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POINTWISE BOUNDS FOR POSITIVE SUPERSOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS INVOLVING THE p-LAPLACIAN

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ABSTRACT. We derive a priori bounds for positive supersolutions of $-\Delta_p u = \rho(x)f(u)$, where $p > 1$ and Δ_p is the p-Laplace operator, in a smooth bounded domain of $\mathbb{R}^N$ with zero Dirichlet boundary conditions. We apply our results to the nonlinear elliptic eigenvalue problem $-\Delta_p u = \lambda f(u)$, with Dirichlet boundary condition, where f is a nondecreasing continuous differentiable function on such that $f(0) > 0$, $f(t)^{1/(p-1)}$ is superlinear at infinity, and give sharp upper and lower bounds for the extremal parameter λ_p^*. In particular, we consider the nonlinearities $f(u) = e^u$ and $f(u) = (1 + u)^m$ $(m > p - 1)$ and give explicit estimates on λ_p^*. As a by-product of our results, we obtain a lower bound for the principal eigenvalue of the p-Laplacian that improves obtained results in the recent literature for some range of p and N.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and $p > 1$. We consider the nonlinear elliptic problem

$$\begin{gathered} -\Delta_p u = \rho(x)f(u) \quad x \in \Omega, \\ u \geqslant 0 \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega \end{gathered} \tag{1.1}$$

where Δ_p is the p-Laplace operator defined by $\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$, $\rho : \Omega \to \mathbb{R}$ is a nonnegative bounded measurable function that is not identically zero and f satisfies

(A1) $f : D_f = [0, a_f) \to \mathbb{R}^+ := [0, \infty)$ $(0 < a_f \leqslant +\infty)$ is a nondecreasing C^1 function with $f(u) > 0$ for $u > 0$.

We say that u is a solution of (1.1) if $u \in W_0^{1,p}(\Omega)$, $u \in [0, a_f)$, $\rho(x)f(u) \in L^1(\Omega)$, and

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi = \int_\Omega \rho(x)f(u)\varphi, \quad \text{for all } \varphi \in C_c^\infty(\Omega),$$

that is, for all C^∞ functions φ with compact support in Ω. Note that, since u is p-superharmonic we have that if $u \not\equiv 0$ then $u > 0$ a.e. in Ω, by the strong

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maximum principle (see [9, 23, 25, 26]). A solution $u \in W_0^{1,p}(\Omega)$ is called a regular solution of (1.1) if $\rho(x)f(u) \in L^\infty(\Omega)$. By the well-known regularity results for degenerate elliptic equations, if u is a regular solution of (1.1) then $u \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1]$ (see for instance [9, 22]). Also, we say that $u \in W_0^{1,p}(\Omega)$ is a supersolution of (1.1) if $u \in [0, a_f)$, $\rho(x)f(u) \in L^1(\Omega)$ and $-\Delta_p u \geqslant \rho(x)f(u)$ in the weak sense. Reversing the inequality one defines the notion of subsolution.

The ball of radius R centered at x_0 in $\mathbb{R}^N$ will be denoted by $B_R(x_0)$. Given a set $\Omega \subseteq \mathbb{R}^N$, we denote by $|\Omega|$ its N-dimensional Lebesgue measure. The *p-torsion* function ψ of a domain Ω is the unique solution of the problem

$$\begin{gathered} -\Delta_p u = 1 \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega. \end{gathered}$$

We shall denote $\psi_M := \sup_{x\in\Omega}\ \psi(x)$.

In this paper, first we consider C^1 positive supersolutions u of (1.1) in section 2 (by a positive solution we mean a solution which is nonnegative and nontrivial) and give explicit pointwise lower bounds for u under the condition that f satisfies $(\mathcal{C})$ and $f^{-1/(p-1)} \in L^1(0,a)$ for all $a \in (0, a_f)$. In particular, we prove that

$$F\big(u(x)\big) \geqslant \frac{p-1}{p}\ \Big(\frac{\rho_x\big(d_\Omega(x)\big) d_\Omega^p(x)}{N}\Big)^{1/(p-1)} \quad \text{for all } x \in \Omega,$$

where

$$\begin{gathered} F(t) = \int_0^t \frac{\mathrm{d}s}{f(s)^{1/(p-1)}}, \quad 0 < t < a_f, \\ \rho_x(r) = \inf\big\{\rho(y) : |y-x| < r\big\}, \quad d_\Omega(x) := \operatorname{dist}(x, \partial\Omega). \end{gathered}$$

As an application, in section 3, we consider the eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda f(u) \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

where f satisfies (A1). We define the extremal parameter

$$\lambda_p^* = \lambda_p^*(f,\Omega) := \sup\big\{\lambda > 0 :\ (1.2) \text{ has at least one positive bounded solution.}\big\}.$$

Under the additional assumption

- (A2) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is C^1, $f(0) > 0$ and $f(t)^{1/(p-1)}$ is superlinear at infinity (i.e., $\lim_{t\to\infty} f(t)/t^{p-1} = \infty$),

Cabré and Sanchón in [9, Theorem 1.4] proved that $\lambda_p^* \in (0,\infty)$ and for every $\lambda \in (0, \lambda_p^*)$ problem (1.2) admits a minimal regular solution u_λ. Minimal means that it is smaller than any other supersolution of the problem. If, in addition, $f(t)^{1/(p-1)}$ is a convex function satisfying $\int_0^\infty f(s)^{-1/(p-1)}\mathrm{d}s < \infty$, then (1.2) admits no solution for $\lambda > \lambda_p^*(f,\Omega)$. Moreover, the family $\{u_\lambda\}$ is increasing in λ and every u_λ is semi-stable in the sense that the second variation of the energy functional associated with (1.2) is nonnegative definite [9, Definition 1.1]. Using this property in [9] the authors established that $u^* = \lim_{\lambda \nearrow \lambda_p^*} u_\lambda$ is a solution of (1.2) with $\lambda = \lambda_p^*$ whenever $\liminf_{t\to\infty} tf'(t)/f(t) > p-1$; u^* is called the extremal solution.

