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GROUND STATE SOLUTIONS FOR p-BIHARMONIC EQUATIONS

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Abstract. In this article we study the p-biharmonic equation

$$\Delta_p^2 u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator. When $V(x)$ and $f(x,u)$ satisfy some conditions, we prove that the above equations have Nehari-type ground state solutions.

1. Introduction and statement of main results

In this article, we study the p-biharmonic equation

$$\Delta_p^2 u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $p \geq 2$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is an operator of fourth order, the so-called p-biharmonic operator. Equation (1.1), especially with $p = 2$, has attracted growing interests and figures in a variety of applications. Many authors studied the existence of at least one solution and infinitely many solutions, ground state solution, sign-changing solutions and least energy nodal solution for biharmonic equations; we refer readers to [4, 5, 13, 14, 16, 18, 20, 21].

When $p > 2$, Equation (1.1) becomes an interesting topic and it arises from mathematical modeling of non Newtonian fluids and elastic mechanics, in particular, the electro-rheological fluids. This important class of fluids is characterized by the change of viscosity which is not easy and depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in elastic mechanics, fluid dynamics etc.. For more information, the reader can refer to Ruzicka [8].

Recently many authors have studied the ground state solutions of various types equations including biharmonic equations, see [5, 9, 10, 11, 12, 17, 19]. The so-called ground state solutions are the solutions that have the least energy. But for p-biharmonic equations, there are just few papers that have studied the existence of nontrivial solutions, see [1, 3, 7] and sign-changing solutions, see [6]. And there is no paper studying the ground state solutions of p-biharmonic equations until now.

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In a recent paper Chen and Tang [2] studied the existence of ground state solutions for p-superlinear p-Laplacian equations by using the following assumptions on $V(x)$ and $f(x,u)$:

(A1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and
$$0 < \inf_{x\in\mathbb{R}^N} V(x) \le \sup_{x\in\mathbb{R}^N} V(x) < +\infty;$$

(A2) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and
$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{p^*-1}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N,$$
where $p^* = pN/(N-p)$ if $N > p$ and $p^* \in (p, +\infty)$ if $N \le p$;

(A3) $\lim_{|t|\to 0} \frac{|f(x,t)|}{|t|^{p-1}} < \gamma_p^{-p}$ uniformly in $x \in \mathbb{R}^N$, where $\gamma_s = \sup_{u\in E, \|u\|=1} \|u\|_s$ for $p \le s \le p^*$ and $tf(x,t) - pF(x,t) = o(|t|^p)$ as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$;

(A4) $\lim_{|t|\to\infty} \frac{|F(x,t)|}{|t|^p} = \infty$ for almost every $x \in \mathbb{R}^N$, and there exists $r_0 \ge 0$ such that $F(x,t) \ge 0$ for $|t| \ge r_0$;

(A5) there exists a $\theta_0 \in (0,1)$ such that
$$\frac{1-\theta^p}{p} tf(x,t) \ge \int_{\theta t}^{t} f(x,s)ds, \quad \forall (x,t) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}, \theta \in [0, \theta_0].$$

The solutions obtained in [2] are in the set $\mathcal{M} = \{u \in E\backslash\{0\} : \Phi'(u) = 0\}$ which may contain only one element. It is a very small subset of the Nehari manifold
$$\mathcal{N} = \{u \in E\backslash\{0\} : \langle \Phi'(u), u\rangle = 0\},$$
which contains infinitely many elements of E. The main difference between our arguments and those in [2] is that their solutions are in $\mathcal{M}$, while ours are in $\mathcal{N}$. In fact, for any $u \in E\backslash\{0\}$, there exists $t = t(u) > 0$ such that $t(u)u \in \mathcal{N}$, see Lemma 2.2. If u_0 is a solution at which Φ (Φ is the corresponding functional) has least "energy" in set $\mathcal{N}$, we shall call it a Nehari-type ground state solution.

