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SOLVABILITY OF A SYSTEM OF TOTALLY CHARACTERISTIC EQUATIONS RELATED TO KÄHLER METRICS

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Abstract. We consider a system of equations composed of a higher order singular partial differential equation of totally characteristic type and several higher order non-Kowalevskian linear equations. This system is a higher order version of a system that arose in Bielawski's investigations on Kähler metrics. We first prove that this system has a unique holomorphic solution. We then show that if the coefficients of the system are in some formal Gevrey class, then the unique solution is also in the same formal Gevrey class.

1. Introduction

In 2002, Bielawski considered the system

$$\begin{aligned} t\partial_t u &= f(t,x,u,\partial_x u, w_1,\dots,w_N) \\ \partial_t w_i &= \ell_i(x;\partial_x)u + h_i(x) \quad \text{for } i=1,\dots,N, \end{aligned} \tag{1.1}$$

where the function f is holomorphic with respect to all its variables and each ℓ_i is a second order linear differential operator whose coefficients are functions of x. He showed that it has a unique holomorphic solution $(u,w_1,\dots,w_N)$ that satisfies $u(0,x)\equiv 0$ and $w_i(0,x)\equiv 0$ for all i. This unique solvability result was necessary in showing that it is possible to extend a Kähler metric on a complex manifold X to a Ricci-flat Kähler metric in a neighborhood of X in a line bundle L, under the condition that the canonical S^1-action on L is Hamiltonian [1].

The first equation in this system is very similar to the one studied by Gérard and Tahara in [3] and in several succeeding papers in the 1990s. In fact, in proving the unique existence of a holomorphic solution to (1.1), Bielawski first converted it into an integro-differential equation and then suitably modified the method of Gérard-Tahara to tackle the resulting equation.

Suppose that the function $g(t,x,u,v)$ is holomorphic in a neighborhood of the origin $(0,0,0,0)\in\mathbb{C}_t\times\mathbb{C}_x^n\times\mathbb{C}_u\times\mathbb{C}_v^n$ and satisfies $g(0,x,0,0)\equiv 0$. The equation of Gérard and Tahara is the nonlinear singular partial differential equation

$$\begin{aligned} t\partial_t u &= g(t,x,u,\partial_x u) \\ &= a(x)t + b(x)u + c(x)\partial_x u + g_2(t,x,u,\partial_x u), \end{aligned} \tag{1.2}$$

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where we have denoted by $g_2(t,x,u,v)$ the collection of all nonlinear terms of $g(t,x,u,v)$ with respect to the variables t, u and v. Under the assumptions that $c(x)\equiv 0$ and $b(0)$ is not a positive integer, Gérard and Tahara showed that (1.2) has a unique solution $u(t,x)$ that satisfies $u(0,x)\equiv 0$. This is also known as the nonlinear Fuchsian case of (1.2).

In 1999, Chen and Tahara considered the *totally characteristic* case of (1.2), that is, instead of assuming that $c(x)\equiv 0$, they assumed that $c(x)$ vanishes at the origin but not identically zero. More precisely, they assumed that $c(x) = x\widetilde{c}(x)$ with $\widetilde{c}(0)\neq 0$. Under some Poincaré condition involving $b(0)$ and $\widetilde{c}(0)$, they were able to show that the totally characteristic case also has a unique solution $u(t,x)$ that satisfies $u(0,x)\equiv 0$.

In [4], we extended the unique solvability result of Bielawski to the following higher order version of (1.1):

$$\begin{aligned}(t\partial_t)^m u &= F\big(t,x,\{(t\partial_t)^j\partial_x^\alpha u\}_{j+|\alpha|\le m, j<m},\{w_i\}_{i=1,\dots,N}\big)\\ \partial_t^q w_i &= L_i(t,x;\partial_x)u + H_i(t,x),\quad i=1,\dots,N,\end{aligned}\tag{1.3}$$

where $m,q\geq 1$ and each L_i is a linear differential operator of order $q+1$ having coefficients which are dependent on both x and t. Note that the first equation of the system is an mth order singular nonlinear Fuchsian equation. We used a family of majorant functions used in [5] and [6] to show that the formal solution is convergent.

In this article, we revisit Bielawski's system of equations, this time under the assumption that the first equation is a singular equation of totally characteristic type. We shall show that the higher order version of the system possesses a unique holomorphic solution under some Poincaré condition. Finally, we generalize our results to the case when the coefficients of the partial Taylor expansion of (1.3) are not holomorphic functions of x but rather belong in some formal Gevrey class. This generalization is inspired by the work of Pongérard [6].

2. Holomorphic Solutions

2.1. **Main results.** We denote the set of all nonnegative integers by $\mathbb{N}$, and set $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. For any $v=(v_1,\dots,v_n)$ and $\alpha=(\alpha_1,\dots,\alpha_n)$, we define $v^\alpha = v_1^{\alpha_1}\cdots v_n^{\alpha_n}$ and $|\alpha| = \alpha_1+\dots+\alpha_n$. Let $(t,x)\in\mathbb{C}_t\times\mathbb{C}_x$ and fix $m,q\in\mathbb{N}^*$. Consider the system of differential equations

$$\begin{aligned}(t\partial_t)^m u &= F\big(t,x,\big\{(t\partial_t)^j\partial_x^\alpha u\big\}_{(j,\alpha)\in\Lambda},\{w_i\}_{i=1}^N\big),\\ \partial_t^q w_i &= \mathcal{L}_i(t,x;\partial_x)u + H_i(t,x)\quad\text{for } i=1,\dots,N,\end{aligned}\tag{2.1}$$

where the function $F(t,x,Y,Z)$ satisfies $F(0,x,0,0)\equiv 0$ and is holomorphic in some neighborhood containing

$$\{(t,x,Y,Z)\in\mathbb{C}^{2+\sharp\Lambda+N} : |t|\le r_0, |x|\le R_0, |Y_{j,\alpha}|\le R_1, |Z_i|\le R_1\},$$

for some positive constants r_0, R_0 and R_1. The index set Λ is defined by $\Lambda = \{(j,\alpha)\in\mathbb{N}^2 : j+\alpha\le m,\ j<m\}$ and its cardinality is denoted by $\sharp\Lambda$. The linear differential operator $\mathcal{L}_i$ is of order $q+m$ and of the form

$$\mathcal{L}_i(t,x;\partial_x) = \sum_{\gamma\le q+m}\mathcal{L}_{i,\gamma}(t,x)\partial_x^\gamma,\tag{2.2}$$

where for all $i = 1, \dots, N$ and $\gamma \le q + m$, the functions $\mathcal{L}_{i,\gamma}(t, x)$ and $H_i(t, x)$ are holomorphic in some neighborhood of $\{(t, x) : |t| \le r_0, |x| \le R_0\}$.