Let $\lambda_1 = \lambda(p,\Omega)$ be the first eigenvalue of p-Laplacian subjected to Dirichlet boundary condition; i.e.,

$$\lambda_1 := \min_{0 \neq v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p \mathrm{d}x}{\int_\Omega |v|^p \mathrm{d}x}. \tag{1.3}$$

Azorero, Peral and Puel [17] showed that if $f(u) = e^u$ then

$$\lambda_p^* \leqslant \max\big\{\lambda_1, \lambda_1\big(\frac{p-1}{e}\big)^{p-1}\big\}.$$

Cabré and Sanchón [9] extended this result for every nonlinearity f satisfying (A2), as

$$\lambda_p^* \leqslant \max\big\{\lambda_1, \lambda_1 \sup_{t\geqslant 0} \frac{t^{p-1}}{f(t)}\big\}. \tag{1.4}$$

In both proofs the authors (by a contradiction argument) used a comparison principle for the p-Laplacian operator to construct, for every $\varepsilon > 0$ sufficiently small, an increasing sequence of functions whose limit is in $W_0^{1,p}(\Omega)$ and solves the problem $-\Delta_p w = (\lambda_1 + \varepsilon) w^{p-1}$, then used the fact that the first eigenvalue for the p-Laplacian is isolated to get a contradiction.

Before presenting our estimates on λ_p^*, first we improve (1.4) as follows (using the homogeneity property of p-Laplacian and (1.4)).

$$\lambda_p^* \leqslant \lambda_1 \sup_{t\geqslant 0} \frac{t^{p-1}}{f(t)}. \tag{1.5}$$

Then we prove the following upper bound, without using the fact that the first eigenvalue for the p-Laplacian is isolated,

$$\lambda_p^* \leqslant \frac{1}{\psi_M^{p-1}} \Big(\int_0^\infty \frac{\mathrm{d}s}{f(s)^{1/(p-1)}}\Big)^{p-1},$$

where ψ_M as defined before is the supremum (maximum) of the p-torsion function on Ω. As we shall see, in many cases, this represents a sharper upper bound than (1.5).

While there is no explicit formula for the lower bound in the literature for the critical parameter λ_p^* ($p \neq 2$), which is very important in application, we shall prove the following lower bound for the extremal parameter of problem (1.2) with general nonlinearity f satisfying (A1), using the method of sub-super solution,

$$\lambda_p^* \geqslant \max\big\{\frac{1}{\psi_M^{p-1}} \sup_{0<t<a_f} \frac{t^{p-1}}{f(t)}, \sup_{0<\alpha<\frac{\|F\|_\infty}{\psi_M}} \alpha^{p-1} - \alpha^p \beta(\alpha)\big\},$$

where

$$\beta(\alpha) := \sup_{x\in\Omega} f'\big(F^{-1}(\alpha\psi(x))\big) f\big(F^{-1}(\alpha\psi(x))\big)^{\frac{2-p}{p-1}} |\nabla\psi(x)|^p,$$

$$\|F\|_\infty = \int_0^{a_f} \frac{\mathrm{d}s}{f(s)^{1/(p-1)}}.$$

In particular, if $\Omega = B$ the unit ball in $\mathbb{R}^N$ centered at the origin, then we have

$$\lambda_p^* \geqslant \max\Big\{N\big(\frac{p}{p-1}\big)^{p-1} \sup_{0<t<a_f} \frac{t^{p-1}}{f(t)}, \big(\frac{p}{p-1}\big)^{p-1} N \sup_{0<\alpha<\|F\|_\infty} \gamma(\alpha)\Big\}, \tag{1.6}$$

where

$$\gamma(\alpha) := \alpha^{p-1}\Big(1 - \frac{p}{(p-1)N} \sup_{0<t<a_f} f'(t) f(t)^{\frac{2-p}{p-1}} \big(\alpha - F(t)\big)\Big).$$

As we shall see, the lower bound (1.6), in some dimensions, gives the exact value of the extremal parameter for the standard nonlinearities $f(u) = e^u$ and $f(u) =$

$(1+u)^m$ with $(m > p-1)$. Moreover, when $p = 2$ the above bounds coincide with those given in [2]. For example for the nonlinearity $f(u) = e^u$ our results give

$$Np^{p-1} \geqslant \lambda_p^*(e^u, B) \geqslant \begin{cases} (\frac{p}{e})^{p-1}N & N \leqslant \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)}, \\ (\frac{p-1}{p})^{p-1}\frac{N^p}{p} & \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)} < N \leqslant \frac{p^2}{p-1}, \\ p^{p-1}(N-p) & N > \frac{p^2}{p-1}. \end{cases}$$

Also we show that our results can be used to estimate the first eigenvalue of p-Laplacian from below. As it mentioned in [15], while upper bounds for $\lambda_1(\Omega)$ can be obtained by choosing particular test function v in (1.3), but lower bounds are more challenging. For more details on estimates and asymptotic behavior of the principal eigenvalue and eigenfunction of the p-Laplacian operator, we refer the reader to [3, 4, 5, 15]. For example when $\Omega = B$ we shall prove the following lower bound, which is better than those given in [3, 4, 15], for some range of p and N (see the end of Section 3).

$$\lambda_1(B) \geqslant \begin{cases} (\frac{p}{p-1})^{p-1}N & N \leqslant \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)}, \\ (\frac{e}{p})^{p-1}\frac{N^p}{p} & \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)} < N \leqslant \frac{p^2}{p-1}, \\ (\frac{pe}{p-1})^{p-1}(N-p) & N > \frac{p^2}{p-1}. \end{cases}$$

Finally in section 4, as an another application, we give a nonexistence result for positive supersolutions of (1.1) and apply this result to obtain upper bound for the pull-in voltage of a simple Micro-Electromechanical-Systems MEMS device.