Motivated by the above facts, we shall use new techniques to establish the existence of Nehari-type ground state solutions of (1.1). To state our results, we make the following assumptions:

(A6) there exists $p < q < p^*$ such that
$$\frac{1-t^q}{q} uf(x,u) \ge \int_{tu}^{u} f(x,s)ds, \quad \forall (x,u) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}, t \ge 0;$$

(A6') there exist $p < q < p^*$ and $K \ge 1$ such that
$$\frac{1-t^q}{q} uf(x,u) \ge F(x,u) - KF(x,tu), \quad \forall (x,u) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}, t \ge 0;$$

(A7) $pF(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}$.

Our main results read as follows.

Theorem 1.1. *Assume that* (A1)–(A4), (A6), (A7) *are satisfied. Then* (1.1) *has a Nehari-type ground state solution.*

Obviously, we see that $(A6')$ implies $(A6)$, then we have the following corollary.

Corollary 1.2. *Assume that* (A1)–(A4), (A6'), (A7) *are satisfied. Then* (1.1) *has a Nehari-type ground state solution.*

The remainder of this article is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of our main results.

2. Preliminaries

Throughout this article, in $L^s(\mathbb{R}^N)$ the norm is $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{1/s}$, and positive constants are denoted C_i. As usual, we let

$$E = \{u \in W^{2,p}(\mathbb{R}^N) \cap W_0^{1,p}(\mathbb{R}^N) | \int_{\mathbb{R}^N} (|\Delta u|^p + V(x)|u|^p) dx < \infty\}.$$

Then, by condition (A1), E is a Sobolev space, with norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\Delta u|^p + V(x)|u|^p) dx \Big)^{1/p}$$

And define

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\Delta u|^p + V(x)|u|^p) dx - \int_{\mathbb{R}^N} F(x,u) dx, \tag{2.1}$$

obviously, the solutions of (1.1) are the critical points of the functional Φ, and it is easy to see that $\Phi \in C^1(E,\mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} (|\Delta u|^{p-2} \Delta u \Delta v + V(x)|u|^{p-2} uv)\, dx - \int_{\mathbb{R}^N} f(x,u) v\, dx, \tag{2.2}$$

and define the Nehari manifold

$$\mathcal{N} = \{u \in E\backslash\{0\} : \langle \Phi'(u), u\rangle = 0\}. \tag{2.3}$$

To prove Theorem 1.1, we use the well known arguments involving the Nehari manifold. So we give the following lemmas.

Lemma 2.1 ([2, Lemma 2.1]). *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma = \{\gamma \in C(M,X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X,\mathbb{R})$ satisfies

$$\infty > b := \inf_{\gamma\in\Gamma} \sup_{t\in M} \varphi(\gamma(t)) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{t\in M_0} \varphi(\gamma_0(t)), \tag{2.4}$$

then there exists a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to b, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.5}$$

Lemma 2.2. *Assume that* (A1)–(A3) *are satisfied. Then for any $u \in E\backslash\{0\}$, there exists $t(u) > 0$ such that $t(u)u \in \mathcal{N}$.*

Proof. Let $u \in E\backslash\{0\}$ be fixed and define the function $g(t) := \Phi(tu)$ on $[0,\infty)$. Obviously, we have

$$g'(t) = 0 \Leftrightarrow tu \in \mathcal{N} \Leftrightarrow \|u\|^p = \frac{1}{t^{p-1}} \int_{\mathbb{R}^N} f(x,tu) u dx.$$

Using (A2) and (A3), fix $p < q < p^*$, there exist $\epsilon_0 > 0$, $\epsilon > 0$ and $C_\epsilon > 0$ such that

$$|f(x,t)| \le \frac{1}{\gamma_p^p + \epsilon_0} |t|^{p-1} + \epsilon |t|^{p^*-1} + C_\epsilon |t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.6}$$

$$|F(x,t)| \le \frac{1}{p(\gamma_p^p + \epsilon_0)} |t|^p + \frac{\epsilon}{p^*} |t|^{p^*} + \frac{C_\epsilon}{q} |t|^q, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.7}$$

Combining this and it is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\in[0,\infty)} g(t)$ is achieved at a $t = t(u)$ so that $g'(t(u)) = 0$ and $t(u)u \in \mathcal{N}$. □