We define the set $\Lambda_0 = \{(j, \alpha) \in \Lambda : \partial_{Y_{j,\alpha}} F(0, x, 0, 0) \not\equiv 0\}$. Under the above assumptions, we can expand $F(t, x, Y, Z)$ as

$$\begin{aligned} F(t,x,Y,Z) = a(x)t &+ \sum_{(j,\alpha)\in\Lambda_0} b_{j,\alpha}(x) Y_{j,\alpha} + \sum_{1\le i\le N} d_i(x) Z_i \\ &+ \sum_{p+|\nu|+|\mu|\ge 2} g_{p,\nu,\mu}(x) t^p Y^\nu Z^\mu. \end{aligned}$$

Now suppose that for all $(j, \alpha) \in \Lambda_0$, we have $b_{j,\alpha}(x) = x^\alpha \lambda_{j,\alpha}(x)$ for some holomorphic function $\lambda_{j,\alpha}$ that satisfies $\lambda_{j,\alpha}(0) \ne 0$. In other words, we are assuming that the first equation of (2.1) is of *totally characteristic type* as defined by Chen-Tahara in [2]. Let

$$P(\tau,\xi) = \tau^m - \sum_{(j,\alpha)\in\Lambda} \lambda_{j,\alpha}(0) \tau^j \xi(\xi-1)\cdots(\xi-\alpha+1). \tag{2.3}$$

We have the following result on the existence and uniqueness of a holomorphic solution.

Theorem 2.1. *If $P(\tau, \xi) \ne 0$ for all $(\tau, \xi) \in \mathbb{N}^* \times \mathbb{N}$, then (2.1) has a unique holomorphic solution $(u, w_1, \dots, w_N)$ that satisfies $u(0, x) \equiv 0$ and $\partial_t^k w_i(0, x) \equiv 0$ for $k = 0, 1, \dots, q-1$ and $i = 1, \dots, N$.*

2.2. **Existence of a unique formal solution.** Under the assumption that $b_{j,\alpha} = x^\alpha \lambda_{j,\alpha}$, we can rewrite each $b_{j,\alpha}(t\partial_t)^j \partial_x^\alpha u$ as $\lambda_{j,\alpha}(t\partial_t)^j (x\partial_x)(x\partial_x - 1)\cdots(x\partial_x - \alpha + 1)u$. Thus if we let $x\lambda^*_{j,\alpha}(x) = \lambda_{j,\alpha}(x) - \lambda_{j,\alpha}(0)$, then the first equation of (2.1) can be written as

$$\begin{aligned} P(t\partial_t, x\partial_x)u = a(x)t &+ \sum_{(j,\alpha)\in\Lambda_0} x\big(x^\alpha \lambda^*_{j,\alpha}(x)\big)(t\partial_t)^j \partial_x^\alpha u + \sum_{1\le i\le N} d_i(x) w_i \\ &+ \sum_{p+|\nu|+|\mu|\ge 2} g_{p,\nu,\mu}(x) t^p \prod_{(j,\alpha)\in\Lambda} \big((t\partial_t)^j \partial_x^\alpha u\big)^{\nu_{j,\alpha}} \prod_{1\le i\le N} w_i^{\mu_i}, \end{aligned} \tag{2.4}$$

where P is the polynomial defined in (2.3).

We wish to find a formal solution $(u, w_1, \dots, w_N)$ of the form $u(t, x) = \sum_{j\ge1,k\ge0} u_{j,k} t^j x^k$ and $w_i(t, x) = \sum_{j\ge q,k\ge0} w_{i,j,k} t^j x^k$ that satisfy (2.1). Let us expand the coefficients as follows: $a(x) = \sum_{k\ge0} a_k x^k$, $\lambda^*_{\ell,\alpha}(x) = \sum_{k\ge0} \lambda^*_{\ell,\alpha,k} x^k$, $d_i(x) = \sum_{k\ge0} d_{i,k} x^k$ and $g_{p,\nu,\mu}(x) = \sum_{k\ge0} g_{p,\nu,\mu,k} x^k$ for the first equation, and $\mathcal{L}_{i,\gamma}(t, x) = \sum_{j\ge0,k\ge0} \mathcal{L}_{i,\gamma,j,k} t^j x^k$ and $H_i(t, x) = \sum_{j\ge0,k\ge0} H_{i,j,k} t^j x^k$ for the second equation of the system. Under the assumption that $P(\tau, \xi) \ne 0$, for any $(\tau, \xi) \in \mathbb{N}^* \times \mathbb{N}$, we see that

$$u_{1,0} = \begin{cases} P(1,0)^{-1} a_0 & \text{if } q > 1 \\ P(1,0)^{-1}\big(a_0 + \sum_{1\le i\le N} d_{i,0} w_{i,1,0}\big) & \text{if } q = 1 \end{cases}$$

$$w_{i,q,0} = (q!)^{-1} H_{i,0,0}.$$

In addition, for $J \geq 1$ and $K \geq 0$, there exists functions $\mathcal{F}_{J,K}$ and $\mathcal{G}_{J,K}$ such that

$$\begin{aligned} w_{i,J+q,K} &= \frac{J!}{(J+q)!}\mathcal{F}_{J,K}\Big(\{\mathcal{L}_{i,\gamma,j,k}\}_{i\leq N, j\leq J, k\leq K, \gamma\leq q+m},\\ &\quad \Big\{\frac{(k+\gamma)!}{k!}u_{j,k+\gamma}\Big\}_{j\leq J, k\leq K, \gamma\leq q+m}, H_{J,K}\Big)\\ u_{J,K} &= \frac{1}{P(J,K)}\mathcal{G}_{J,K}\Big(a_K, \{\lambda^*_{\ell,\alpha,k}\}_{k<K}, \Big\{\frac{J^\ell k!}{(k-\alpha)!}u_{J,k}\Big\}_{k<K,(\ell,\alpha)\in\Lambda_0},\\ &\quad \{d_{i,k}\}_{i\leq N, k\leq K}, \{w_{i,J,k}\}_{i\leq N,k\leq K}, \{g_{p,\nu,\mu,k}\}_{k\leq K, p+|\nu|+|\mu|\geq 2},\\ &\quad \Big\{\frac{J^\ell(k+\alpha)!}{k!}u_{j,k+\alpha}\Big\}_{j<J, k<K, (\ell,\alpha)\in\Lambda}, \{w_{i,j,k}\}_{j<J,k\leq K}\Big). \end{aligned}$$

Observe that for any $k \geq 0$, the coefficients $w_{i,q,k}$ are uniquely determined by the function H_i. These will enable us to solve for $u_{1,k}$. We move forward by solving for $w_{i,q+1,k}$ for any $k \geq 0$, which will allow us to solve for $u_{2,k}$. We follow these steps to recursively and uniquely determine all the coefficients of the formal solution.

We have thus shown that there exists a unique formal solution to (2.1). It remains to show that this formal solution is convergent.

2.3. **Convergence of the formal solution.** To show convergence, we use the majorant method. For power series $a(z) = \sum_{|\alpha|\geq 0} a_\alpha z^\alpha$ and $A(z) = \sum_{|\alpha|\geq 0} A_\alpha z^\alpha$, we say that A majorizes a, written as $a \ll A$, if for all $|\alpha| \geq 0$, we have $|a_\alpha| \leq A_\alpha$. We construct a system of majorant relations whose solution majorizes the formal solution to (2.1).