2. Bounds for positive supersolutions of problem (1.1)

In this section we consider positive supersolutions of problem (1.1) and give pointwise lower bounds independent of any given supersolution under consideration. The following simple lemma is useful in making bounds for solutions. The case $p = 2$ is a variant of Kato's inequality used in [6, 7], see [6, Lemma 1.7] and [7, Lemma 2].

Lemma 2.1. *Let $G : (0, a) \to \mathbb{R}^+$ $(a \leqslant \infty)$ be an increasing concave C^2 function and u a continuously differentiable function on Ω with $0 < u(x) < a$ for $x \in \Omega$. Then we have*

$$-\Delta_p G(u) \geq G'(u)^{p-1}(-\Delta_p u), \ x \in \Omega,$$

in the weak sense.

Proof. For simplicity, we assume that u is a C^2 function in Ω. By smoothing u and a standard argument one can prove it for a C^1 function u. Using the definition of Δ_p, the product rule for the divergence of product of a scalar valued function and a vector field, $G' > 0$ and $G'' \leq 0$ we simply compute

$$\begin{aligned} \Delta_p G(u) &= \operatorname{div}\Big(|\nabla G(u)|^{p-2}\nabla G(u)\Big) \\ &= \operatorname{div}\Big(G'(u)^{p-1}|\nabla u|^{p-2}\nabla u\Big) \\ &= \nabla\Big(G'(u)^{p-1}\Big)\cdot|\nabla u|^{p-2}\nabla u + G'(u)^{p-1}\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) \\ &= (p-1)G''(u)G'(u)^{p-2}\nabla u\cdot|\nabla u|^{p-2}\nabla u + G'(u)^{p-1}\Delta_p u \end{aligned}$$

$$= (p-1)G''(u)G'(u)^{p-2}|\nabla u|^p + G'(u)^{p-1}\Delta_p u \le G'(u)^{p-1}\Delta_p u$$

as desired. □

Now let ψ_ρ be the unique solution of the equation

$$\begin{gathered} -\Delta_p u = \rho(x) \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{gathered} \tag{2.1}$$

where $\rho(x)$ is a bounded measurable function. If $\rho \equiv 1$ then $\psi_1 = \psi$ is the *p-torsion* function of Ω as in Section 1. Recall the definition

$$\rho_x(r) := \inf_{y \in B_r(x)} \rho(y) \quad 0 < r \leqslant d_\Omega(x) = \operatorname{dist}(x, \partial\Omega).$$

Theorem 2.2. *Let u be a C^1 positive supersolution of problem* (1.1) *where f satisfies* (A1) *and $f^{1/(p-1)} \in L^1(0,a)$ for all $0 < a < a_f$. Then*

$$F\big(u(x)\big) \geqslant \psi_\rho(x), \quad x \in \Omega, \tag{2.2}$$

where $F(0) = 0$ and $F(t) = \int_0^t \frac{\mathrm{d}s}{f(s)^{1/(p-1)}}$, $t \in (0, a_f)$, and ψ_ρ defined in (2.1). *Moreover, we have*

$$F\big(u(y)\big) \geqslant \frac{p-1}{p} \rho_x\big(d_\Omega(x)\big)^{1/(p-1)} \frac{d_\Omega(x)^{\frac{p}{p-1}} - |x-y|^{\frac{p}{p-1}}}{N^{1/(p-1)}}, \quad |y-x| < d_\Omega(x). \tag{2.3}$$

In particular,

$$F\big(u(x)\big) \geqslant \frac{p-1}{p} \Big(\frac{\rho_x\big(d_\Omega(x)\big) d_\Omega^p(x)}{N} \Big)^{1/(p-1)} \quad \text{for all } x \in \Omega. \tag{2.4}$$

Proof. First note that by the assumptions on f and definition of F we have $F'(t) = \frac{1}{f(t)^{1/(p-1)}} > 0$ and $F''(t) = \frac{-f'(t)}{(p-1)f(t)^{\frac{p}{p-1}}} \leqslant 0$, $0 < t < a_f$, thus using Lemma 2.1 (with $G = F$ and $a = a_f$) and the fact that u is a supersolution, we can write

$$-\Delta_p F(u) \ge F'(u)^{p-1}(-\Delta_p u) = \frac{1}{f(u)}(-\Delta_p u) \ge \rho(x) = -\Delta_p \psi_\rho.$$

Now since we have $F(u) = \psi_\rho = 0$ on $\partial\Omega$, by the maximum principle we obtain $F\big(u(x)\big) \geqslant \psi_\rho(x)$, $x \in \Omega$ that proves (2.2).

To prove (2.3) we need to estimate ψ_ρ from below. Let $x \in \Omega$. Then for $y \in B_{d_\Omega(x)}(x)$, from (2.1), we obtain

$$-\Delta_p \psi_\rho(y) = \rho(y) \geqslant \rho_x\big(d_\Omega(x)\big). \tag{2.5}$$

Now consider the auxiliary function

$$w(y) = \big(\frac{p-1}{p}\big) \frac{d_\Omega(x)^{\frac{p}{p-1}} - |x-y|^{\frac{p}{p-1}}}{N^{1/(p-1)}}$$

which satisfies $-\Delta_p w = 1$ in $B_{d_\Omega(x)}(x)$ and $w = 0$ on $\partial B_{d_\Omega(x)}(x)$. Then from (2.5) we obtain

$$-\Delta_p \psi_\rho(y) \geqslant -\Delta_p \Big(\rho_x\big(d_\Omega(x)\big)^{1/(p-1)} w(y) \Big),$$

hence by the maximum principle $\psi_\rho(y) \geqslant \rho_x\big(d_\Omega(x)\big)^{1/(p-1)} w(y)$ in $B_{d_\Omega(x)}(x)$ that with the aid of (2.2) proves (2.3). Taking $y = x$ in (2.3) gives (2.4). □

3. Application to eigenvalue problems

3.1. **Lower and upper bounds for $\lambda_p^*(f,\Omega)$.** Consider the nonlinear eigenvalue problem (1.2). Before presenting our results based on Theorem 2.2, first we improve the upper bound (1.4) for the extremal parameter $\lambda_p^*(f,\Omega)$ where f satisfies (A2), in the following lemma using the homogeneity property of p-Laplacian and (1.4).