Lemma 2.3. *Assume that* (A1)–(A3), (A6) *are satisfied. Then for* $u \in \mathcal{N}$*, it holds*

$$\Phi(u) \geq \Phi(tu), \qquad t \in [0, \infty). \tag{2.8}$$

Proof. For $u \in \mathcal{N}$, one has

$$\|u\|^p = \int_{\mathbb{R}^N} f(x, u)u dx. \tag{2.9}$$

Thus, by (2.1), (2.9) and (A6), there exists $p < q < p^*$ such that

$$\begin{aligned}
\Phi(u) - \Phi(tu) &= \frac{1 - t^p}{p}\|u\|^p + \int_{\mathbb{R}^N} [F(x, tu) - F(x, u)]dx \\
&\geq \frac{1 - t^p}{p}\|u\|^p - \frac{1 - t^q}{q}\int_{\mathbb{R}^N} f(x, u)udx \\
&= \frac{1 - t^p}{p}\|u\|^p + \frac{t^q - 1}{q}\|u\|^p \\
&= (\frac{1 - t^p}{p} + \frac{t^q - 1}{q})\|u\|^p.
\end{aligned}$$

It is easy to verify that

$$\frac{1 - t^p}{p} + \frac{t^q - 1}{q} \geq 0, \quad \forall t \geq 0, 2 \leq p < q < p^*.$$

Then (2.8) holds. □

Define

$$c_1 := \inf_{\mathcal{N}} \Phi, \quad c_2 := \inf_{u\in E\setminus\{0\}} \max_{t\geq 0} \Phi(tu), \quad c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}$.

Lemma 2.4. *Assume that* (A1)–(A3), (A7) *are satisfied. Then* $c_1 = c_2 = c > 0$ *and there exists a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.10}$$

Proof. (1) Both Lemmas 2.2 and 2.3 imply that $c_1 = c_2$. Next, we prove that $c = c_1 = c_2$. By the definition of c_2, we can choose a sequence $\{u_n\} \in E\setminus\{0\}$ such that

$$c_2 \leq \max_{t\geq 0} \Phi(tu_n) < c_2 + \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{2.11}$$

For $u \in E\setminus\{0\}$ and t large enough, we have $\Phi(tu) < 0$, and then there exist $t_n = t(u_n) > 0$ and $s_n > t_n$ such that

$$\Phi(t_n u_n) = \max_{t\geq 0} \Phi(tu_n), \quad \Phi(s_n u_n) < 0, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

Let $\gamma_n(t) = ts_n u_n$, $t \in [0, 1]$, then $\gamma_n \in \Gamma$. And it follows from (2.11) and (2.12) that

$$\sup_{t\in[0,1]} \Phi(\gamma_n(t)) = \max_{t\geq 0} \Phi(tu_n) < c_2 + \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

which implies that $c \leq c_2$. On the other hand, the manifold $\mathcal{N}$ separates E into two components: $E^+ = \{u \in E : \langle \Phi'(u), u\rangle > 0\} \cup \{0\}$ and $E^- = \{u \in E : \langle \Phi'(u), u\rangle < 0\}$.

Combining (A7) with (2.1) and (2.2), we obtain

$$\Phi(u) \geq \frac{1}{p}\langle \Phi'(u), u\rangle, \quad \forall u \in E.$$

It follows that $\Phi(u) \geq 0$ for all $u \in E^+$. Since (A2) and (A3), it follows that (2.6) implies that E^+ contains a small ball around the origin. Thus every $\gamma \in \Gamma$ has to cross $\mathcal{N}$, because $\gamma(0) \in E^+$ and $\gamma(1) \in E^-$. So $c_1 \leq c$. The proof of part (1) is complete.