For simplicity suppose that constants r_0, R_0 and R_1 are all less than 1. We choose M large enough such that the functions $F(t,x,Y,Z)$, $\mathcal{L}_{i,\gamma}(t,x)$ (for $i = 1,\dots,N, \gamma \leq q+m$), $H_i(t,x)$ (for $i = 1,\dots,N$) and $\lambda^*_{j,\alpha}(x)$ (for $(j,\alpha) \in \Lambda_0$) appearing in (2.1), (2.2) and (2.4), are bounded by M for all $|x| \leq R_0$, $|t| \leq r_0$, $|Y_{j,\alpha}| \leq R_1$ and $|Z_i| \leq R_1$. In addition, we fix a sufficiently large $A > 0$ such that for all $(\tau,\xi) \in \mathbb{N}^* \times \mathbb{N}$, we have

$$\frac{1}{|P(\tau,\xi)|} \leq \frac{A}{1+\tau^m+\xi^m}. \tag{2.5}$$

We can choose such a constant since $P(\tau,\xi) \neq 0$ for all $(\tau,\xi) \in \mathbb{N}^*\times\mathbb{N}$ and since the polynomial $P(\tau,\xi)$ is of degree m in τ and ξ.

Proposition 2.2. *Consider the system of majorant relations*

$$\begin{aligned} &\Big[(t\partial_t)^m + (x\partial_x)^m + 1\Big]U(t,x)\\ &\gg \frac{AM}{1-x/R_0}\Big[\frac{t}{r_0} + x\sum_{(j,\alpha)\in\Lambda_0} x^\alpha (t\partial_t)^j\partial_x^\alpha U + \sum_{1\leq i\leq N}\frac{W_i}{R_1}\Big]\\ &\quad + \sum_{p+|\nu|+|\mu|\geq 2}\frac{AM}{1-x/R_0}\frac{t^p}{r_0^p R_1^{|\nu|+|\mu|}}\prod_{(j,\alpha)\in\Lambda}\Big((t\partial_t)^j\partial_x^\alpha U\Big)^{\nu_{j,\alpha}}\prod_{1\leq i\leq N} W_i^{\mu_i}, \end{aligned} \tag{2.6}$$

$$\partial_t^q W_i \gg \frac{M}{1-x/R_0-t/r_0}\Big[\sum_{\gamma\leq q+m}\partial_x^\gamma U + 1\Big] \quad \textit{for } i = 1,\dots,N, \tag{2.7}$$

$$U(0,x) \gg 0, \quad \partial_t^k W_i(0,x) \gg 0 \quad \textit{for } i = 1,\dots,N,\ k = 0,\dots,q-1. \tag{2.8}$$

Then for any $(U, W_1, \dots, W_N)$ *that satisfies the above relations, we have* $u \ll U$ *and* $w_i \ll W_i$ *for* $i = 1, \dots, N$*, where* $(u, w_1, \dots, w_N)$ *is the unique formal solution of* (2.1).

The above proposition implies that the task of proving the convergence of the formal solution is reduced to finding holomorphic functions $U(t, x)$ and $W_i(t, x)$ that satisfy the above relations. The proof of this proposition is an easy calculation and will be omitted here.

2.4. **Majorant functions.** The same family of majorant functions found in [4] will be used in the proof of our main result. Let $S = 1 + 1/2^2 + \cdots = \pi^2/6$. For $i \in \mathbb{N}$ and $s \in \mathbb{N}^*$, define the following family of functions:

$$\varphi_i(x) = \frac{1}{4S} \sum_{n \geq 0} \frac{x^n}{(n+1)^{2+i}} \quad \text{and} \quad \Phi_i^s(t, x) = \sum_{p \geq 0} t^p \frac{D^{sp}\varphi_i(x)}{(sp)!}.$$

Note that for any $i \in \mathbb{N}$ and $s \in \mathbb{N}^*$, $\varphi_i(x)$ converges when $|x| < 1$, and $\Phi_i^s(t, x/R)$ converges when $|t|^{1/s} + |x| < R$. We enumerate some useful properties of φ_i and Φ_i^s.

Proposition 2.3. *The following hold for all* $i \in \mathbb{N}$, $s \in \mathbb{N}^*$:

1. $\varphi_i(x)\varphi_i(x) \ll 2^i \varphi_i(x)$
2. $\varphi_{i+1}(x) \ll \varphi_i(x)$
3. $2^{-3-i}\varphi_i(x) \ll \varphi'_{i+1}(x) \ll \varphi_i(x)$
4. *For any* $k \in \mathbb{N}$, $xD^k\varphi_i(x) \ll 2^{2+i}D^k\varphi_i(x)$
5. $\Phi_i^s(x)\Phi_i^s(x) \ll 2^i\Phi_i^s(x)$
6. *For any* $\epsilon \in (0, 1)$*, there exists a constant* $B_{i,\epsilon} > 0$ *such that for all* $j \in \mathbb{N}$, $(1 - \epsilon x)^{-1}D^j\varphi_i(x) \ll B_{i,\epsilon}D^j\varphi_i(x)$.
7. *For a sufficiently small* $R(< 2^{-2-i})$,
   (a) $(p + q)!D^p\varphi_i(x/R) \ll p!D^{p+q}\varphi_i(x/R)$
   (b) $\varphi_i(t + x/R) \ll \Phi_i^s(t, x/R)$
   (c) *For any* $\epsilon \in (0, 1)$*, there exists a constant* $B_{i,\epsilon} > 0$ *such that*

$$\frac{\Phi_i^s(t, x/R)}{1 - \epsilon(t + x/R)} \ll B_{i,\epsilon}\Phi_i^s(t, x/R).$$

Proof. The first four assertions easily follow from the definition of φ_i. The proof for (5) may be found in [6] but essentially rests on (1) and the fact that

$$\frac{D^k(\varphi_i^2)}{k!} = \sum_{j=0}^{k} \frac{D^{k-j}\varphi_i}{(k-j)!} \frac{D^j\varphi_i}{j!}.$$

Item (6) follows from the estimates $4S\epsilon^n(n + 1)^{2+i} \leq B_{i,\epsilon}$ for any $n \geq 0$, and $D^j[(1 - \epsilon x)^{-1}\varphi_i(x)] \gg (1 - \epsilon x)^{-1}D^j\varphi_i(x)$ for any $j \geq 0$. The proof for (7a) is also found in [6] where it was shown by induction that $(p + 1)D^p\varphi_i(x/R) \ll D^{p+1}\varphi_i(x/R)$. This is equivalent in showing that for all $k, p, i \in \mathbb{N}$,

$$R\Big(\frac{k+p+2}{k+p+1}\Big)^{2+i} \frac{p+1}{k+p+1} < 1$$

for some R. This is achieved when R is chosen to be less than 2^{-2-i}. Item (7b) easily follows from (7a), and (7c) follows from (6) and (7b). $\square$

We now present holomorphic functions that satisfy all the relations in Proposition 2.2.

Proposition 2.4. *Let $r \in (0, r_0)$ and $\eta = m+q$. Then there exist positive constants L_1, L_2, L_3, c and $R(< 2^{-2-\eta}R_0)$ such that the functions*

$$U(t,x) = L_1 t\Phi_\eta^\eta\Big(\frac{t}{cr}, \frac{x}{R}\Big), \tag{2.9}$$

$$W_i(t,x) = L_2(cr)^{q+1}\sum_{k=1}^{\infty}(k+1)\big(\frac{t}{cr}\big)^{k+q}\frac{D^{\eta k}\varphi_\eta(x/R)}{(\eta k)!} + L_3 t^q\Phi_\eta^\eta\Big(\frac{t}{cr}, \frac{x}{R}\Big) \tag{2.10}$$

for $i = 1, \dots, N$ satisfy (2.6)*,* (2.7) *and* (2.8)*.*

Proof. For brevity, assume that the argument of Φ_i^s is always $(t/cr, x/R)$ and the argument of φ_i is always x/R, and thus omit these from our notations. We also choose the constant K to be sufficiently large such that $(1 - x/R_0 - t/r_0)^{-1}\Phi_i^\eta \ll K\Phi_i^\eta$ and $(1 - x/R_0)^{-1}D^j\varphi_i \ll KD^j\varphi_i$ for all $i \le \eta$, $0 \le j$ and $R < 2^{-2-\eta}R_0$.