Lemma 3.1. *For the extremal parameter of problem* (1.2) *where f satisfies* (A2), *we have*

$$\lambda_p^* \leqslant \lambda_1 \sup_{t\geqslant 0} \frac{t^{p-1}}{f(t)}. \tag{3.1}$$

Proof. Assume that for some $\lambda > 0$, u_λ is the minimal solution of (1.2) and take an arbitrary positive number $M \in (0,\infty)$. Then it is easy to see that the function $w := Mu_\lambda$ is a bounded solution of the equation

$$\begin{gathered} -\Delta_p w = M^{p-1}\lambda g(w) \quad x \in \Omega, \\ w = 0 \quad x \in \partial\Omega, \end{gathered}$$

where $g(u) := f(\frac{u}{M})$. Hence from (1.4) we must have

$$M^{p-1}\lambda \leqslant \max\big\{\lambda_1, \lambda_1 \sup_{t\geqslant 0} \frac{t^{p-1}}{g(t)}\big\}. \tag{3.2}$$

However, we have

$$\sup_{t\geqslant 0} \frac{t^{p-1}}{g(t)} = M^{p-1} \sup_{t\geqslant 0} \frac{t^{p-1}}{f(t)},$$

thus from (3.2) we obtain

$$\lambda \leqslant \max\big\{\frac{\lambda_1}{M^{p-1}}, \lambda_1 \sup_{t\geqslant 0} \frac{t^{p-1}}{f(t)}\big\}. \tag{3.3}$$

Now for M sufficiently large, from (3.3), we obtain

$$\lambda \leqslant \lambda_1 \sup_{t\geqslant 0} \frac{t^{p-1}}{f(t)},$$

which proves (3.1). □

Theorem 3.2. *Let λ_p^* be the extremal parameter of problem* (1.2) *where f satisfies* (A1) *and $f(0) > 0$. Then*

$$\lambda_p^* \leqslant \frac{1}{\psi_M^{p-1}}\Big(\int_0^{a_f} \frac{\mathrm{d}s}{f(s)^{1/(p-1)}}\Big)^{p-1}, \tag{3.4}$$

$$\lambda_p^* \geqslant \max\big\{\frac{1}{\psi_M^{p-1}} \sup_{0<t<a_f} \frac{t^{p-1}}{f(t)}, \sup_{0<\alpha<\frac{\|F\|_\infty}{\psi_M}} \alpha^{p-1} - \alpha^p\beta(\alpha)\big\}, \tag{3.5}$$

where $\beta(\alpha) := \sup_{x\in\Omega} f'\Big(F^{-1}\big(\alpha\psi(x)\big)\Big) f\Big(F^{-1}\big(\alpha\psi(x)\big)\Big)^{\frac{2-p}{p-1}} |\nabla\psi(x)|^p$.

In particular, if $\Omega = B$ the unit ball in $\mathbb{R}^N$, then we have

$$\lambda_p^* \geqslant \max\big\{N\big(\frac{p}{p-1}\big)^{p-1} \sup_{0<t<a_f} \frac{t^{p-1}}{f(t)}, \big(\frac{p}{p-1}\big)^{p-1} N \sup_{0<\alpha<\|F\|_\infty} \gamma(\alpha)\big\}, \tag{3.6}$$

where

$$\gamma(\alpha) := \alpha^{p-1}\Big(1 - \frac{p}{(p-1)N} \sup_{0<s<F^{-1}(\alpha)} f'(s) f(s)^{\frac{2-p}{p-1}}\big(\alpha - F(s)\big)\Big).$$

Proof. From Theorem 2.2 (and, of course, with $\rho \equiv 1$ and f replaced by λf) we have $F\big(u_\lambda(x)\big) \geqslant \lambda^{1/(p-1)}\psi(x)$, $x \in \Omega$, thus

$$\lambda^{1/(p-1)} \leqslant \frac{1}{\psi_M}\int_0^{u_\lambda(x_0)} \frac{\mathrm{d}s}{f(t)^{1/(p-1)}} \leqslant \frac{1}{\psi_M}\int_0^{a_f} \frac{\mathrm{d}s}{f(t)^{1/(p-1)}},$$

that proves (3.4).

We prove (3.5) by the method of sub-supersolution. We construct a supersolution of (1.2) in the form $\bar u = \alpha\psi$ where $\alpha > 0$ is a scalar to be chosen later. We require that

$$\Delta_p \bar u + \lambda f(\bar u) = -\alpha^{p-1} + \lambda f(\alpha\psi) \leqslant 0, \quad \text{in } \Omega.$$

Since f is nondecreasing this is satisfied if $\lambda \leqslant \frac{\alpha^{p-1}}{f(\alpha\psi_M)}$ and making the optimal choice of α we obtain the sufficient condition that

$$\lambda \leqslant \frac{1}{\psi_M^{p-1}} \sup_{0<t<a_f} \frac{t^{p-1}}{f(t)}.$$

On the other hand, $\underline{u} = 0$ is an allowable subsolution (note that we have $f(0) > 0$), now [9, Proposition 2.1] implies that problem (1.2) has a positive bounded solution, hence

$$\lambda_p^* \geqslant \frac{1}{\psi_M^{p-1}} \sup_{0<t<a_f} \frac{t^{p-1}}{f(t)}. \tag{3.7}$$