(2) To prove the second part of Lemma 2.4, we apply Lemma 2.1 with $M = [0, 1]$, $M_0 = \{0, 1\}$, and $\Gamma_0 = \{\gamma_0 : \{0, 1\} \to E : \gamma_0(0) = 0, \Phi(\gamma_0(1)) < 0\}$. By (A2) and (A3), it is easy to verify that there exists $r > 0$ such that $\min_{\|u\|\leq r} \Phi(u) = 0$, $\inf_{\|u\|=r} \Phi(u) > 0$. Hence we obtain

$$c \geq \inf_{\|u\|=r} \Phi(u) > 0 = \sup_{\gamma_0\in\Gamma_0} \sup_{t\in M_0} \Phi(\gamma_0(t)).$$

As a consequence, all assumptions of Lemma 2.1 are satisfied. Therefore, there exists a sequence $\{u_n\} \subset E$ satisfying (2.10). □

Lemma 2.5. *Assume* (A1)–(A4), (A6) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \quad \langle \Phi'(u_n), u_n\rangle \to 0 \tag{2.13}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, we argue by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $p \leq s < p^*$ and $v_n \to v$ almost everywhere on $\mathbb{R}^N$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^p dx = 0,$$

then by Lions' concentration compactness principle [15, Lemma 1.21], $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for $p < s < p^*$. Fix $R > [p(c + 1)(\gamma_p^p + \epsilon_0)/\epsilon_0]^{1/p}$, $\epsilon = p^*/[4(R\gamma_{p^*})^{p^*}] > 0$, it follows from (2.7) that

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n)dx &\leq \frac{(R\gamma_p)^p}{p(\gamma_p^p + \epsilon_0)} + \frac{\epsilon(R\gamma_{p^*})^{p^*}}{p^*} + \frac{R^q C_\epsilon}{q} \lim_{n\to\infty} \|v_n\|_q^q \\ &= \frac{(R\gamma_p)^p}{p(\gamma_p^p + \epsilon_0)} + \frac{1}{4}.\end{aligned} \tag{2.14}$$

Since $\|u_n\| \to \infty$, $R/\|u_n\| \in [0, 1)$ for large $n \in \mathbb{N}$. Hence using (2.13), (2.14) and Lemma 2.3,

$$\begin{aligned}c + o(1) = \Phi(u_n) &\geq \Phi(Rv_n) \\ &= \frac{R^p}{p} - \int_{\mathbb{R}^n} F(x, Rv_n)dx \\ &\geq \frac{\epsilon_0 R^p}{p(\gamma_p^p + \epsilon_0)} - \frac{1}{4} + o(1) \\ &> \frac{3}{4} + c + o(1),\end{aligned}$$

which is a contradiction. Thus, $\delta > 0$.

Going if necessary to a subsequence, we assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |v_n|^p dx > \delta/2$. Let $\omega_n(x) = v_n(x + k_n)$. Then

$$\int_{B_{1+\sqrt{N}}(0)} |\omega_n|^p dx > \frac{\delta}{2}. \tag{2.15}$$

Now we define $\widetilde{u}_n(x) = u_n(x + k_n)$, then $\|\widetilde{u}_n\| = \|u_n\|$ and $\widetilde{u}_n/\|u_n\| = \omega_n$. Passing to a subsequence, we have $\omega_n \rightharpoonup \omega$ in E, $\omega_n \to \omega$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $p \le s < p^*$ and $\omega_n \to \omega$ almost everywhere on $\mathbb{R}^N$. Thus (2.15) implies that $\omega \neq 0$.

By (A2) and (A3), there exists $C_1 > 0$ such that

$$|f(x,t)| \le \frac{1}{\gamma_p^p}|t|^{p-1} + C_1|t|^{p^*-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

which implies

$$|F(x,t)| \le \frac{1}{p\gamma_p^p}|t|^p + \frac{C_1}{p^*}|t|^{p^*}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.16}$$