Let us begin with (2.7). Using Proposition 2.3(3) and the fact that $\partial_t^j(t^jV) \gg V$ for any $V \gg 0$, the left-hand side can be estimated as follows:

$$\begin{aligned}
\partial_t^q W_i &\gg L_2 cr\sum_{k=0}^{\infty}\frac{(k+2)(k+q+1)!}{(k+1)!}\big(\frac{t}{cr}\big)^{k+1}\frac{D^{\eta k+\eta}\varphi_\eta}{(\eta k+\eta)!} + L_3\Phi_\eta^\eta \\
&\gg \frac{L_2 t}{2^{\eta(\eta+2)}\eta^{q+1}}\sum_{k=0}^{\infty}\big(\frac{t}{cr}\big)^k\frac{D^{\eta k}\varphi_0}{(\eta k)!} + L_3\Phi_\eta^\eta = \frac{L_2 t\Phi_0^\eta}{2^{\eta(\eta+2)}\eta^{q+1}} + L_3\Phi_\eta^\eta.
\end{aligned} \tag{2.11}$$

As for the corresponding right-hand side, we again use Proposition 2.3(3) to obtain

$$\partial_x^\gamma U \ll \frac{L_1 t}{R^\gamma}\sum_{k=0}^{\infty}\big(\frac{t}{cr}\big)^k\frac{D^{\eta k+\gamma}\varphi_\eta}{(\eta k)!} \ll \frac{L_1}{R^\eta}t\Phi_0^\eta.$$

Here, we used the fact that $\gamma \le \eta$. Since $1 \ll 4S\Phi_\eta^\eta$, the right-hand side of relation (2.7) will be majorized by

$$KM\Big(\frac{L_1\eta}{R^\eta}t\Phi_0^\eta + 4S\Phi_\eta^\eta\Big), \tag{2.12}$$

where we used Proposition 2.3(7c) and K is the constant defined at the beginning of the proof. Comparing (2.11) and (2.12), we obtain the following conditions for the constants L_1, L_2, L_3 and R:

$$\frac{KML_1\eta}{R^\eta} \le \frac{L_2}{2^{\eta(\eta+2)}\eta^{q+1}} \quad \text{and} \quad 4SKM \le L_3. \tag{2.13}$$

Let us now turn our attention to (2.6). Since $\eta \geq m$, we have

$$
\begin{aligned}
(t\partial_t)^m U &= L_1 cr \sum_{k=0}^{\infty} (k+1)^m \big(\frac{t}{cr}\big)^{k+1} \frac{D^{\eta k}\varphi_\eta}{(\eta k)!} \\
&\gg \frac{L_1 cr}{\eta^m} \sum_{k=1}^{\infty} \big(\frac{t}{cr}\big)^{k+1} \frac{D^{\eta k}\varphi_\eta}{(\eta k - m)!} \\
&\gg \frac{L_1 cr}{2^{m(\eta+2)}\eta^m} \sum_{k=1}^{\infty} \big(\frac{t}{cr}\big)^{k+1} \frac{D^{\eta k - m}\varphi_{\eta-m}}{(\eta k - m)!} \\
&= \frac{L_1 cr}{2^{m(\eta+2)}\eta^m} \sum_{k=0}^{\infty} \big(\frac{t}{cr}\big)^{k+2} \frac{D^{\eta k - m+\eta}\varphi_{\eta-m}}{(\eta k - m+\eta)!} \\
&\gg \frac{L_1 cr}{2^{m(\eta+2)}\eta^m} \sum_{k=0}^{\infty} \big(\frac{t}{cr}\big)^{k+2} \frac{D^{\eta k}\varphi_{\eta-m}}{(\eta k)!} \\
&= \frac{L_1 t^2}{2^{m(\eta+2)}\eta^m cr} \Phi^\eta_{\eta-m}.
\end{aligned} \tag{2.14}
$$

We used Proposition 2.3(3) in the third line, while in the last simplification, we used Proposition 2.3(7a) and the fact that $\eta - m \geq 0$. Now, consider the other terms on the left-hand side. Since $m \geq 1$ and $(x\partial_x)^m V \gg x\partial_x V$ for any holomorphic function $V \gg 0$, we have

$$
\begin{aligned}
(x\partial_x)^m U &\gg \frac{L_1 t}{R} \sum_{k=0}^{\infty} \big(\frac{t}{cr}\big)^{k} \frac{x D^{\eta k+1}\varphi_\eta}{(\eta k)!} \\
&\gg \frac{L_1 t}{2^{\eta+2} R} \sum_{k=0}^{\infty} \big(\frac{t}{cr}\big)^{k} \frac{x D^{\eta k}\varphi_{\eta-1}}{(\eta k)!} \\
&\gg \frac{L_1 t x}{2^{\eta+2} R} \Phi^\eta_{\eta-m}.
\end{aligned} \tag{2.15}
$$

To majorize the third summation on the right-hand side of (2.6), we will estimate the term $((t\partial_t)^m + 1)U$ in the following manner:

$$
\begin{aligned}
\big((t\partial_t)^m + 1\big)U &\gg L_1 cr \sum_{k=1}^{\infty} (k+1)^m \big(\frac{t}{cr}\big)^{k+1} \frac{D^{\eta k}\varphi_\eta}{(\eta k)!} + L_1 t \Phi^\eta_\eta \\
&\gg L_1 cr \sum_{k=0}^{\infty} (k+2) \big(\frac{t}{cr}\big)^{k+2} \frac{D^{\eta k+\eta}\varphi_\eta}{(\eta k+\eta)!} + L_1 t \Phi^\eta_\eta.
\end{aligned} \tag{2.16}
$$

Therefore, using (2.14), (2.15) and (2.16), the left-hand side of (2.6) will majorize

$$
\begin{aligned}
&\frac{1}{2}\Big(\frac{L_1 t^2}{2^{m(\eta+2)}\eta^m cr} \Phi^\eta_{\eta-m} + L_1 t \Phi^\eta_\eta\Big) + \frac{L_1 t x}{2^{\eta+2} R} \Phi^\eta_{\eta-m} \\
&+ \frac{1}{2}\Big[L_1 cr \sum_{k=0}^{\infty} (k+2) \big(\frac{t}{cr}\big)^{k+2} \frac{D^{\eta k+\eta}\varphi_\eta}{(\eta k+\eta)!} + L_1 t \Phi^\eta_\eta\Big].
\end{aligned} \tag{2.17}
$$

