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PERTURBATIONAL SELF-SIMILAR SOLUTIONS FOR MULTI-DIMENSIONAL CAMASSA-HOLM-TYPE EQUATIONS

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Abstract. In this article, we sutdy a multi-component Camassa-Holm-type system. By employing the characteristic method, we obtain a class of perturbational self-similar solutions with elliptical symmetry, whose velocity components are governed by the generalized Emden equations. In particular, when $n = 1$, these solutions constitute a generalization of that obtained by Yuen in [38]. Interestingly, numerical simulations show that the analytical solutions obtained can be used to describe the drifting phenomena of shallow water flows. In addition, the method proposed can be extended to other mathematical physics models such as higher-dimensional Hunter-Saxton equations and Degasperis-Procesi equations.

1. Introduction

Here, our concern is the investigation of analytical solutions of the multi-dimensional Camassa-Holm-type system:

$$\begin{aligned} \rho_t &= -\nabla\rho\cdot u - \rho(\nabla\cdot u), \\ m_t &= -u\cdot\nabla m - (\nabla u)^T\cdot m - m(\nabla\cdot u) - (\nabla\rho)^T\rho\,, \end{aligned} \tag{1.1}$$

where u, m and ρ are vector fields on the n-torus $\mathbb{S}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$. Here, it is assumed that there exists a linear operator A such that $m = Au$, and that A takes a form of $\alpha\mu + \beta - \Delta$ with $\{\alpha, \beta\} = \{0, 1\}$ and $\alpha + \beta \neq 2$. While $\mu(u) = \int_{\mathbb{S}^n} u(x)dx$ denotes the mean value operator.

Interest in such a system may trace back to the Camassa-Holm equation

$$u_t - u_{xxt} = -3uu_x + 2u_xu_{xx} + uu_{xxx}\,, \tag{1.2}$$

which was originally derived in [12] as an abstract equation with a bi-Hamiltonian structure, and independently in [4] as a shallow water approximation. Due to the distinguished features of complete integrability, there has been an extensive literature devoted to the analysis of the Camassa-Holm equation. For example, peakon solutions and their stability were investigated in [6, 7, 9, 34]. Geometric properties were discussed in [8, 23, 33]. A condensed account of the geometric

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picture was found in [22, 27]. Subsequently, the two-component Camassa-Holm equations

$$\begin{aligned} \rho_t &= -\rho u_x - \rho_x u\,, \\ m_t &= -m_x u - 2u_x m - \rho\rho_x\,, \end{aligned} \tag{1.3}$$

with $m = u - u_{xx}$ was introduced by Chen *et al* [5] and independently by Falqui [11]. Investigations showed that the two-component generalization of (1.2) version shared analogous properties with the Camassa-Holm equation (see [10, 14, 16, 17]). The closest relatives of the Camassa-Holm equation are the Hunter-Saxton (HS) equation

$$-u_{xxt} = 2u_x u_{xx} + u u_{xxx}\,, \tag{1.4}$$

and the (μ)-Hunter-Saxton (μHS) equation

$$2\mu(u)u_x - u_{xxt} = 2u_x u_{xx} + u u_{xxx}\,. \tag{1.5}$$

The former first appeared in [18] as an asymptotic equation for rotators in liquid crystals. While the latter was proposed by Khesin et al [20] and also appeared in the work of Lenells et al wherein it was called the (μ)-Camassa-Holm equation [30]. The μHS equation is usually used to describe evolution of rotators in liquid crystals with an external magnetic field and self-interaction. Similar to the Camassa-Holm equation, both the HS and μHS equation have their associated two-component analogue, which can be considered as the system (1.3) with $m = -u_{xx}$ and $m = \mu(u) - u_{xx}$, respectively. All these equations have attracted great attentions among the integrable systems and the partial differential equations communities due to their nice mathematical properties and meaningful physical interpretations [18, 19, 20, 26, 28, 29, 30, 35, 36, 37, 41].

What needs to point out is that most works mentioned above concern with the case $n = 1$ or 2. However, except for the work in [13, 21, 30], there has not been many investigations carried out on the higher dimensional variations of the Camassa-Holm-type system named above. Especially, no related work has been done on construction of the analytical solutions. Since such equations are of interest from the physical and the mathematical point of view as explained in, e.g.[13, 15, 24], this indeed gives us a strong push to pursue our present investigation.