For $0 \le a < b$, let $\Omega_n(a,b) = \{x \in \mathbb{R}^N : a \le |\widetilde{u}_n(x)| < b\}$. Set $A := \{x \in \mathbb{R}^N : \omega(x) \neq 0\}$, then $\operatorname{meas}(A) > 0$. For almost every $x \in A$, we have $\lim_{n\to\infty} |\widetilde{u}_n(x)| = \infty$. Hence $A \subset \Omega_n(r_n, \infty)$ for large $n \in \mathbb{N}$, it follows from (2.1), (2.13), (2.16), (A4) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^p} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^p} \\
&= \lim_{n\to\infty} [\frac{1}{p} - \int_{\mathbb{R}^N} \frac{F(x,\widetilde{u}_n)}{|\widetilde{u}_n|^p} |\omega_n|^p dx] \\
&= \lim_{n\to\infty} [\frac{1}{p} - \int_{\Omega_n(0,r_0)} \frac{F(x,\widetilde{u}_n)}{|\widetilde{u}_n|^p} |\omega_n|^p dx - \int_{\Omega_n(r_0,\infty)} \frac{F(x,\widetilde{u}_n)}{|\widetilde{u}_n|^p} |\omega_n|^p dx] \\
&\le \limsup_{n\to\infty} [\frac{1}{p} + (\frac{1}{p\gamma_p^p} + \frac{C_1}{p^*} r_0^{p^*-p}) \int_{\mathbb{R}^N} |\omega_n|^p dx - \int_{\Omega_n(r_0,\infty)} \frac{F(x,\widetilde{u}_n)}{|\widetilde{u}_n|^p} |\omega_n|^p dx] \\
&\le \frac{1}{p} + (\frac{1}{p\gamma_p^p} + \frac{C_1}{p^*} r_0^{p^*-p}) \gamma_p^p - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{F(x,\widetilde{u}_n)}{|\widetilde{u}_n|^p} |\omega_n|^p dx \\
&= \frac{1}{p} + (\frac{1}{p\gamma_p^p} + \frac{C_1}{p^*} r_0^{p^*-p}) \gamma_p^p - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,\widetilde{u}_n)}{|\widetilde{u}_n|^p} [\chi_{\Omega_n(r_0,\infty)}(x)] |\omega_n|^p dx \\
&\le \frac{1}{p} + (\frac{1}{p\gamma_p^p} + \frac{C_1}{p^*} r_0^{p^*-p}) \gamma_p^p - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,\widetilde{u}_n)}{|\widetilde{u}_n|^p} [\chi_{\Omega_n(r_0,\infty)}(x)] |\omega_n|^p dx \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. □

For the proof of Theorem 1.1, we need one more lemma.

Lemma 2.6. *Under assumptions* (A1)–(A4), (A6), (A7)*, equation* (1.1) *has a nontrivial solution, that is,* $\mathcal{N} \neq \emptyset$.

The proof of the above lemma is similar to that of [2, Lemma 2.8] so it is omitted.

3. Proof of main results

Proof of Theorem 1.1. Lemma 2.6 shows that $\mathcal{N}$ is not empty. By Lemma 2.3 and $c_1 = \inf_{\mathcal{N}} \Phi$, one has $\Phi(u) \geq \Phi(0) = 0$ for all $u \in \mathcal{N}$. Let $\{u_n\} \subset \mathcal{N}$ such that $\Phi(u_n) \to c$, then $\langle \Phi'(u_n), u_n \rangle = 0$. In view of the proof of Lemma 2.5, $\{u_n\}$ is bounded in E, and

$$\|u_n\|^p = \int_{\mathbb{R}^N} f(x, u_n) u_n \, dx.$$

Let $\inf_{n\in\mathbb{N}} \|u_n\| = \delta_0$. If $\delta_0 = 0$, going if necessary to a subsequence, we may assume that $\|u_n\| \to 0$. Fix $q \in (p, p^*)$, by $(A2)$ and $(A3)$, there exist $\epsilon_0 > 0$ and $C_2 > 0$ such that

$$|f(x,t)| \leq \frac{1}{\gamma_p^p + \epsilon_0} |t|^{p-1} + |t|^{p^*-1} + C_2 |t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus,

$$\begin{aligned}
\|u_n\|^p &= \int_{\mathbb{R}^N} f(x, u_n) u_n \, dx \\
&\leq \frac{1}{\gamma_p^p + \epsilon_0} \|u_n\|_p^p + \|u_n\|_{p^*}^{p^*} + C_2 \|u_n\|_q^q \\
&\leq \frac{\gamma_p^p}{\gamma_p^p + \epsilon_0} \|u_n\|^p + \gamma_{p^*}^{p^*} \|u_n\|^{p^*} + C_2 \gamma_q^q \|u_n\|^q,
\end{aligned}$$

which implies

$$\frac{\epsilon_0}{\gamma_p^p + \epsilon_0} \leq \gamma_{p^*}^{p^*} \|u_n\|^{p^*-p} + C_2 \gamma_q^q \|u_n\|^{q-p} = o(1).$$

