \in [0,T].$$

Let $c_2 := \phi(\lambda c_1 + \mu)$. By (3.2), we have $|x'(t)| \le c_2$ for all $t \in [0,T]$. Obviously, $c_2 < c_1$. Repeating this argument we have a sequence $\{c_n\}_{n\in\mathbb{N}}$ defined by $c_n = \phi(\lambda c_{n-1} + \mu)$.

Since ϕ is an increasing homeomorphism and $c_2 < c_1$, we know that $c_3 = \phi(\lambda c_2 + \mu) < \phi(\lambda c_1 + \mu) = c_2, \dots, c_n = \phi(\lambda c_{n-1} + \mu) < \phi(\lambda c_{n-2} + \mu) = c_{n-1}, \dots$. That is, $\{c_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. On the other hand, $|c_n| = |\phi(\lambda c_{n-1} + \mu)| < c$. Hence $\{c_n\}_{n\in\mathbb{N}}$ converges to some value $c_* < \infty$. Since ϕ is continuous, by passing we have $c_* = \phi(\lambda c_* + \mu)$, that is

$$\begin{aligned} c_* &= \phi(\lambda c_* + \mu) \\ &= \frac{c\big(kTc_* + \beta + k\alpha + (a + \|p\|_\infty)T\big)}{\sqrt{c^2 + \big(kTc_* + \beta + k\alpha + (a + \|p\|_\infty)T\big)^2}}. \end{aligned}$$

□

Proof of Theorem 1.2. Let $\gamma = \frac{3}{2}k\pi + (a + \|p\|_\infty)T$ and $c_* = c_*(3\pi/2, \gamma)$. Let us construct a rectangle as follows

$$D_1 = \{(x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} \le x \le \frac{\pi}{2}, -\gamma \le y \le \gamma\}.$$

The boundary is of D_1 is given by

$$\begin{gathered} V_-^1 = \{(x,y) \in D_1 : x = -\frac{\pi}{2}\}, \\ V_+^1 = \{(x,y) \in D_1 : x = \frac{\pi}{2}\} \\ V_-^2 = \{(x,y) \in D_1 : y = -\gamma\}, \\ V_+^2 = \{(x,y) \in D_1 : y = \gamma\}. \end{gathered}$$

Let $(x(t;x_0,y_0), y(t;x_0,y_0))$ be the solution of (3.1) and (3.2) with the initial value $(x_0,y_0) \in D_1$. Define the continuous mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x_0,y_0) = \begin{pmatrix} F_1(x_0,y_0) \\ F_2(x_0,y_0) \end{pmatrix} = (P - \mathrm{id})(x_0,y_0),$$

where P denotes the Poincaré mapping associated with system (3.1)-(3.2).

(i) When $(x_0,y_0) \in V_-^1$, using Lemma 3.1 we know that

$$|x'(t)| < c_*, \quad \forall t \in [0,T].$$

Then it follows that

$$-\frac{\pi}{2} - c_* t \le x(t) \le -\frac{\pi}{2} + c_* t, \quad \forall t \in [0,T].$$

When $t \in [0, \pi/c_*]$, we know that

$$-\frac{\pi}{2} \le -\frac{c_* t}{2} \le \frac{x(t) + \frac{\pi}{2}}{2} \le \frac{c_* t}{2} \le \frac{\pi}{2}, \quad \forall t \in [0,T].$$

Then it follows that, for any $t \in [0, \pi/c_*]$,

$$\Big(\cos\frac{c_* t}{2}\Big)^2 \le \Big(\cos\frac{x(t)+\frac{\pi}{2}}{2}\Big)^2 \le 1.$$