Now we show that for $\alpha \in (0, \frac{\|F\|_\infty}{\psi_M})$ the function $\bar{\bar u}(x) = F^{-1}\big(\alpha\psi(x)\big)$ is a supersolution of (1.2) for $\lambda = \alpha^{p-1} - \alpha^p\beta(\alpha)$. To do this we simply compute $\Delta_p \bar{\bar u}(x)$, using the facts that if we take $y(t) := F^{-1}(\alpha t)$ then $\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha f(y)^{1/(p-1)}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\alpha^2}{p-1} f'(y) f(y)^{\frac{3-p}{p-1}}$. We have

$$\begin{aligned}
\Delta_p \bar{\bar u}(x) &= \Big(\alpha^p f'(\bar{\bar u}) f(\bar{\bar u})^{\frac{2-p}{p-1}} |\nabla\psi(x)|^p - \alpha^{p-1}\Big) f(\bar{\bar u}) \\
&\leqslant \Big(\alpha^p \sup_{x\in\Omega} f'(\bar{\bar u}) f(\bar{\bar u})^{\frac{2-p}{p-1}} |\nabla\psi(x)|^p - \alpha^{p-1}\Big) f(\bar{\bar u}) \\
&= -\Big(\alpha^{p-1} - \alpha^p\beta(\alpha)\Big) f(\bar{\bar u}).
\end{aligned}$$

In other words, $\Delta_p \bar{\bar u}(x) + \big(\alpha^{p-1} - \alpha^p\beta(\alpha)\big) f(\bar{\bar u}) \leqslant 0$, and since we have $\bar{\bar u}(x) = 0$, $x \in \partial\Omega$, this shows that $\bar{\bar u}$ is a supersolution of (1.2) for $\lambda = \alpha^{p-1} - \alpha^p\beta(\alpha)$. Using again the fact that $\underline{u} = 0$ is an allowable subsolution and [9, Proposition 2.1], we infer that problem (1.2) with $\lambda = \alpha^{p-1} - \alpha^p\beta(\alpha)$ has a positive bounded solution, hence

$$\lambda_p^* \geqslant \alpha^{p-1} - \alpha^p\beta(\alpha).$$

Taking the supremum over $\alpha \in (0, \frac{\|F\|_\infty}{\psi_M})$ and combining it with (3.7), we obtain (3.5).

If $\Omega = B$ the unit ball of $\mathbb{R}^N$, then we have the explicit formula $\psi(x) = (\frac{p-1}{p})\frac{1}{N^{1/(p-1)}}(1 - |x|^{\frac{p}{p-1}})$, hence $\psi_M = \frac{p-1}{p} N^{-1/(p-1)}$ and $|\nabla\psi(x)|^p = N^{\frac{-p}{p-1}}|x|^{\frac{p}{p-1}}$. Taking $s = F^{-1}(\alpha\psi(x))$ and make the change $\alpha \to \frac{pN^{1/(p-1)}}{p-1}\alpha$ in (3.5) we arrive at (3.6). $\square$

Now we compare (3.1) with the upper bound for λ_p^* in Theorem 3.2. First note that from (3.1) and (3.5) we obtain

$$\frac{1}{\psi_M^{p-1}} \leqslant \lambda_1. \tag{3.8}$$

Also, since f is nondecreasing we have

$$\|F\|_\infty^{p-1} = \Big(\int_0^{a_f} \frac{\mathrm{d}s}{f(s)^{1/(p-1)}}\Big)^{p-1} \geqslant \sup_{0<t<a_f} \frac{t^{p-1}}{f(t)} := \alpha_{f,p}.$$

Thus generally (3.4) is better than (3.1) if $\|F\|_\infty^{p-1} < \lambda_1 \alpha_{f,p} \psi_M^{p-1}$. However, in high dimension (3.4) is much better than (3.1), as one can show by the known results that $\lambda_1 \psi_M^{p-1} \to \infty$ when $N \to \infty$. For example, from [15, 21] if Ω is a ball B_R of radius R then $\lambda_1(B_R) \geqslant (\frac{N}{pR})^p$, and since $\psi_M(B_R) = R^{\frac{p}{p-1}}(\frac{p-1}{p})N^{\frac{-1}{p-1}}$, then we have

$$\lambda_1 \psi_M^{p-1} \geqslant \frac{(p-1)^{p-1}}{p^{2p-1}} N^{p-1} \to \infty \quad \text{as } N \to \infty.$$

Another way to illustrate the sharpness of our results, we consider the quasilinear elliptic problem

$$\begin{gathered} -\Delta_p u = \lambda f(u^q) \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{gathered} \tag{3.9}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (A1) and $f(0) > 0$. The next theorem shows that (3.4) and (3.5) become sharp when $q \to \infty$. We omit the proof as it follows along the same lines as that in the proof of the similar result for the case $p = 2$ in recent joint work of the authors with Ghoussoub [2].

Theorem 3.3. *The extremal parameter $\lambda_p^* = \lambda_p^*(f, \Omega, q)$ of problem (3.9) satisfies*

$$\lim_{q\to\infty} \lambda_p^* = \frac{1}{f(0)\psi_M^{p-1}}.$$

In particular, when $f(0) = 1$ and Ω is the unit ball B then

$$\lim_{q\to\infty} \lambda_p^* = \big(\frac{p}{p-1}\big)^{p-1} N.$$

Example 3.4. Consider problem (1.2) with $f(u) = e^u$ and $\Omega = B$. Here, we have $\sup_{0<t<\infty} \frac{t^{p-1}}{f(t)} = \frac{(p-1)^{p-1}}{e^{p-1}}$ and $\|F\|_\infty = p - 1$, thus from (3.4) we obtain

$$\lambda_p^* \leqslant N p^{p-1}.$$

Moreover, it is easy to see that the function $f'(t) f(t)^{\frac{2-p}{p-1}}\big(\alpha - F(t)\big)$ is decreasing, hence takes its maximum value at $t = 0$. Thus, $\gamma(\alpha) = \alpha^{p-1} - \frac{p}{(p-1)N}\alpha^p$. Now from (3.6) we obtain

$$\lambda_p^*(e^u, B) \geqslant \begin{cases} (\frac{p}{e})^{p-1} N & N \leqslant \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)}, \\ (\frac{p-1}{p})^{p-1} \frac{N^p}{p} & \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)} < N \leqslant \frac{p^2}{p-1}, \\ p^{p-1}(N - p) & N > \frac{p^2}{p-1}. \end{cases} \tag{3.10}$$