This contradiction shows that $\inf_{n\in\mathbb{N}} \|u_n\| = \delta_0 > 0$. Choose a constant $C_3 > 0$ such that $\|u_n\|_{p^*} \leq C_3$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^p dx = 0,$$

then by Lions' concentration compactness principle [15, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $p < s < p^*$. Fix $q \in (p, p^*)$, by (A2) and (A3), for $\epsilon = \epsilon_0 \delta_0^p / [2(\gamma_p^p + \epsilon_0) C_3^{p^*}] > 0$, there exists $C_\epsilon > 0$ such that

$$|f(x,t)| \leq \frac{1}{\gamma_p^p + \epsilon_0} |t|^{p-1} + \epsilon |t|^{p^*-1} + C_\epsilon |t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus,

$$\|u_n\|^p = \int_{\mathbb{R}^N} f(x, u_n) u_n \, dx \leq \frac{\gamma_p^p}{\gamma_p^p + \epsilon_0} \|u_n\|^p + \epsilon \|u_n\|_{p^*}^{p^*} + C_\epsilon \|u_n\|_q^q,$$

which yields

$$\begin{aligned}
\frac{\epsilon_0 \delta_0^p}{\gamma_p^p + \epsilon_0} &\leq \frac{\epsilon_0}{\gamma_p^p + \epsilon_0} \|u_n\|^p \\
&\leq \epsilon \|u_n\|_{p^*}^{p^*} + C_\epsilon \|u_n\|_q^q \leq \epsilon C_3^{p^*} + o(1) \\
&= \frac{\epsilon_0 \delta_0^p}{2(\gamma_p^p + \epsilon_0)} + o(1).
\end{aligned}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^p dx > \delta/2$. Let us define $v_n(x) = u_n(x+k_n)$ so that

$$\int_{B_{1+\sqrt{N}}(0)} |v_n|^p dx > \frac{\delta}{2}. \tag{3.1}$$

Since $V(x)$ and $f(x,u)$ are periodic, we have $\|v_n\| = \|u_n\|$ and by (2.1), (2.2) and (2.10), we have

$$\Phi(v_n) \to c, \quad \Phi'(v_n) = 0. \tag{3.2}$$

Passing to a subsequence, we have $v_n \rightharpoonup v_0$ in E, $v_n \to v_0$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $p \le s < p^*$ and $v_n \to v_0$ almost everywhere on $\mathbb{R}^N$. Thus (3.1) implies that $v_0 \ne 0$. For every $\omega \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \Phi'(v_0), \omega\rangle = \lim_{n\to\infty} \langle \Phi'(v_n), \omega\rangle = 0.$$

Hence $\Phi'(v_0) = 0$. This shows that $v_0 \in \mathcal{N}$ and so $\Phi(v_0) \ge c$. On the other hand, by (2.1), (2.2), (3.2) and Fatou's lemma,

$$\begin{aligned}
c &= \lim_{n\to\infty} [\Phi(v_n) - \frac{1}{p}\langle \Phi'(v_n), v_n\rangle] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} [\frac{1}{p} f(x,v_n)v_n - F(x,v_n)] dx \\
&\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} [\frac{1}{p} f(x,v_n)v_n - F(x,v_n)] dx \\
&= \int_{\mathbb{R}^N} [\frac{1}{p} f(x,v_0)v_0 - F(x,v_0)] dx \\
&= \Phi(v_0) - \frac{1}{p}\langle \Phi'(v_0), v_0\rangle = \Phi(v_0).
\end{aligned}$$

This shows that $\Phi(v_0) \le c$ and so $\Phi(v_0) = c = \inf_{\mathcal{N}} \Phi$. The proof is complete. □

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