Therefore,

$$\begin{aligned}\int_0^{\pi/c_*}[-\sin x(t)]\,\mathrm{d}t &= \int_0^{\pi/c_*}\cos\big[x(t)+\frac{\pi}{2}\big]\,\mathrm{d}t\\ &= \int_0^{\pi/c_*}\big[2\big(\cos\frac{x(t)+\frac{\pi}{2}}{2}\big)^2-1\big]\,\mathrm{d}t\\ &\ge \int_0^{\pi/c_*}\big[2\big(\cos\frac{c_* t}{2}\big)^2-1\big]\,\mathrm{d}t = 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}y(T)-y(0) &= \int_0^T[-a\sin x(t)]\,\mathrm{d}t\\ &= \int_0^{\pi/c_*}[-a\sin x(t)]\,\mathrm{d}t + \int_{\pi/c_*}^T[-a\sin x(t)]\,\mathrm{d}t\\ &\ge -a\big(T-\frac{\pi}{c_*}\big) \ge 0.\end{aligned} \tag{3.3}$$

The above inequality is obtained by the hypothesis $2c_*T \le 2\pi$.

When $(x_0, y_0) \in V_+^1$, using the same arguments, we have

$$\frac{\pi}{2} - c_* t < x(t) < \frac{\pi}{2} + c_* t, \quad \forall t \in [0, T].$$

When $t \in [0, \pi/c_*]$, we know that

$$\frac{\pi}{2} \le \pi - \frac{c_* t}{2} \le \frac{x(t)+\frac{3\pi}{2}}{2} \le \pi + \frac{c_* t}{2} \le \frac{3\pi}{2}, \quad \forall t \in [0, T].$$

Similarly, for any $t \in [0, \pi/c_*]$, we have

$$\big(\cos\frac{c_* t}{2}\big)^2 \le \big(\cos\frac{x(t)+\frac{3\pi}{2}}{2}\big)^2 \le 1.$$

Therefore,

$$\begin{aligned}\int_0^{\pi/c_*}[-\sin x(t)]\,\mathrm{d}t &= -\int_0^{\pi/c_*}\cos\big[x(t)+\frac{3\pi}{2}\big]\,\mathrm{d}t\\ &= -\int_0^{\pi/c_*}\big[2\big(\cos\frac{x(t)+\frac{3\pi}{2}}{2}\big)^2-1\big]\,\mathrm{d}t\\ &\le -\int_0^{\pi/c_*}\big[2\big(\cos\frac{c_* t}{2}\big)^2-1\big]\,\mathrm{d}t = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}y(T)-y(0) &= \int_0^T[-a\sin x(t)]\,\mathrm{d}t\\ &= \int_0^{\pi/c_*}[-a\sin x(t)]\,\mathrm{d}t + \int_{\pi/c_*}^T[-a\sin x(t)]\,\mathrm{d}t\\ &\le a\big(T-\frac{\pi}{c_*}\big) \le 0.\end{aligned} \tag{3.4}$$

The last inequality is obtained by the hypothesis $2c_*T \leq 2\pi$. From (3.3) and (3.4), we have that $F_2(V_-^1)F_2(V_+^1) \leq 0$.

(ii) When $(x_0, y_0) \in V_-^2$, using the inequality $|x'(t)| \leq c_*$ we know that, for all $t \in [0,T]$,

$$-\frac{3\pi}{2} \leq -c_*T - \frac{\pi}{2} \leq x(t) \leq \frac{\pi}{2} + c_*T \leq \frac{3\pi}{2}.$$

Then using (3.2) we know that

$$\begin{aligned} y(t) - kx(t) &= y_0 + \int_0^t (-a\sin x(s) + p(s))\,\mathrm{d}s - kx(t) \\ &\leq -\gamma + (a + \|p\|_\infty)T + \frac{3}{2}k\pi = 0, \quad t \in [0,T]. \end{aligned}$$

Since ϕ is a continuous homeomorphism such that $\phi(0) = 0$, we have $\phi(u)u \geq 0$. Then it follows that $x'(t) = \phi(y(t) - kx(t)) \leq 0$ for all $t \in [0,T]$, which yields

$$x(T) - x(0) = \int_0^T x'(\tau)\,\mathrm{d}\tau \leq 0. \tag{3.5}$$

When $(x_0, y_0) \in V_+^2$, we also know that for all $t \in [0,T]$, $|x(t)| \leq \frac{3\pi}{2}$, and

$$\begin{aligned} y(t) - kx(t) &= y_0 + \int_0^t (-a\sin x(s) + p(s))\,\mathrm{d}s - kx(t) \\ &\geq \gamma - (a + \|p\|_\infty)T - \frac{3}{2}k\pi = 0, \quad t \in [0,T]. \end{aligned}$$