In this paper, we would like to seek self-similar solutions for the multi-component Camassa-Holm-type system. It is known that it is a hot topic to investigate such type solutions in various models and many works have been done. For instance, Barna [3] presented the self-similar solutions for the 3-dimensional Navier-Stokes equations via a group theoretical method. Yuen [39] derived the self-similar solutions to the compressible Euler equations by perturbing the velocity term. Yuen [38, 40] also found that this kind of solutions existed in the two-component Camassa-Holm equations. Recently, An and Yuen [1] obtained drifting solutions of self-similar type for the compressible Navier-Stokes equations with density-dependent viscosity. As we notice that the multi-component Camassa-Holm systems (1.1) share some similarities with the equations given in [1, 3, 38, 39] in the form, it is nature to inquire whether one can construct the self-similar solutions of perturbational type for (1.1). In this paper, by using a characteristic method, we successfully obtain the perturbational self-similar solutions for the multi-component Camassa-Holm system. The main result is described in the following theorem.

Theorem 1.1. *For the multi-component Camassa-Holm system, there exists a family of perturbational self-similar solutions:*

$$\begin{aligned} \rho &= \frac{F(\eta)}{\prod_{k=1}^{n} a_k^{1/3}(s)}\,, \\ u_i &= \frac{a_i'(s)}{a_i(s)}\left(x_i - d_i^*(t)\right) + \dot{d}_i^*(t), \quad \text{for } i = 1, 2, \dots, n \end{aligned} \tag{1.6}$$

where

$$s = 3t, \quad \eta = \sum_{k=1}^{n}\Big(\frac{x_k - d_k^*(t)}{a_k^{1/3}(s)}\Big)^2, \tag{1.7}$$

$$F(\eta) = \sqrt{c_0^2 - 3\xi\eta(\alpha\mu - \beta)}. \tag{1.8}$$

In the above, c_0 and ξ are constants of integration. While, the auxiliary functions $a_i(s)$ and $d_i^*(t)$ are determined by the following equations.
(I) In the case $n = 1$, the function $a_i(s) = a(s) = a(3t)$ is governed by the canonical Emden equation [25]

$$\begin{aligned} a''(s) &= \frac{\xi}{a^{1/3}(s)}\,, \\ a(0) &= a_0 > 0, \quad \dot{a}(0) = a_1\,. \end{aligned} \tag{1.9}$$

While, the perturbational function $d_i^*(t) = d^*(t) \in C^2$ is given by

$$d^*(t) = d_0 + \frac{1}{3}\int_0^{3t} \frac{d_1 a_0^{2/3}}{a^{2/3}(s)}\mathrm{d}s\,, \tag{1.10}$$

where d_0 and d_1 are integration constants.
(II) In the case $n > 1$, the functions $a_i(s) = a_i(3t)$ are determined by a modified Emden equation with damping

$$\begin{gathered} a_i''(s) + \frac{1}{3}a_i'(s)\sum_{k\neq i}^{n}\frac{a_k'(s)}{a_k(s)} = \frac{\xi}{a_i^{-1/3}(s)\prod_{k=1}^{n} a_k^{2/3}(s)}\,, \quad \text{for } i = 1, 2, \dots, n \ (n > 1)\,, \\ a_i(0) = a_{i0} > 0, \quad \dot{a}_i(0) = a_{i1}\,. \end{gathered} \tag{1.11}$$

While, the perturbational variables $d_i^*(t) \in C^2$ are given by an integral equation

$$d_i^*(t) = d_{i0} + \frac{1}{3}\int_0^{3t} \frac{d_{i0} a_{i0}^{1/3}\prod_{k=1}^{n} a_{k0}^{1/3}}{{a_i}^{1/3}(s)\prod_{k=1}^{n} {a_k}^{1/3}(s)}\mathrm{d}s\,, \tag{1.12}$$

where d_{i0} and d_{i1} are initial values.

Remark 1.2. It is noticed that when $n = 1$ and the function $d^*(t)$ degenerates to zero, namely $d_0 = d_1 = 0$, the solution given by (1.6) coincides with that obtained by Yuen [38]. However, in other cases, we conclude that the solution is more general than Yuen's.

Remark 1.3. It is known that exact solutions are rare in compressible flows [32], and, in fact, for fluid mechanics are general. Hence, results obtained in this paper may provide valuable physical insight and may serve as benchmarks for testing numerical solutions.

Remark 1.4. We want to point out here that the method proposed can also be extended to construct perturbational self-similar solutions of the higher-dimensional HS, μHS and Degasperis-Procesi equations as well as other models whose forms are analogous to the Camassa-Holm equation.