Remark 3.5. Garcia-Azorero, Peral and Puel [16, 17] considered problem (1.2) for $f(u)=e^u$ in a general bounded domain Ω and proved that if $N<p+\frac{4p}{p-1}$ then the extremal solution u^* is bounded. Also, if $N\geqslant p+\frac{4p}{p-1}$ and $\Omega=B$ they showed that

$$u^*(x)=-p\ln|x| \quad\text{and}\quad \lambda_p^*=p^{p-1}(N-p),$$

Hence the extremal solution is unbounded in this range, implies that $\lambda_p^*\geqslant p^{p-1}(N-p)$ in every dimension N. So from (3.10) we see that our formula gives the exact value of λ_p^* as a lower bound (without knowing the exact formula of u^*) when $N>p^2/(p-1)$, and also gives a better lower bound when $N<p+\frac{4p}{p-1}$.

Example 3.6. Consider problem (1.2) with $f(u)=\big(1+u\big)^m$, $m>p-1$ and $\Omega=B$. Then from (3.4) we obtain

$$\lambda_p^*\leqslant(\frac{p}{p-1})^{p-1}N\Big(\int_0^\infty(1+s)^{\frac{-m}{p-1}}\Big)^{p-1}=\Big(\frac{p}{m+1-p}\Big)^{p-1}N.$$

Also, we have $\sup_{0<t<\infty}\frac{t^{p-1}}{f(t)}=\big(p-1\big)^{p-1}\big(m+1-p\big)^{m+1-p}m^{-m}$ and $\|F\|_\infty=\frac{p-1}{m+1-p}$. Moreover, it is easy to see that the function $f'(t)f(t)^{\frac{2-p}{p-1}}\big(\alpha-F(t)\big)$ is decreasing, hence takes the maximum at $t=0$. So $\gamma(\alpha)=\alpha^{p-1}-\frac{pm}{(p-1)N}\alpha^p$. Now from (3.6) we obtain

$$\lambda_p^*\big((1+u)^m,B\big)\geqslant\begin{cases}Nm^{-m}p^{p-1}(m+1-p)^{m+1-p}\\ \quad\text{if } N\leqslant\frac{p^{\frac{2p-1}{p-1}}}{p-1}(\frac{m+1-p}{m})^{\frac{m+1-p}{p-1}},\\ (\frac{p-1}{m})^{p-1}(\frac{N}{p})^p\\ \quad\text{if } \frac{p^{\frac{2p-1}{p-1}}}{p-1}(\frac{m+1-p}{m})^{\frac{m+1-p}{p-1}}<N\leqslant\frac{mp^2}{(p-1)(m+1-p)},\\ (\frac{p}{m+1-p})^{p-1}\ \frac{m(N-p)-N(p-1)}{m+1-p}\\ \quad\text{if } N>\frac{mp^2}{(p-1)(m+1-p)}.\end{cases}\tag{3.11}$$

Remark 3.7. By introducing the exact formula of u^*, i.e., the radial function $u^*(x)=|x|^{-\frac{p}{m-p+1}}-1$ corresponding to $\tilde\lambda=(\frac{p}{m+1-p})^{p-1}\ \frac{m(N-p)-N(p-1)}{m+1-p}$, Ferrero [14] (see also [9]), proved that if $N>p4p/(p-1)$ and $m>m_\sharp$ (see [14, 9] for definition of $m_\sharp$) then $\lambda_p^*=\tilde\lambda$. Hence from (3.11) we see that our formula as a lower bound gives the exact value of λ_p^* when $\frac{mp^2}{(p-1)(m+1-p)}<N$, and better bounds for all other cases.

Example 3.8. Considered problem (1.2) with $f(u)=\big(1-u\big)^{-m}$ and $\Omega=B$. Then from (3.4) we obtain

$$\lambda_p^*\leqslant(\frac{p}{p-1})^{p-1}N\Big(\int_0^1(1-s)^{\frac{m}{p-1}}\Big)^{p-1}=\Big(\frac{p}{m+p-1}\Big)^{p-1}N.$$

Also, we have

$$\sup_{0<t<1}\frac{t^{p-1}}{f(t)}=\big(p-1\big)^{p-1}\big(m+p-1\big)^{1-m-p}m^m$$

and $\|F\|_\infty=\frac{p-1}{m+p-1}$. Moreover, it is easy to see that the function $f'(t)f(t)^{\frac{2-p}{p-1}}\big(\alpha-F(t)\big)$ is decreasing, hence takes the maximum at $t=0$. So $\gamma(\alpha)=\alpha^{p-1}-\frac{pm}{(p-1)N}\alpha^p$.