With the same arguments we have $x(T) - x(0) \geq 0$. Therefore,

$$F_1(V_-^2)F_1(V_+^2) \leq 0.$$

We have verified that F satisfies the bend-twist condition on D_1. By Theorem 1.1, there exists at least one point $(x_1, y_1) \in D_1$ such that $F(x_1, y_1) = 0$, which is corresponding to a fixed point of the Poincaré mapping.

Similarly, we can construct the rectangle

$$D_2 = \{(x,y) \in \mathbb{R}^2 : \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}, -\gamma \leq y \leq \gamma\}.$$

With the same arguments, we can verify that F satisfies the bend-twist condition on D_2 and obtain another fixed point of the Pincaré mapping in D_2.

Let $V = D_1 \cap D_2$. To prove that such two fixed points of F are distinct, it is sufficient to prove that there is no T-periodic solution with the initial value on V. Assume that $(x(t; \frac{\pi}{2}, y_0), y(t; \frac{\pi}{2}, y_0))$ is a T-periodic solution of (3.1) and (3.2). Then we know that $\{x(t; \frac{\pi}{2}, y_0)\big| t \in [0,T]\}$ is contained in $[0,\pi]$, since the maximum of the derivative of $x(t)$ is c_* and $c_*T \leq \pi$. Then we have

$$y(T) - y(0) = \int_0^T [-a\sin x(t)]\,\mathrm{d}t \leq \int_0^{\pi/(3c_*)} [-a\sin x(t)]\,\mathrm{d}t < 0.$$

Therefore, we obtain two distinct fixed points, which are corresponding to two distinct T-periodic solutions of equation (1.1). □

4. Numerical examples and proof of Theorem 1.3

First we prove Theorem 1.3 by using the method of averaging. Recall that

$$\Big(\frac{x'}{\sqrt{1-\frac{x'^2}{c^2}}}\Big)' + \varepsilon^2 kx' + a\sin x = \varepsilon^3\gamma_0 \sin(\omega t), \tag{4.1}$$

where $\omega^2 = a + \varepsilon^2\beta_0$ and ε is a small parameter.

Equation (4.1) is equivalent to the plane system

$$\begin{gathered} x' = \frac{c(y-\varepsilon^2 kx)}{\sqrt{c^2+(y-\varepsilon^2 kx)^2}}, \\ y' = -a\sin x + \varepsilon^3\gamma_0\sin(\omega t). \end{gathered} \tag{4.2}$$

Let $x = \varepsilon u$, $y = \varepsilon v$ and $\epsilon = \varepsilon^2$. We expand system (4.2) into the form of power series by

$$\begin{gathered} u' = v + \epsilon f_1(u,v,t) = v - \Big(ku + \frac{1}{2}c^{-2}v^3\Big)\epsilon + O(\epsilon^2), \\ y' = -\omega^2 u + \epsilon f_2(u,v,t) = -\omega^2 u + \beta_0 u\epsilon + \frac{1}{6}\omega^2 u^3\epsilon + \epsilon\gamma_0\sin\omega t + O(\epsilon^2). \end{gathered} \tag{4.3}$$

Using the van der Pol transformation

$$u = q\sin\omega t + p\cos\omega t, \quad v = \omega(q\cos\omega t - p\sin\omega t),$$

we obtain

$$\begin{gathered} q' = \epsilon\Big(f_1(u,v,t)\sin\omega t + \frac{\cos\omega t}{\omega} f_2(u,v,t)\Big) + O(\epsilon^2), \\ p' = \epsilon\Big(f_1(u,v,t)\cos\omega t - \frac{\sin\omega t}{\omega} f_2(u,v,t)\Big) + O(\epsilon^2). \end{gathered} \tag{4.4}$$