We will deal separately with each summation on the right-hand side of (2.6). Using Proposition 2.3 items (7a), (3) and (4) in this order, we have for any $j+\alpha \leq m$:

$$
x^\alpha (t\partial_t)^j \partial_x^\alpha U = \frac{L_1 cr}{R^\alpha} \sum_{k=0}^{\infty} (k+1)^j \big(\frac{t}{cr}\big)^{k+1} \frac{x^\alpha D^{\eta k+\alpha}\varphi_\eta}{(\eta k)!}
$$

$$\begin{aligned}
&\ll \frac{L_1 cr}{R^\alpha}\sum_{k=0}^{\infty}(k+1)^j\big(\frac{t}{cr}\big)^{k+1}\frac{x^\alpha D^{\eta k+\alpha+j}\varphi_\eta}{(\eta k+j)!}\\
&\ll \frac{L_1 cr}{R^\alpha}\sum_{k=0}^{\infty}\big(\frac{t}{cr}\big)^{k+1}\frac{x^\alpha D^{\eta k}\varphi_{\eta-\alpha-j}}{(\eta k)!}\\
&\ll 2^{\alpha(2+\eta)}L_1 t\Phi^\eta_{\eta-m} \ll 2^{m(2+\eta)}L_1 t\Phi^\eta_{\eta-m},
\end{aligned}$$

and therefore by Proposition 2.3(7c),

$$\frac{AMx}{1-x/R_0}\sum_{(j,\alpha)\in\Lambda_0} x^\alpha (t\partial_t)^j(\partial_x)^\alpha U \ll (2^{m(2+\eta)}\sharp\Lambda_0 AKM)L_1 tx\Phi^\eta_{\eta-m},$$

where $\sharp\Lambda_0$ is the cardinality of Λ_0. Comparing this to the term with $tx\Phi^\eta_{\eta-m}$ in (2.17), we then see that we need to satisfy the inequality

$$2^{m(2+\eta)}\sharp\Lambda_0 AKM \le \frac{1}{2^{\eta+2}R}. \tag{2.18}$$

Now, we consider the linear terms containing W_i. Using Proposition 2.3(7a), we have

$$\begin{aligned}
W_i &\ll L_2(cr)^{q+1}\ \sum_{k=0}^{\infty}(k+2)\big(\frac{t}{cr}\big)^{k+q+1}\frac{D^{\eta(k+1)+\eta(q-1)}\varphi_\eta}{(\eta(k+1)+\eta(q-1))!}+L_3t^q\Phi^\eta_\eta\\
&= L_2(cr)^{q+1}\sum_{k=q-1}^{\infty}(k-q+3)\big(\frac{t}{cr}\big)^{k+2}\frac{D^{\eta(k+1)}\varphi_\eta}{(\eta(k+1))!}+L_3t^q\Phi^\eta_\eta\\
&\ll L_2(cr)^{q+1}\ \sum_{k=0}^{\infty}(k+2)\big(\frac{t}{cr}\big)^{k+2}\frac{D^{\eta k+\eta}\varphi_\eta}{(\eta k+\eta)!}+L_3t^q\Phi^\eta_\eta.
\end{aligned}\tag{2.19}$$

Note that $t^{q-1}\ll(1-t/r_0)^{-1}\ll(1-t/r_0-x/R_0)^{-1}\ll 4SK\Phi^\eta_\eta$ since $r_0<1$. Thus, the summation of the W_i's on the right-hand side is majorized by

$$\frac{4AKMNS}{R_1}\Big[L_2(cr)^2\sum_{k=0}^{\infty}(k+2)\big(\frac{t}{cr}\big)^{k+2}\frac{D^{\eta k+\eta}\varphi_\eta}{(\eta k+\eta)!}+KL_3t\Phi^\eta_\eta\Big], \tag{2.20}$$

and by comparing this with (2.17), we obtain the inequalities:

$$\frac{4AKMNSL_2cr}{R_1}\le\frac{L_1}{2}\quad\text{and}\quad\frac{4AK^2MNSL_3}{R_1}\le\frac{L_1}{2}. \tag{2.21}$$

As for the nonlinear terms, note that for $j+\alpha\le m$, $j<m$, we have

$$\begin{aligned}
(t\partial_t)^j(\partial_x^\alpha)U &\ll \frac{L_1 cr}{R^\alpha}\sum_{k=0}^{\infty}(k+1)^j\big(\frac{t}{cr}\big)^{k+1}\frac{D^{\eta k+\alpha+j}\varphi_\eta}{(\eta k+j)!}\\
&\ll \frac{L_1 cr}{R^\alpha}\sum_{k=0}^{\infty}(k+1)^j\big(\frac{t}{cr}\big)^{k+1}\frac{D^{\eta k}\varphi_{\eta-\alpha-j}}{(\eta k+j)!}\\
&\ll \frac{L_1 cr}{R^\alpha}\sum_{k=0}^{\infty}\big(\frac{t}{cr}\big)^{k+1}\frac{D^{\eta k}\varphi_{\eta-m}}{(\eta k)!}=\frac{L_1t}{R^m}\Phi^\eta_{\eta-m}.
\end{aligned}$$

Note that we again used (7a) and (3) of Proposition 2.3. Since $cr < 1$ and $r_0 < 1$, we see that

$$\begin{aligned} W_i &\ll L_2(cr)^{q+1}\sum_{k=1}^{\infty}\Big(\frac{t}{cr}\Big)^{k+q}\frac{D^{\eta k-1}\varphi_{\eta-1}}{(\eta k-1)!}+L_3t^q\Phi_\eta^\eta \\ &\ll L_2crt^q\sum_{k=0}^{\infty}\Big(\frac{t}{cr}\Big)^{k}\frac{D^{\eta k}\varphi_{\eta-m}}{(\eta k)!}+L_3t^q\Phi_{\eta-m}^\eta \\ &\ll 4KS(L_2+L_3)t\Phi_{\eta-m}^\eta. \end{aligned}\tag{2.22}$$

Using the above estimates for $(t\partial_t)^j(\partial_x^\alpha)U$ and W_i, we can now majorize the remaining terms on the right-hand side of (2.6). Setting $\kappa = 2^{\eta-m}\max\{\sharp\Lambda, N\}$, the remaining terms of the right-hand side will be majorized by

$$\begin{aligned} &\frac{AM}{1-x/R_0}\Big\{\sum_{k=1}^{\infty}\Big(\frac{t}{r_0}\Big)^k+\sum_{\substack{k+|\nu|+|\mu|\ge 2\\ |\nu|+|\mu|\ge 1}}\Big(\frac{t}{r_0}\Big)^k\Big(\frac{L_1t\Phi_{\eta-m}^\eta}{R^mR_1}\Big)^{|\nu|} \\ &\quad\times\Big(\frac{4KS(L_2+L_3)t\Phi_{\eta-m}^\eta}{R_1}\Big)^{|\mu|}\Big\} \\ &\ll \frac{AM}{1-x/R_0}\Big\{\frac{1}{1-t/r_0}\cdot\frac{t}{r_0}+\Phi_{\eta-m}^\eta\sum_{\substack{k+i+j\ge 2\\ i+j\ge 1}}\Big(\frac{t}{r_0}\Big)^k\Big(\frac{\kappa L_1t}{R^mR_1}\Big)^i \\ &\quad\times\Big(\frac{\kappa 4KS(L_2+L_3)t}{R_1}\Big)^j\Big\} \\ &\ll \frac{AM}{1-x/R_0}\Big\{\frac{4St\Phi_\eta^\eta}{r_0(1-t/r_0)}+\Big(\frac{1}{r_0}+\frac{\kappa L_1}{R^mR_1}+\frac{\kappa 4KS(L_2+L_3)}{R_1}\Big)^2 \\ &\quad\times\frac{t^2\Phi_{\eta-m}^\eta}{1-\frac{t}{r_0}-\frac{\kappa L_1t}{R^mR_1}-\frac{\kappa 4KS(L_2+L_3)t}{R_1}}\Big\} \\ &\ll AMK\Big\{\frac{4S}{r_0}t\Phi_\eta^\eta+\Big(\frac{1}{r_0}+\frac{\kappa L_1}{R^mR_1}+\frac{\kappa 4KS(L_2+L_3)}{R_1}\Big)^2t^2\Phi_{\eta-m}^\eta\Big\}, \end{aligned}\tag{2.23}$$

where the last simplification is possible if

$$\frac{1}{r_0}+\frac{\kappa L_1}{R^mR_1}+\frac{\kappa 4KS(L_2+L_3)}{R_1}\le\frac{1}{cr_0}.\tag{2.24}$$