2. Solutions of the multi-dimensional CH-type system

Inspired by the work of [38], we present a particular class of analytical perturbational solutions of the multi-dimensional Camassa-Holm-type equations (1.1). This progress is achieved mainly due to the following novel important lemma.

Lemma 2.1. *For the continuity equation of the multi-dimensional Camassa-Holm-type system*

$$\rho_t = -(\nabla\rho)\cdot\boldsymbol{u} - \rho(\nabla\cdot\boldsymbol{u}), \tag{2.1}$$

there exist solutions

$$\begin{gathered}\rho = \frac{F\big(\frac{x_1-d_1^*(t)}{a_1^{1/3}(3t)}, \frac{x_2-d_2^*(t)}{a_2^{1/3}(3t)}, \dots, \frac{x_n-d_n^*(t)}{a_n^{1/3}(3t)}\big)}{\prod_{i=1}^n a_i^{1/3}(3t)}, \\ u_i = \frac{a_i'(3t)}{a_i(3t)}(x_i - d_i^*(t)) + \dot{d}_i^*(t), \quad \text{for } i = 1, 2, \dots, n\end{gathered} \tag{2.2}$$

where $\prime = \frac{d}{ds}$*,* $a_i(s) > 0$ *and* F *is an arbitrary non-negative* C^1 *function.*

Proof. According to the work of An and Yuen's in [39], we perturb the velocity in the form

$$\rho = \rho(t,x), \quad u_i = \frac{a_i'(3t)}{a_i(3t)}(x_i - d_i^*(t)) + \dot{d}_i^*(t)\,. \tag{2.3}$$

Substitution of this ansatz into (2.1), yields

$$\begin{aligned}&\rho_t + (\nabla\rho)\cdot\mathbf{u} + \rho(\nabla\cdot\mathbf{u}) \\ &= \frac{\partial}{\partial t}\rho + \sum_{i=1}^n \frac{\partial}{\partial x_i}\rho\Big[\frac{a_i'(3t)}{a_i(3t)}\,(x_i - d_i^*(t)) + \dot{d}_i^*(t)\Big] + \rho\sum_{i=1}^n \frac{a_i'(3t)}{a_i(3t)} = 0\,.\end{aligned} \tag{2.4}$$

According to the classical characteristic method [31], we have

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x_i}{\frac{a_i'(3t)}{a_i(3t)}\,(x_i - d_i^*(t)) + \dot{d}_i^*(t)} = \frac{\mathrm{d}\rho}{-\sum_{i=1}^n \frac{\rho a_i'(3t)}{a_i(3t)}}, \tag{2.5}$$

whence, the solution is

$$\Psi\Big(\prod_{i=1}^n \rho a_i^{1/3}(3t), \frac{x_1 - d_1^*(t)}{a_1^{1/3}(3t)}, \frac{x_2 - d_2^*(t)}{a_2^{1/3}(3t)}, \dots, \frac{x_n - d_n^*(t)}{a_n^{1/3}(3t)}\Big) = 0 \tag{2.6}$$

with an arbitrary function $\Psi \in C^1$ such that $\rho \geq 0$.

For convenience, we rewrite (2.6) in the explicit form

$$\rho = \frac{F\big(\frac{x_1-d_1^*(t)}{a_1^{1/3}(3t)}, \frac{x_2-d_2^*(t)}{a_2^{1/3}(3t)}, \dots, \frac{x_n-d_n^*(t)}{a_n^{1/3}(3t)}\big)}{\prod_{i=1}^n a_i^{1/3}(3t)}. \tag{2.7}$$

The proof is complete. $\square$

Remark 2.2. It needs to point out that the negative symbol in the perturbational non-constant functions d_i for the velocity in (2.3) is critical to guarantee the use of the characteristic method.