Now from (3.6) we obtain

$$\lambda_p^*\big((1-u)^{-m},B\big) \geqslant \begin{cases} Nm^m p^{p-1}(m+p-1)^{1-m-p} \\ \quad \text{if } N \leqslant \frac{p^{\frac{2p-1}{p-1}}}{p-1}\big(\frac{m}{m+p-1}\big)^{\frac{m+p-1}{p-1}}, \\ \big(\frac{p-1}{m}\big)^{p-1}\big(\frac{N}{p}\big)^p \\ \quad \text{if } \frac{p^{\frac{2p-1}{p-1}}}{p-1}\big(\frac{m}{m+p-1}\big)^{\frac{m+p-1}{p-1}} < N \leqslant \frac{mp^2}{(p-1)(m+p-1)}, \\ \big(\frac{p}{m+p-1}\big)^{p-1}\frac{m(N-p)-N(p-1)}{m+p-1} \\ \quad \text{if } N > \frac{mp^2}{(p-1)(m+p-1)}. \end{cases}$$

To obtain more explicit formulas for λ_p^*, here we give explicit upper and lower bounds for ψ_M. Let

$$r_\Omega := \sup_{x\in\Omega} d_\Omega(x), \tag{3.12}$$

be the Chebyshev radius of $\Omega \subseteq \mathbb{R}^N$. Also, let $d := \frac{1}{2}\operatorname{diam}(\Omega)$. Find $x_0 \in \Omega$ and $x_1 \in \mathbb{R}^N$ such that $B_{r_\Omega}(x_0) \subseteq \Omega \subseteq B_d(x_1)$. Then by comparing the p-torsion function ψ of Ω with the p-torsions of $B_{r_\Omega}(x_0)$ and $B_d(x_1)$, i.e., functions

$$\big(\frac{p-1}{p}\big)N^{-1/(p-1)}\big(r_\Omega^{\frac{p}{p-1}} - |x-x_0|^{\frac{p}{p-1}}\big), \quad \big(\frac{p-1}{p}\big)N^{-1/(p-1)}\big(d^{\frac{p}{p-1}} - |x-x_0|^{\frac{p}{p-1}}\big),$$

respectively, we obtain

$$\big(\frac{p-1}{p}\big)N^{-1/(p-1)}r_\Omega^{\frac{p}{p-1}} \leqslant \psi_M \leqslant \big(\frac{p-1}{p}\big)N^{-1/(p-1)}\Big(\frac{\operatorname{diam}(\Omega)}{2}\Big)^{\frac{p}{p-1}}. \tag{3.13}$$

Also, the following lower bound for ψ_M from [12] is better than that in (3.13) whenever r_Ω is small with respect to the volume $|\Omega|$ of Ω. Let $\tau_p(\Omega)$ be the p-torsional rigidity

$$\tau_p(\Omega) := \int_\Omega \psi(x)\mathrm{d}x,$$

then from [12, Theorem 5.1] we have

$$\tau_p(\Omega) \geqslant \big(\frac{p-1}{2p-1}\big)\frac{|\Omega|^{\frac{2p-1}{p-1}}}{P(\Omega)^{\frac{p}{p-1}}}, \tag{3.14}$$

where $P(\Omega)$ is the perimeter of Ω. Now using $\tau_p(\Omega) \leqslant \psi_M|\Omega|$, then from (3.14) we obtain

$$\psi_M \geqslant \frac{p-1}{2p-1}\Big(\frac{|\Omega|}{P(\Omega)}\Big)^{\frac{p}{p-1}}.$$

Hence from Theorem 3.2 we obtain the following explicit bounds for λ_p^*.

Corollary 3.9. *Let λ_p^* be the extremal parameter of problem* (1.2) *where f satisfies* (A1). *Then*

$$\big(\frac{p}{p-1}\big)^{p-1}\frac{2^pN}{\operatorname{diam}(\Omega)^p}\sup_{0<t<a_f}\frac{t^{p-1}}{f(t)} \leqslant \lambda_p^* \leqslant \theta_{p,\Omega}\Big(\int_0^{a_f}\frac{\mathrm{d}s}{f(s)^{1/(p-1)}}\Big)^{p-1},$$

where

$$\theta_{p,\Omega} := \min\Big\{\big(\frac{p}{p-1}\big)^{p-1}\frac{N}{r_\Omega^p},\ \big(\frac{2p-1}{p-1}\big)^{p-1}\big(\frac{P(\Omega)}{|\Omega|}\big)^p\Big\}.$$

3.2. **Lower bound for the first eigenvalue of the p-Laplacian.** Here we show that how our results can be applied to estimate the first eigenvalue of p-Laplacian from below. First we recall some results from the literature. Let $h(\Omega)$ be the Cheeger constant of Ω, i.e.,

$$h(\Omega) := \inf_{D} \frac{|\partial D|}{|D|},$$

with D varying over all smooth domain of Ω whose boundary ∂D does not touch $\partial\Omega$ and with $|\partial D|$ and $|D|$ denoting $(n-1)$- and n-dimensional measure of ∂D and D, see [15]. The following lower bound from [21] is the extension of the same result for $p = 2$ proved by Cheeger, see [10].

$$\lambda_1(\Omega) \geqslant \big(\frac{h(\Omega)}{p}\big)^p, \quad p \in (1, \infty). \tag{3.15}$$

If Ω is a ball we know that $h(\Omega) = \frac{N}{R}$, (see [15]) hence from (3.15) we have

$$\lambda_1(B_R) \geqslant \big(\frac{N}{pR}\big)^p, \quad p \in (1, \infty). \tag{3.16}$$

The lower bound (3.16) becomes sharp when $p \to 1$, as it is shown by Friedman and Kawhol [15] that $\lambda_1(\Omega)$ converges to the Cheeger constant $h(\Omega)$ when $p \searrow 1$. However, it is not sharp when $p \to \infty$, as from [20] we know that

$$\lim_{p\to\infty} \lambda_1^{\frac{1}{p}}(\Omega) = \frac{1}{r_\Omega},$$

where r_Ω is defined in (3.12). Hence, $\lim_{p\to\infty} \lambda_1^{\frac{1}{p}}(B_R) = \frac{1}{R}$, while the p-th root of the right hand side of (3.16) appraoches zero when $p \to \infty$.