Comparing (2.17) and (2.23), we obtain the conditions

$$\frac{4SAMK}{r_0}\le\frac{L_1}{2},\tag{2.25}$$

$$AMK\Big(\frac{1}{r_0}+\frac{\kappa L_1}{R^mR_1}+\frac{\kappa 4KS(L_2+L_3)}{R_1}\Big)^2\le\frac{L_1}{2^{m(\eta+2)+1}\eta^mcr}.\tag{2.26}$$

By choosing a small enough R and fixing it, and then choosing sufficiently large L_3, L_1 and L_2 in that order, and lastly choosing c small enough, we can satisfy conditions (2.13), (2.18), (2.21), (2.24), (2.25) and (2.26) so that U and W_i satisfy (2.6), (2.7) and (2.8). $\square$

In view of Proposition 2.2, the functions U and W_i defined in (2.9) and (2.10) are majorants of u and w_i, respectively. Since $W_i \ll 4KS(L_2+L_3)t\Phi_{\eta-m}^\eta$ from

(2.22), we know that the formal solution $(u, w_1, \dots, w_n)$ converges on

$$\{(t,x) : |t/(cr)|^{1/\eta} + |x| \le R\},$$

and this completes our proof for Theorem 2.1.

3. Gevrey class solutions

3.1. **Formulation and result.** In this section, we consider the case when the coefficients of (2.1) are in some formal Gevrey class. This space is defined as follows: for $d \ge 1$, the formal Gevrey class G_x^d of index d is defined to be the space of all formal series $u(x) = \sum_{\alpha \ge 0} u_\alpha x^\alpha$ such that

$$\sum_{\alpha \ge 0} \frac{u_\alpha x^\alpha}{(\alpha!)^{d-1}}$$

is a convergent power series. Similarly, we define the formal Gevrey class $G_x^d[Z]$ to be the space of all formal expansions $u(x,Z) = \sum_{\alpha \ge 0} u_\alpha(Z) x^\alpha$ such that

$$\sum_{\alpha \ge 0} \frac{u_\alpha(Z) x^\alpha}{(\alpha!)^{d-1}}$$

is a convergent power series. The function $u(x, Z)$ is said to be (formal) Gevrey of index d in the variable x and holomorphic in the variable Z.

We now state the problem. Fix $d \ge 1$ and consider the system

$$\begin{aligned} (t\partial_t)^m u &= F\big(t, x, \big\{(t\partial_t)^j \partial_x^\alpha u\big\}_{(j,\alpha)\in\Lambda^d}, \{w_i\}_{i=1}^N\big), \\ \partial_t^q w_i &= \mathcal{L}_i(t, x; \partial_x) u + H_i(t,x) \quad \text{for } i = 1, \dots, N, \end{aligned} \tag{3.1}$$

where $\Lambda^d = \big\{(j,\alpha) \in \mathbb{N}^2 : j + |d\alpha| \le m,\ j < m\big\}$ and $\mathcal{L}_i$ is a linear operator given by

$$\mathcal{L}_i(t,x;\partial_x) = \sum_{|d\gamma| \le q+m} \mathcal{L}_{i,\gamma}(t,x)\partial_x^\gamma. \tag{3.2}$$

In addition, assume that

$$\begin{gathered} F(t,x,Y,Z) \in G_x^d[t,Y,Z] \\ \mathcal{L}_{i,\gamma}(t,x), H_i(t,x) \in G_x^d[t]. \end{gathered}$$

Note that Λ^d is a subset of the index set Λ defined in Section 2 and is equal to Λ if $d = 1$. In addition, the linear operator $\mathcal{L}_i$ is of degree at most $\lfloor (q+m)/d \rfloor$, where $\lfloor x \rfloor$ is the floor function of x.

Define $\Lambda_0^d = \{(j,\alpha) \in \Lambda^d; \partial_{Y_{j,\alpha}} F(0,x,0,0) \not\equiv 0\}$. Assume that for all $(j,\alpha) \in \Lambda_0^d$, there exists $\lambda_{j,\alpha}(x) \in G_x^d$ such that $b_{j,\alpha}(x) = x^\alpha \lambda_{j,\alpha}(x)$ and $\lambda_{j,\alpha}(0) \ne 0$. Define the polynomial

$$P^d(\tau,\xi) = \tau^m - \sum_{(j,\alpha)\in\Lambda_0^d} \lambda_{j,\alpha}(0)\tau^j \xi(\xi-1)\cdots(\xi-\alpha+1).$$

We have the following result.

Theorem 3.1. *If $P^d(\tau,\xi) \ne 0$ for all $(\tau,\xi) \in \mathbb{N}^* \times \mathbb{N}$, then* (3.1) *has a unique formal solution $(u, w_1, \dots, w_N)$ of class $G_x^d[t]$ that satisfies $u(0,x) \equiv 0$ and $\partial_t^k w_i(0,x) \equiv 0$ for $k = 0, 1, \dots, q-1$ and $i = 1, \dots, N$.*

Under the assumptions on F and $\mathcal{L}_i$, (3.1) can be rewritten as

$$\begin{aligned}(t\partial_t)^m u &= a(x)t + \sum_{(j,\alpha)\in\Lambda_0^d} b_{j,\alpha}(x)(t\partial_t)^j\partial_x^\alpha u + \sum_{1\le i\le N} d_i(x)w_i \\ &\quad + \sum_{p+|\nu|+|\mu|\ge 2} g_{p,\nu,\mu}(x)t^p \prod_{(j,\alpha)\in\Lambda^d}\big((t\partial_t)^j\partial_x^\alpha u\big)^{\nu_{j,\alpha}} \prod_{1\le i\le N} w_i^{\mu_i} \\ \partial_t^q w_i &= \sum_{|d\gamma|\le q+m} \mathcal{L}_{i,\gamma}(t,x)\partial_x^\gamma u + H_i(t,x) \quad \text{for } i=1,\dots,N.\end{aligned} \tag{3.3}$$

As in the holomorphic case, the existence of a formal solution directly follows from the assumption that $P^d(\tau,\xi)\neq 0$ for all $(\tau,\xi)\in\mathbb{N}^*\times\mathbb{N}$.

3.2. **Preparatory lemmas.** To study the formal series of class $G_x^d[Z]$, we use the following notation: given $u(x,Z)=\sum_{j\ge 0} u_j(Z)x^j$, define

$$u^{(d)}(x,Z) = \sum_{j\ge 0} u_j(Z)(j!)^{d-1}x^j.$$

We have a similar definition for formal series of class G_x^d. In view of this notation, observe that if $u(x,Z)$ is of class $G_x^d[Z]$, then there is a unique holomorphic function $w(x,Z)$ such that $w^{(d)}=u$. We shall denote this w by $\hat{u}$. We shall do the same for formal series of class G_x^d.