On application of the above lemma, we obtain a class of perturbational self-similar solutions for the multi-dimensional Camassa-Holm-type system. The result is described as follows:

Theorem 2.3. *For the multi-dimensional Camassa-Holm-type system* (1.1)*, there exists a class of analytical perturbational self-similar solutions*

$$\begin{gathered} \rho = \frac{F(\eta)}{\prod_{k=1}^{n} a_k^{1/3}(s)}\,, \\ u_i = \frac{a_i'(s)}{a_i(s)}\,(x_i - d_i^*(t)) + \dot{d}_i^*(t), \quad \text{for } i = 1, 2, \dots, n, \end{gathered} \tag{2.8}$$

where

$$s = 3t\,, \quad \eta = \sum_{k=1}^{n} \Big(\frac{x_k - d_k^*(t)}{a_k^{1/3}(s)}\Big)^2, \tag{2.9}$$

$$F(\eta) = \sqrt{c_0^2 - 3\xi\eta(\alpha\mu - \beta)}\,. \tag{2.10}$$

In the above, c_0 and ξ are constants of integration. While, the auxiliary functions $a_i(s) = a_i(3t)$ and the perturbational functions $d_i^*(t)$ are determined by the following equations:

(I) In the case $n = 1$, the function $a_i(s) = a(s) = a(3t)$ is governed by the canonical Emden equation [25]

$$\begin{gathered} a''(s) = \frac{\xi}{a^{1/3}(s)}\,, \\ a(0) = a_0 > 0, \quad \dot{a}(0) = a_1\,. \end{gathered} \tag{2.11}$$

While, the perturbational function $d_i^*(t) = d^*(t) \in C^2$ is given by

$$d^*(t) = d_0 + \frac{1}{3}\int_0^{3t} \frac{d_1 a_0^{2/3}}{a^{2/3}(s)} \mathrm{d}s\,, \tag{2.12}$$

where d_0 and d_1 are constants of integration.

(II) In the case $n > 1$, the functions $a_i(s) = a_i(3t)$ are determined by the generalized Emden dynamical system

$$\begin{gathered} a_i''(s) + \frac{1}{3}a_i'(s)\sum_{k\neq i}^{n} \frac{a_k'(s)}{a_k(s)} = \frac{\xi}{a_i^{-1/3}(s)\prod_{k=1}^{n} a_k^{2/3}(s)}\,, \quad \text{for } i = 1, 2, \dots, n \ (n > 1) \\ a_i(0) = a_{i0} > 0, \quad \dot{a}_i(0) = a_{i1}\,. \end{gathered} \tag{2.13}$$

While, the perturbational variables $d_i^*(t) \in C^2$ are given by the integral relations

$$d_i^*(t) = d_{i0} + \frac{1}{3}\int_0^{3t} \frac{d_{i0} a_{i0}^{1/3} \prod_{k=1}^{n} a_{k0}^{1/3}}{{a_i}^{1/3}(s)\prod_{k=1}^{n} {a_k}^{1/3}(s)} \mathrm{d}s\,, \tag{2.14}$$

where d_{i0} and d_{i1} are initial values.

Proof. It is clear, from the the above lemma, that the functions (2.8) indeed satisfy the continuity equation of the multi-dimensional Camassa-Holm-type system (1.1). In what follows, we only need to validate the functions (2.8) also hold for the second equation.

For the i-th momentum equation of the multi-dimensional Camassa-Holm-type system (1.1), by defining an elliptically symmetric variable via