Then it follows that

$$\begin{aligned} q' &= \epsilon F_1(q,p,t,\epsilon) \\ &= \frac{1}{48c^2\omega}\Big(\omega\Big(9p(p^2+q^2)\omega^3 + 3c^2\big(-8kq + p(p^2+q^2)\omega\big) \\ &\quad + 4\big(-3p^3\omega^3 + c^2(6kq + p^3\omega)\big)\cos(2\omega t) + p(p^2-3q^2)\omega(c^2+3\omega^2)\cos(4\omega t) \\ &\quad + 2\big(-3q(3p^2+q^2)\omega^3 + c^2(-12kp + 3p^2q\omega + q^3\omega)\big)\sin(2\omega t) \\ &\quad - q(-3p^2+q^2)\omega(c^2+3\omega^2)\sin(4\omega t)\Big) \\ &\quad + 24c^2\big((p + p\cos(2\omega t) + q\sin(2\omega t))\beta_0 + \sin(2\omega t)\gamma_0\big)\Big)) + O(\epsilon^2), \end{aligned}$$

and

$$\begin{aligned} p' &= \epsilon F_2(q,p,t,\epsilon) \\ &= \frac{1}{48c^2\omega}\Big(\omega\Big(-9q(p^2+q^2)\omega^3 - 3c^2\big(8kp + q(p^2+q^2)\omega\big) \\ &\quad + 4\big(-3q^3\omega^3 + c^2(-6kp + q^3\omega)\big)\cos(2\omega t) - q(-3p^2+q^2)\omega(c^2+3\omega^2)\cos(4\omega t) \\ &\quad - 2\Big(-3p(p^2+3q^2)\omega^3 + c^2\big(p^3\omega + 3q(4k + pq\omega)\big)\Big)\sin(2\omega t) \\ &\quad - p(p^2-3q^2)\omega(c^2+3\omega^2)\sin(4\omega t)\Big) \\ &\quad - 48c^2\sin(\omega t)\big(p\cos(\omega t)\beta_0 + \sin(\omega t)(q\beta_0+\gamma_0)\big)\Big) + O(\epsilon^2). \end{aligned}$$

It is not difficult to obtain the averaging system

$$\begin{aligned}
\bar{q}' &= \epsilon G_1(\bar{q},\bar{p}) \\
&= \epsilon \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} F_1(\bar{q},\bar{p},t,0)\,\mathrm{d}t \\
&= \epsilon \frac{1}{16}\omega\Big(p(p^2+q^2) - \frac{8kq}{\omega} + \frac{3p(p^2+q^2)\omega^2}{c^2} + \frac{8p\beta_0}{\omega^2}\Big), \\
\bar{p}' &= \epsilon G_2(\bar{q},\bar{p}) \\
&= \epsilon \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} F_2(\bar{q},\bar{p},t,0)\,\mathrm{d}t \\
&= -\epsilon \frac{\omega\big(3q(p^2+q^2)\omega^3 + c^2\big(8kp + q(p^2+q^2)\omega\big)\big) + 8c^2(q\beta_0+\gamma_0)}{16c^2\omega}.
\end{aligned} \tag{4.5}$$

The other equilibrium points of system (4.5) correspond to the solutions of

$$G_1(\bar{q},\bar{p}) = \frac{1}{16c^2}\Big(\omega c^2 pr^2 - 8kc^2 q + 3\omega^3 pr^2 + \frac{8c^2\beta_0}{\omega}p\Big) = 0,$$
$$G_2(\bar{q},\bar{p}) = -\frac{1}{16c^2}\Big(\omega c^2 qr^2 + 8kc^2 p + 3\omega^3 qr^2 + \frac{8c^2\beta_0}{\omega}q + \frac{8c^2\gamma_0}{\omega}\Big) = 0,$$

where $r^2 = q^2 + p^2$, which is equivalent to

$$q = -\Big(\frac{\omega^2(c^2+3\omega^2)}{8c^2\gamma_0}r^2 + \frac{\beta_0}{\gamma_0}\Big)r^2, \quad p = -\frac{k\omega}{\gamma_0}r^2.$$