It is easy to show that for formal series u and w, $u \ll w \iff u^{(d)} \ll w^{(d)}$ and $u^{(d)}w^{(d)} \ll (uw)^{(d)}$. These two results imply that by replacing φ_i and Φ_i^s by $(\varphi_i)^{(d)}$ and $(\Phi_i^s)^{(d)}$ respectively, the results for Proposition 2.3 will all hold except for (4), (6), (7a) and (7c). The next lemma states the modified results.

Lemma 3.2. *The following hold for all $i\in\mathbb{N}$:*

(1) $x(D^k\varphi_i(x))^{(d)} \ll 2^{2+i}(D^k\varphi_i(x))^{(d)}$;
(2) *For any $\epsilon\in(0,1)$, there exists a constant $B_{i,\epsilon}>0$ such that for all $j\in\mathbb{N}$,* $((1-\epsilon x)^{-1})^{(d)}(D^j\varphi_i(x))^{(d)} \ll B_{i,\epsilon}(D^j\varphi_i(x))^{(d)}$;
(3) *For a sufficiently small $R<2^{-2-i}$,*
$$(p+q)!(D^p\varphi_i(x/R))^{(d)} \ll (p!)(D^{p+q}\varphi_i(x/R))^{(d)};$$
(4) *Let $p,q\in\mathbb{N}$. There exists a constant C_d (dependent on d but not on p and q) such that*
$$(D^{p+q}\varphi_i)^{(d)} \ll D^p(D^q\varphi_i)^{(d)} \ll C_d(D^{\lceil dp\rceil+q}\varphi_i)^{(d)}.$$
Here, $\lceil x\rceil$ is the ceiling of x;
(5) *For any $\epsilon\in(0,1)$ and $R<2^{-2-i}$, there exists a constant $B_{i,\epsilon}>0$ such that*
$$\Big(\frac{1}{1-\epsilon(t+x/R)}\Big)^{(d)}(\Phi_i^s)^{(d)}(t,x/R) \ll B_{i,\epsilon}(\Phi_i^s)^{(d)}(t,x/R).$$

Proof. The proofs for (2), (3) and (5) follow from our two previous assertions and the proof for (1) uses the fact that $x=(x)^{(d)}$. To prove (4), we note that $(D^{p+q}\varphi_i)^{(d)} \ll D^p(D^q\varphi_i)^{(d)}$ by inspection. For the second majorant relation, it is sufficient to show that for all j, p, $q\ge 0$, the quantity

$$\frac{(j+p+q)!}{(j+\lceil dp\rceil+q)!}\Big(\frac{j+\lceil dp\rceil+q+1}{j+p+q+1}\Big)^{2+i}\Big(\frac{(j+p)!}{j!}\Big)^{d-1}$$

is bounded, which easily follows from the definition of $\lceil x\rceil$. $\square$

Proof of Theorem 3.1. Let $\hat{F}(t,x,Y,Z)$, $\hat{\mathcal{L}}_{i,\gamma}(t,x)$ and $\hat{H}_i(t,x)$ be the holomorphic functions derived from $F(t,x,Y,Z)$, $\mathcal{L}_{i,\gamma}(t,x)$ and $H_i(t,x)$ in (3.1) and (3.2). Define $\lambda^*_{j,\alpha}$ by $x\lambda^*_{j,\alpha}(x)=\lambda_{j,\alpha}(x)-\lambda_{j,\alpha}(0)$. Let $\hat{\lambda}^*_{j,\alpha}(x)$ be the holomorphic function derived from $\lambda^*_{j,\alpha}$.

Suppose that the functions $\hat{F}(t,x,Y,Z)$, $\hat{\mathcal{L}}_{i,\gamma}(t,x)$, $\hat{H}_i(t,x)$, and $\hat{\lambda}^*_{j,\alpha}(x)$ are bounded by some constant $M>0$ for $|x|\le R_0$, $|t|\le r_0$, $|Y_{j,\alpha}|\le R_1$ and $|Z_i|\le R_1$. In addition, fix a constant $A_d>0$ such that

$$\frac{1}{|P^d(\tau,\xi)|}\le\frac{A_d}{1+\tau^m+\xi^{\lfloor m/d\rfloor}} \tag{3.4}$$

for all $(\tau,\xi)\in\mathbb{N}^*\times\mathbb{N}$. We present this proposition as a counterpart of Proposition 2.2.

Proposition 3.3. *Consider the following majorant system:*

$$\begin{aligned}&\Big[(t\partial_t)^m+(x\partial_x)^{\lfloor m/d\rfloor}+1\Big]U(t,x)\\&\gg\Big(\frac{AM}{1-x/R_0}\Big)^{(d)}\Big[\frac{t}{r_0}+x\sum_{(j,\alpha)\in\Lambda_0}x^\alpha(t\partial_t)^j\partial_x^\alpha U+\sum_{1\le 1\le N}\frac{W_i}{R_1}\Big]\\&\quad+\sum_{p+|\nu|+|\mu|\ge 2}\Big(\frac{AM}{1-x/R_0}\Big)^{(d)}\frac{t^p}{r_0^pR_1^{|\nu|+|\mu|}}\prod_{(j,\alpha)\in\Lambda}\Big((t\partial_t)^j\partial_x^\alpha U\Big)^{\nu_{j,\alpha}}\prod_{1\le i\le N}W_i^{\mu_i}\end{aligned} \tag{3.5}$$

$$\partial_t^qW_i\gg\Big(\frac{M}{1-x/R_0-t/r_0}\Big)^{(d)}\Big[\sum_{|d\gamma|\le q+m}\partial_x^\gamma U+1\Big] \tag{3.6}$$

$$U(0,x)\gg 0\quad\text{and}\quad \partial_t^kW_i(0,x)\gg 0\quad\text{for } i=1,\dots,N,\ k=1,\dots,q. \tag{3.7}$$

If the formal series $U(t,x)$ and $W_i(t,x)$ $(i=1,\dots,N)$ satisfy the above relations, then $u\ll U$ and $w_i\ll W_i$ for $i=1,\dots,N$, where $(u,w_1,\dots,w_N)$ is the formal solution to (3.1).

The task of proving the Gevrey regularity of the formal solution is now reduced to finding a formal solution $(U,W_1,\dots,W_N)$ of class G^d_x that satisfies the above relations.

Proposition 3.4. *Let $r\in(0,r_0)$ and $\eta=m+q$. Then there exist positive constants L_1, L_2, L_3, c and $R(<2^{-2-\eta}R_0)$ such that*

$$U(t,x)=L_1t(\Phi^\eta_\eta)^{(d)}\Big(\frac{t}{cr},\frac{x}{R}\Big) \tag{3.8}$$

and

$$W_i(t,x)=L_2(cr)^{q+1}\sum_{k=1}^\infty(k+1)\Big(\frac{t}{cr}\Big)^{k+q}\frac{(D^{\eta k}\varphi_\eta(x/R))^{(d)}}{(\eta k)!}+L_3t^q(\Phi^\eta_\eta)^{(d)}\Big(\frac{t}{cr},\frac{x}{R}\Big) \tag{3.9}$$

for $i=1,\dots,N$, satisfy (3.5), (3.6) *and* (3.7).