$$\eta=\sum_{k=1}^{n}\Big(\frac{x_k-d_k^*(t)}{a_k^{1/3}(s)}\Big)^2 \tag{2.15}$$

and on using (2.8), we have

$$\begin{aligned}
&\frac{\partial m_i}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial m_i}{\partial x_k}+\sum_{k=1}^{n}m_k\frac{\partial u_i}{\partial x_k}+m_i\sum_{k=1}^{n}\frac{\partial u_k}{\partial x_k}+\rho\frac{\partial \rho}{\partial x_i}\\
&=\frac{\partial Au_i}{\partial t}+\sum_{k=1}^{n}u_k\frac{\partial Au_i}{\partial x_k}+\sum_{k=1}^{n}Au_k\frac{\partial u_i}{\partial x_k}+Au_i\sum_{k=1}^{n}\frac{\partial u_k}{\partial x_k}+\rho\frac{\partial \rho}{\partial x_i}\\
&=(\alpha\mu-\beta)\frac{\partial}{\partial t}\Big[\frac{a_i'(s)}{a_i(s)}(x_i-d_i^*)+\dot d_i^*\Big]+2(\alpha\mu-\beta)\Big[\frac{a_i'(s)}{a_i(s)}(x_i-d_i^*)+\dot d_i^*\Big]\\
&\quad\times\frac{\partial}{\partial x_i}\Big[\frac{a_i'(s)}{a_i(s)}(x_i-d_i^*)+\dot d_i^*\Big]\\
&\quad+(\alpha\mu-\beta)\Big[\frac{a_i'(s)}{a_i(s)}(x_i-d_i^*)+\dot d_i^*\Big]\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\Big[\frac{a_k'(s)}{a_k(s)}(x_k-d_k^*)+\dot d_k^*\Big]\\
&\quad+\frac{F(\eta)}{\prod_{k=1}^{n}a_k^{1/3}(s)}\frac{\partial}{\partial x_i}\frac{F(\eta)}{\prod_{k=1}^{n}a_k^{1/3}(s)}\\
&=(\alpha\mu-\beta)\Big[\Big(\frac{3a_i''(s)}{a_i(s)}-\frac{3{a_i'}^2(s)}{a_i^2(s)}\Big)(x_i-d_i^*)-\frac{\dot d_i^* a_i'(s)}{a_i(s)}+\ddot d_i^*+\frac{2{a_i'}^2(s)}{a_i^2(s)}(x_i-d_i^*)\\
&\quad+\frac{2\dot d_i^* a_i'(s)}{a_i(s)}+\dot d_i^*\sum_{k=1}^{n}\frac{a_k'(s)}{a_k(s)}+\frac{a_i'(s)}{a_i(s)}\sum_{k=1}^{n}\frac{a_k'(s)}{a_k(s)}(x_i-d_i^*)\Big]+\frac{2F(\eta)F'(\eta)}{\prod_{k=1}^{n}a_k^{\frac23}(s)}\frac{(x_i-d_i^*)}{a_i^{\frac23}(s)}\\
&=(\alpha\mu-\beta)\Big\{\frac{3(x_i-d_i^*)}{a_i(s)}\Big[a_i''(s)+\frac13 a_i'(s)\Big(\sum_{k=1}^{n}\frac{a_k'(s)}{a_k(s)}-\frac{a_i'(s)}{a_i(s)}\Big)\\
&\quad+\frac{2}{3(\alpha\mu-\beta)}\frac{F(\eta)F'(\eta)}{a_i^{-1/3}(s)\prod_{k=1}^{n}a_k^{\frac23}(s)}\Big]+\ddot d_i^*+\dot d_i^*\Big[\frac{a_i'(s)}{a_i(s)}+\sum_{k=1}^{n}\frac{a_k'(s)}{a_k(s)}\Big]\Big\}\\
&=(\alpha\mu-\beta)\Big\{\frac{3(x_i-d_i^*)}{a_i^{2/3}(s)\prod_{k=1}^{n}a_k^{\frac23}(s)}\Big[\xi+\frac{2}{3(\alpha\mu-\beta)}F(\eta)F'(\eta)\Big]\\
&\quad+\ddot d_i^*+\dot d_i^*\Big[\frac{a_i'(s)}{a_i(s)}+\sum_{k=1}^{n}\frac{a_k'(s)}{a_k(s)}\Big]\Big\}
\end{aligned}$$

if we require the arbitrary functions $a_i(s)$ satisfy the Emden equations:

$$\begin{gathered}
a''(s)=\frac{\xi}{a^{1/3}(s)},\quad \text{for } n=1,\ a_1(s)=a(s),\\
a_i''(s)+\frac13 a_i'(s)\sum_{k\neq i}^{n}\frac{a_k'(s)}{a_k(s)}=\frac{\xi}{a_i^{-1/3}(s)\prod_{k=1}^{n}a_k^{2/3}(s)},\\
\text{for } i=1,2,3,\dots,n\ (n\neq 1)\\
a_i(0)=a_{i0}>0,\quad \dot a_i(0)=a_{i1}
\end{gathered}\tag{2.16}$$

and $d_i(t)$ governed by the equation

$$\ddot{d}_i^* + \dot{d}_i^* \Big[\frac{a_i'(s)}{a_i(s)} + \sum_{k=1}^n \frac{a_k'(s)}{a_k(s)}\Big] = 0\,, \quad d_i^*(0) = d_{i0}\,, \quad \dot{d}_i^*(0) = d_{i1}\,, \tag{2.17}$$

whence produces

$$d_i^*(t) = d_{i0} + \frac{1}{3}\int_0^{3t} \frac{d_{i0} a_{i0}^{1/3} \prod_{k=1}^n a_{k0}^{1/3}}{{a_i}^{1/3}(s) \prod_{k=1}^n {a_k}^{1/3}(s)} ds\,. \tag{2.18}$$