Thus, r satisfies

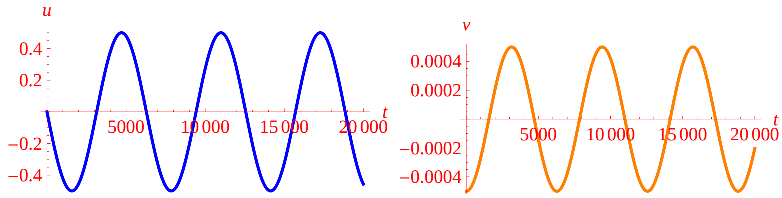
$$\Phi(r) = \Big(\frac{\omega^2(c^2+3\omega^2)}{8c^2\gamma_0}r^2 + \frac{\beta_0}{\gamma_0}\Big)^2 r^2 + \big(\frac{k\omega}{\gamma_0}\big)r^2 - 1 = 0.$$

Since $\Phi(0) = -1 < 0$ and $\Phi(+\infty) = +\infty$, by the intermediate value theorem we know that there is a $r_* \in (0,+\infty)$ such that $\Phi(r_*) = 0$. The Jacobi determinant of (G_1, G_2) at r_* is

$$\begin{aligned}
J &= \frac{\partial(G_1,G_2)}{\partial(q,p)}\Big|_{(q,p)=(q(r_*),p(r_*))} \\
&= \frac{k^2}{4} + \frac{3p^4\omega^2}{256} + \frac{3}{128}p^2q^2\omega^2 + \frac{3q^4\omega^2}{256} + \frac{9p^4\omega^4}{128c^2} \\
&\quad + \frac{9p^2q^2\omega^4}{64c^2} + \frac{9q^4\omega^4}{128c^2} + \frac{27p^4\omega^6}{256c^4} + \frac{27p^2q^2\omega^6}{128c^4} \\
&\quad + \frac{27q^4\omega^6}{256c^4} + \frac{p^2\beta_0}{8} + \frac{q^2\beta_0}{8} + \frac{3p^2\omega^2\beta_0}{8c^2} + \frac{3q^2\omega^2\beta_0}{8c^2} + \frac{\beta_0^2}{4\omega^2} > 0,
\end{aligned}$$

for $\beta_0 > 0$. The Jacobi Matrix has a pair of conjugate imaginary eigenvalues $\lambda_{1,2}$ which satisfy that $\mathfrak{Re}(\lambda_{1,2}) = -k/2$. With the classical arguments of averaging theory, we know that system (4.4) has a $2\pi/\omega$-periodic solution (q_ϵ, p_ϵ) such that $(q_\epsilon, p_\epsilon) \to (q(r_*), p(r_*))$ as $\epsilon \to 0$, which yields a $2\pi/\omega$-periodic solution $x(t)$ of (4.1). The periodic solution $x(t)$ is stable for $k > 0$, while it is unstable for $k < 0$. Now we have finished the proof of Theorem 1.3.

To support our analytical work, we numerically simulate the $2\pi/\omega$-periodic solution of (4.3) for $\omega = 0.001, k = 1, \beta_0 = 2, \gamma_0 = 1, c = 100$. We obtain that the rest point of (4.4) is $(q(r_*), p(r_*)) = (-0.50000, -0.00025)$, the Jacobi determinant of (G_1, G_2) at $(q(r_*), p(r_*))$ is 1.0×10^6 and the corresponding eigenvalues are



(a) (b)

FIGURE 1. Profiles of the $2\pi/\omega$-periodic solution (u, v) of (4.3) with $\omega = 10^{-3}$, $k = 1$, $\beta_0 = 2$, $\gamma_0 = 1$, $c = 100$, $\epsilon = 10^{-5}$.

$\lambda_{1,2} = -0.5 \pm 1000\mathrm{i}$. We depict the corresponding stable $2\pi/\omega$-periodic solution of (4.3) in figure 1.

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