Here, we give some lower bounds for λ_1 using our results. First note that from (3.8) and (3.13) we have

$$\lambda_1(\Omega) \geqslant \frac{1}{\psi_M^{p-1}} \geqslant \big(\frac{p}{p-1}\big)^{p-1}\big(\frac{2}{\operatorname{diam}(\Omega)}\big)^p N. \tag{3.17}$$

In particular, in the special case when Ω is the ball B_R then

$$\lambda_1(B_R) \geqslant \frac{1}{\psi_M^{p-1}} = \big(\frac{p}{p-1}\big)^{p-1} \frac{N}{R^p}, \tag{3.18}$$

which is recently obtained by Benedikt and Drábek [3].

The lower bound (3.18) is better than (3.16) when $N < \frac{p^{\frac{2p-1}{p-1}}}{p-1}$, and also becomes sharp in both critical cases $p \searrow 1$ and $p \to \infty$. Also, the following lower bound for λ_1, which is a consequence of Example 3.4 and (3.1), gives better bound on $\lambda_1(B)$, for more values of p and N.

$$\lambda_1(B) \geqslant \begin{cases} (\frac{p}{p-1})^{p-1} N & N \leqslant \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)}, \\ (\frac{e}{p})^{p-1} \frac{N^p}{p} & \frac{p^{\frac{2p-1}{p-1}}}{e(p-1)} < N \leqslant \frac{p^2}{p-1}, \\ (\frac{pe}{p-1})^{p-1}(N-p) & N > \frac{p^2}{p-1}. \end{cases} \tag{3.19}$$

Benedikt and Drábek [4] also presented upper and lower bounds for $\lambda_1(\Omega)$ on a bounded domain $\Omega \subseteq \mathbb{R}^N$. In particular, when $\Omega = B$ they proved that

$$\lambda_1(B) \geqslant Np. \tag{3.20}$$

Comparing (3.19) and (3.20), one can easily check that when $1 < p \leqslant 2$ the lower bound (3.19) is better than (3.20) in every dimension N. Also, when $p > 2$ the same is true when $N \geqslant \dfrac{p^{\frac{p+1}{p-1}}}{e}$.

4. Nonexistence results

Here we show that how one can apply Theorem 2.2 to prove nonexistence of positive solutions of differential inequalities involving p-Laplacian. Consider the differential inequality

$$\begin{gathered} -\Delta_p u \geqslant \lambda\rho(x) f(u) \quad x \in \Omega, \\ u \geqslant 0 \quad x \in \Omega, \\ u \in W_0^{1,p}(\Omega). \end{gathered} \tag{4.1}$$

Theorem 4.1. *Let f satisfy* (A1), *and $\rho : \Omega \to \mathbb{R}$ be a nonnegative bounded measurable function that is not identically zero. Then*

(i) *Inequality* (4.1) *has no positive C^1 solution if*

$$\lambda > \big(\frac{p}{p-1}\big)^{p-1} \frac{N\|F\|_\infty^{p-1}}{\sup_{x\in\Omega}\left\{\rho_x\big(d_\Omega(x)\big)d_\Omega^p(x)\right\}}. \tag{4.2}$$

(ii) *If $\rho(x) = |x|^\alpha$, $\alpha > 0$ and $\Omega = B_R$, then the same is true if*

$$\lambda > \Big(\frac{\alpha+p}{p-1}\|F\|_\infty\Big)^{p-1}(\alpha+N)R^{-(\alpha+p)}.$$

Proof. (i) If (4.1) has a positive solution u, then from (2.4) in Theorem 2.2 (by replacing f with λf) we obtain

$$N\Big(\int_0^{u(x)} \frac{\mathrm{d}s}{f(s)^{1/(p-1)}}\Big)^{p-1} \geqslant \lambda\big(\frac{p-1}{p}\big)^{p-1}\rho_x\big(d_\Omega(x)\big)d_\Omega^p(x), \quad x \in \Omega,$$

and taking supremum on both sides over Ω we arrive at a contradiction with (4.2).

(ii) Now, let $\rho(x) = |x|^\alpha$ and $\Omega = B_R$. In this case we can use (2.2) directly. Indeed, it is easy to see that the function

$$\psi_\rho(x) = C\big(R^{\frac{\alpha+p}{p-1}} - |x|^{\frac{\alpha+p}{p-1}}\big), \quad \text{with } C := \big(\frac{p-1}{\alpha+p}\big)\big(\alpha+N\big)^{-1/(p-1)},$$

is the solution of (2.1) with $\rho(x) = |x|^\alpha$, hence from (2.2) we must have

$$F\big(u(x)\big) \geqslant \lambda^{1/(p-1)}\psi_\rho(x), \quad x \in B_R.$$

Taking the supremum over B_R we obtain the desired result. □

As an application of this result, for $\alpha > 0$ consider the eigenvalue problem

$$\begin{gathered} -\Delta u = \lambda\frac{|x|^\alpha}{(1-u)^2} \quad x \in B_R, \\ u = 0 \quad x \in \partial B_R, \end{gathered}$$

which in two dimension models a simple *Micro-Electromechanical-Systems* MEMS device, see [11, 13, 18, 19]. Let λ^* (called pull-in voltage) be the extremal parameter of the above eigenvalue problem, then from Theorem 4.1, we have

$$\lambda^* \leqslant \frac{(\alpha+2)(\alpha+N)}{3}R^{-(\alpha+2)}.$$

This upper bound improves the ones obtained in [2, 18, 19]. It could be interesting to compare this bound to the lower bound for λ^* given in [13], then we have

$$\max\Big\{\frac{4(\alpha+2)(\alpha+N)}{27}, \frac{(\alpha+2)(3N+\alpha-4)}{9}\Big\}R^{-(\alpha+2)}$$
$$\leqslant \lambda^* \leqslant \frac{(\alpha+2)(\alpha+N)}{3}R^{-(\alpha+2)}.$$

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