So that the function $F(\eta)$ is determined by

$$\begin{gathered} \xi + \frac{2}{3(\alpha\mu - \beta)} F(\eta)F'(\eta) = 0\,, \\ F(0) = c_0\,, \end{gathered} \tag{2.19}$$

whence the solution is

$$F(\eta) = \sqrt{c_0^2 - 3\xi\eta(\alpha\mu - \beta)} \quad \text{with} \quad \eta = \sum_{k=1}^n \Big(\frac{x_k - d_k^*(t)}{a_k^{1/3}(s)}\Big)^2\,. \tag{2.20}$$

The proof is complete. □

To understand the possible behaviors that the solutions obtained may exhibit, we perform some numerical simulations. Here we take the dimension of the equation $N = 2$. Figure 1 shows the time evolutions of density function ρ given by (1.6) at regular time intervals when $\alpha = 0$, $\beta = 1$. From the figures, we can see that the function ρ continuously move forward in the xy-plane with time changing. Such phenomena can also be clearly noticed from the contour Figure 2. It is known that the multi-component Camassa-Holm equations can be used to describe the shallow water flows. Therefore, we conclude that the moving behaviors our solution reveals are nothing but the drifting phenomena of the flow. When the parameters are changed to $\alpha = 1, \beta = 0$ and $\mu = 0.1$, analogous behaviors can be seen in Figures 3 and 4.

Conclusion. It is known that the Camassa-Holm equation is an important integrable model, which has attracted great attention in mathematical physics. However, unlike the large amount of papers referring to the case $n = 1, 2$, here we have investigated the multi-variable variation of the Camassa-Holm-type equation, namely the system (1.1). Such a system also has nice mathematical properties and extensive physical applications (see references [13, 15, 24]). By using the characteristic method, we have constructed a class of perturbational self-similar solutions of the system (1.1) wherein the velocity components are governed by the generalized Emden equations. It is noticed that when $n = 1$, these solutions are more general than that was obtained by Yuen in [38]. What's more important is that the method proposed can be extended to construct perturbational solutions of other mathematical physics models like higher-dimensional Hunter-Saxton equations and Degasperis-Procesi equations. In addition, we hope that the perturbational self-similar solutions obtained can be helpful in the physical areas such as fluid and hydrodynamics. However, there are still some problems that needs further consideration, for example, what are the properties of the generalized Emden equations? Is the multi-component Camassa-Holm-type system, likewise the Camassa-Holm equation, completely integrable? Does it admit peakon solutions? All these interesting questions are worthy of our deep investigations in the future.

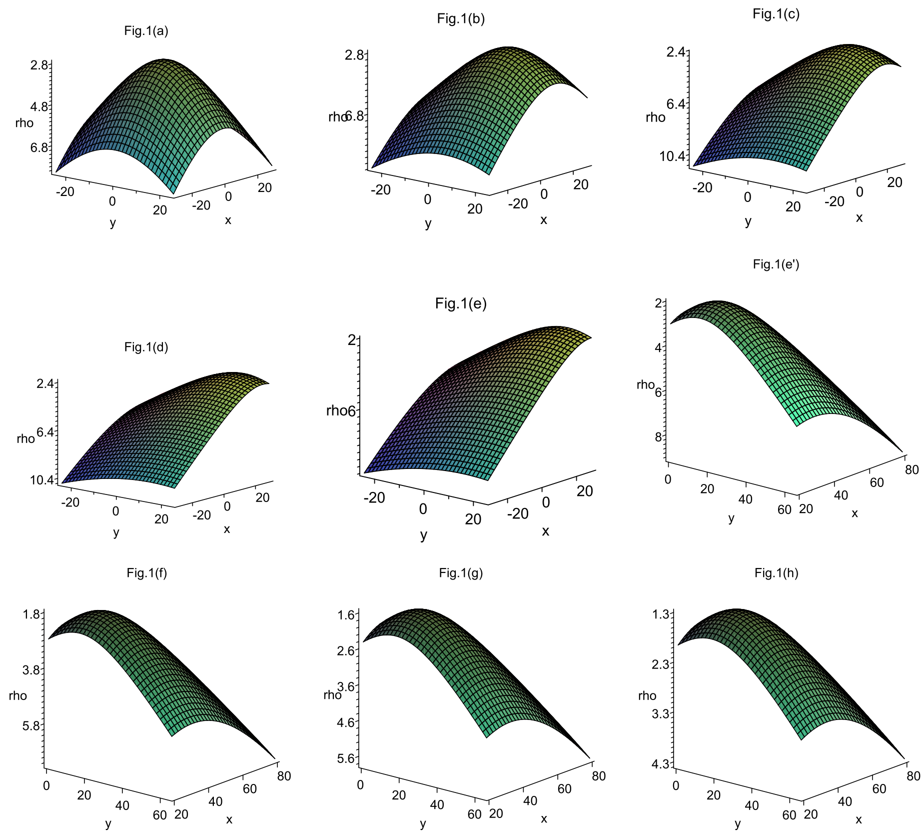

FIGURE 1. Time evolutions of the density function ρ at a regular time intervals $\Delta t = 3$, with $c_0 = 20$, $\xi = 5$, $\alpha = 0$, $\beta = 1$.

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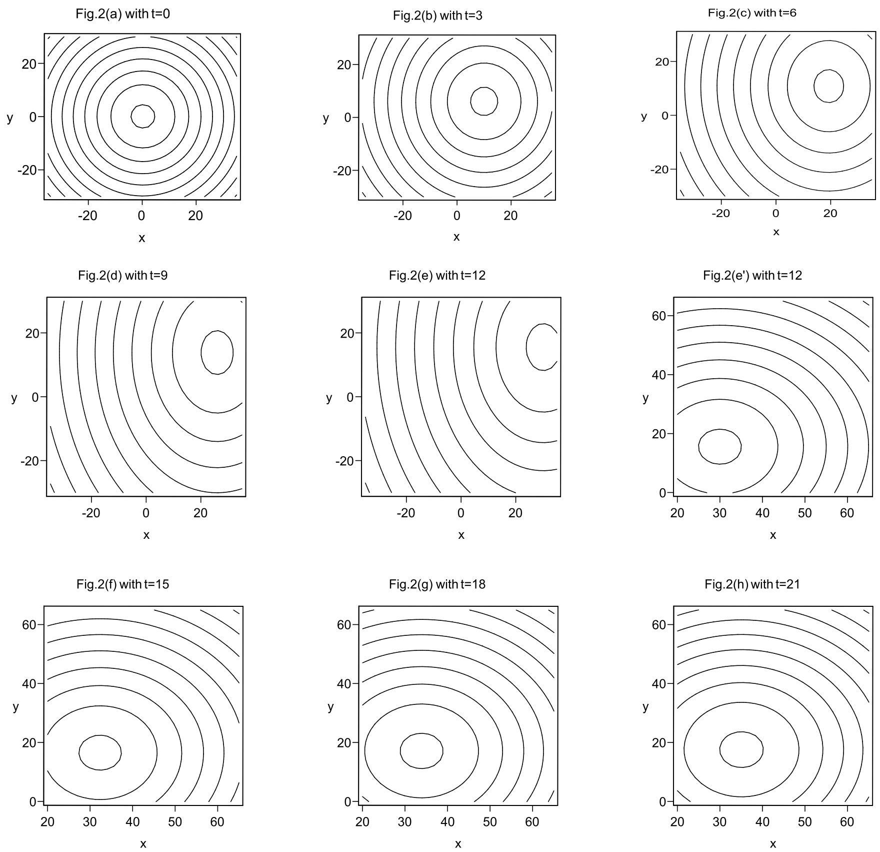

FIGURE 2. Contour figures of the density function ρ with the time intervals $\Delta t = 3$, with $c_0 = 20$, $\xi = 5$, $\alpha = 0$, $\beta = 1$.

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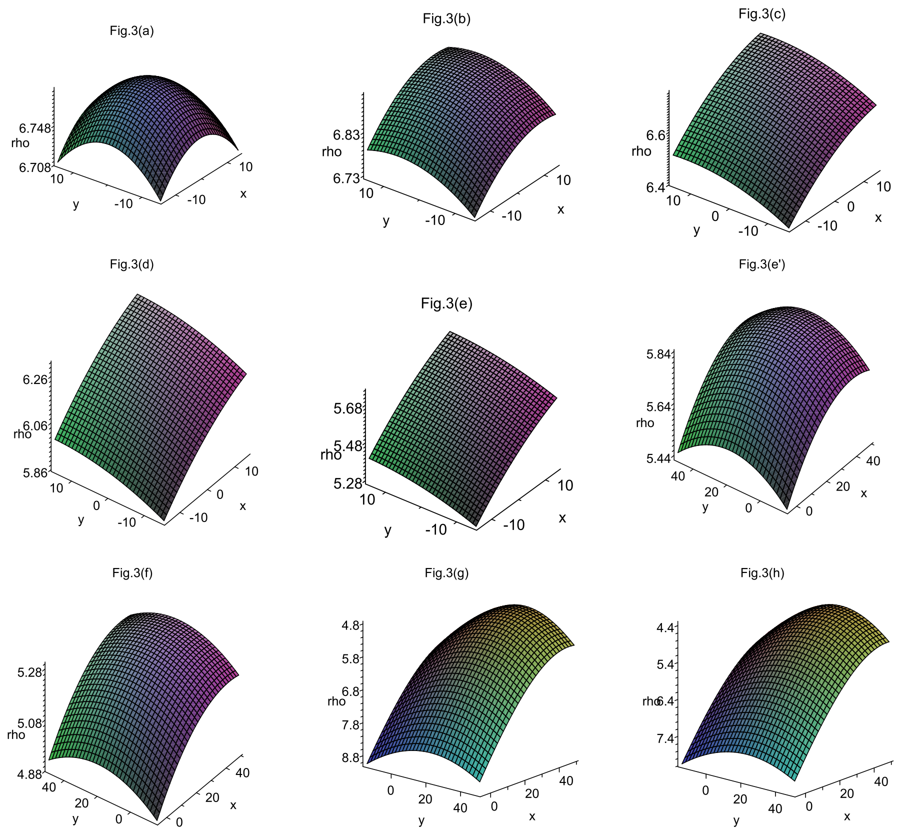

FIGURE 3. Time evolutions of the density function ρ at a regular time intervals $\Delta t = 3$, with $c_0 = 50$, $\xi = 3$, $\alpha = 1$, $\mu = 0.1$, $\beta = 0$.

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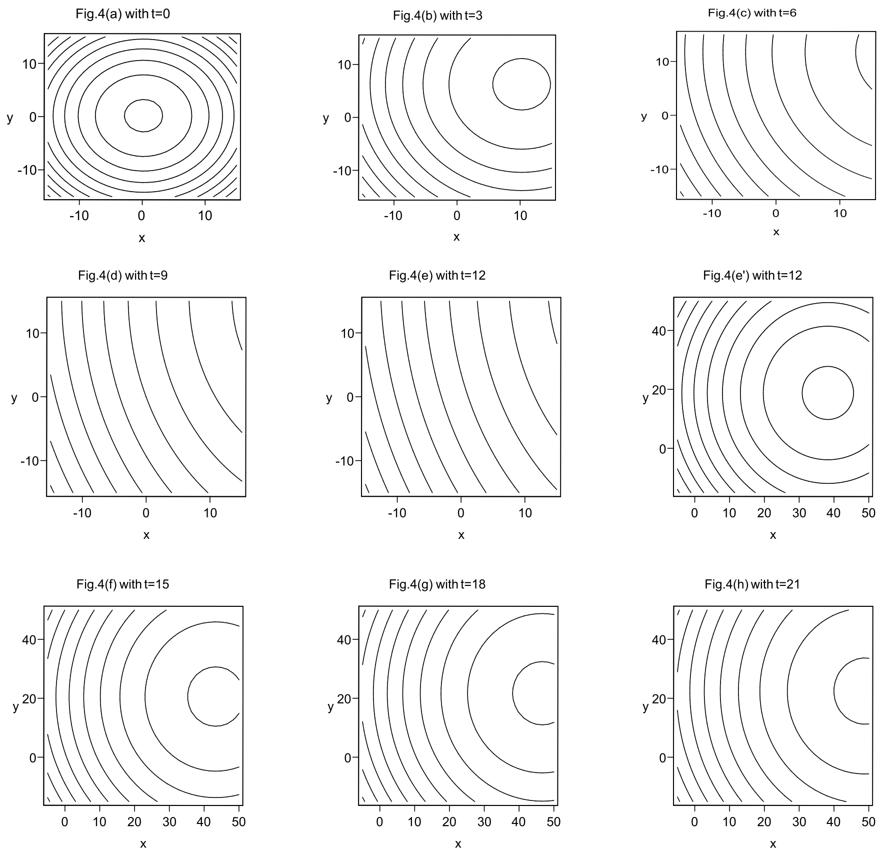

FIGURE 4. Contour figures of the density function ρ with the time intervals $\Delta t = 3$, with $c_0 = 50$, $\xi = 3$, $\alpha = 1$, $\mu = 0.1$, $\beta = 0$.

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