The proof of this proposition is very similar to the proof of the holomorphic case. The only notable difference is when we deal with derivatives of $(\varphi_i)^{(d)}$ and $(\Phi^s_i)^{(d)}$. For brevity, we shall omit the arguments of Φ^s_i and φ^s_i in our notations. Choose the constant K to be sufficiently large such that

$$\Big(\frac{1}{1-x/R_0-t/r_0}\Big)^{(d)}(\Phi_i^\eta)^{(d)} \ll K(\Phi_i^\eta)^{(d)},$$
$$\Big(\frac{1}{1-x/R_0}\Big)^{(d)}(D^j\varphi_i^\eta)^{(d)} \ll K(D^j\varphi_i^\eta)^{(d)},$$

for any $i \le \eta$, $0 \le j$ and $R < 2^{-2-\eta}R_0$. Choose also a constant C_d to be sufficiently large to satisfy Lemma 3.2(4).

As in the result for the holomorphic case, we have the following estimate for the left-hand side of (3.6):

$$\partial_t^q W_i \gg \frac{L_2 t(\Phi_0^\eta)^{(d)}}{2^{\eta(\eta+2)}\eta^{q+1}} + L_3(\Phi_\eta^\eta)^{(d)}. \tag{3.10}$$

For the corresponding right-hand side, note that if $|d\gamma| \le q+m = \eta$ then $\lceil d\gamma\rceil \le \eta$. By Lemma 3.2(4), we have

$$\begin{aligned}
\partial_x^\gamma U &\ll \frac{C_d L_1 t}{R^\gamma}\sum_{k=0}^\infty \big(\frac{t}{cr}\big)^k \frac{(D^{\eta k+\lceil d\gamma\rceil}\varphi_\eta)^{(d)}}{(\eta k)!}\\
&\ll \frac{C_d L_1 t}{R^\eta}\sum_{k=0}^\infty \big(\frac{t}{cr}\big)^k \frac{(D^{\eta k}\varphi_0)^{(d)}}{(\eta k)!}\\
&\ll \frac{C_d L_1}{R^\eta} t(\Phi_0^\eta)^{(d)}.
\end{aligned}$$

Therefore by Lemma 3.2(5), the right-hand side of (3.6) will be majorized by

$$KM\Big(\frac{C_d L_1\lfloor \eta/d\rfloor}{R^\eta} t(\Phi_0^\eta)^{(d)} + 4S(\Phi_\eta^\eta)^{(d)}\Big). \tag{3.11}$$

Comparing (3.10) and (3.11), we obtain the conditions

$$\frac{C_d KML_1\lfloor \eta/d\rfloor}{R^\eta} \le \frac{L_2}{2^{\eta(\eta+2)}\eta^{q+1}} \quad \text{and} \quad 4SKM \le L_3. \tag{3.12}$$

Now for left-hand side of (3.5), we have a similar result as in the holomorphic case given by

$$(t\partial_t)^m U \gg \frac{L_1 t^2}{2^{m(\eta+2)}\eta^m cr}(\Phi_{\eta-m}^\eta)^{(d)}. \tag{3.13}$$

For the term with $(x\partial_x)^{\lfloor m/d\rfloor}$, we use the same technique as in the previous section to obtain

$$\begin{aligned}
(x\partial_x)^{\lfloor m/d\rfloor} U &\gg L_1 t\sum_{k=0}^\infty \big(\frac{t}{cr}\big)^k \frac{(x\partial_x)(D^{\eta k}\varphi_\eta)^{(d)}}{(\eta k)!}\\
&\gg \frac{L_1 t}{R}\sum_{k=0}^\infty \big(\frac{t}{cr}\big)^k \frac{x(D^{\eta k+1}\varphi_\eta)^{(d)}}{(\eta k)!}\\
&\gg \frac{L_1 tx}{2^{\eta+2}R}(\Phi_{\eta-m}^\eta)^{(d)}.
\end{aligned} \tag{3.14}$$

Thus by using (3.13) and (3.14), the left-hand side of (3.5) will majorize

$$\frac{L_1 t^2}{2^{m(\eta+2)}\eta^m cr}(\Phi_{\eta-m}^\eta)^{(d)} + \frac{L_1 tx}{2^{\eta+2}R}(\Phi_{\eta-m}^\eta)^{(d)} + L_1 t(\Phi_\eta^\eta)^{(d)}. \tag{3.15}$$

Similarly, we will majorize the summation with W_i's separately using a stronger majorant derived from $((t\partial_t)^m+1)U$ which is given by

$$\big((t\partial_t)^m+1\big)U \gg L_1 cr\sum_{k=0}^{\infty}(k+2)\big(\frac{t}{cr}\big)^{k+2}\frac{(D^{\eta k+\eta}\varphi_\eta)^{(d)}}{(\eta k+\eta)!}+L_1 t(\Phi_\eta^\eta)^{(d)}. \quad (3.16)$$

For the corresponding right-hand side, note that for all $j+\lceil d\alpha\rceil \le m$,

$$\begin{aligned} x^\alpha(t\partial_t)^j\partial_x^\alpha U &\ll \frac{C_dL_1cr}{R^\alpha}\sum_{k=0}^{\infty}(k+1)^j\big(\frac{t}{cr}\big)^{k+1}\frac{x^\alpha(D^{\eta k+\lceil d\alpha\rceil}\varphi_\eta)^{(d)}}{(\eta k)!}\\ &\ll \frac{C_dL_1cr}{R^\alpha}\sum_{k=0}^{\infty}\big(\frac{t}{cr}\big)^{k+1}\frac{x^\alpha(D^{\eta k}\varphi_{\eta-\lceil d\alpha\rceil-j})^{(d)}}{(\eta k)!}\\ &\ll 2^{\alpha(2+\eta)}C_dL_1t(\Phi_{\eta-m}^\eta)^{(d)} \ll 2^{m(2+\eta)}C_dL_1t(\Phi_{\eta-m}^\eta)^{(d)}, \end{aligned}$$

where we applied Lemma 3.2(1) to obtain the last line. Thus the linear terms with x^α will have the estimate

$$\Big(\frac{AM}{1-x/R_0}\Big)^{(d)}x\sum_{(j,\alpha)\in\Lambda_0}x^\alpha(t\partial_t)^j(\partial_x)^\alpha U \ll (2^{m(2+\eta)}\sharp\Lambda_0AC_dKM)L_1tx(\Phi_{\eta-m}^\eta)^{(d)}.$$

Comparing this to the term with $tx(\Phi_{\eta-m}^\eta)^{(d)}$ in (3.15), we have the necessary condition:

$$2^{m(2+\eta)}\sharp\Lambda_0AC_dKM \le \frac{1}{2^{\eta+2}R}. \quad (3.17)$$

For the linear terms containing W_i, we follow a similar process as in the previous section by using $t^{q-1}\ll 4SK(\Phi_\eta^\eta)^{(d)}$ to obtain

$$\frac{4AKMNS}{R_1}\Big(L_2(cr)^2\sum_{k=0}^{\infty}(k+2)\big(\frac{t}{cr}\big)^{k+2}\frac{(D^{\eta k+\eta}\varphi_\eta)^{(d)}}{(\eta k+\eta)!}+KL_3t(\Phi_\eta^\eta)^{(d)}\Big). \quad (3.18)$$

By comparing this to (2.16), we obtain the following requirements:

$$\frac{4AKMNSL_2cr}{R_1}\le\frac{L_1}{2}\quad\text{and}\quad\frac{4AK^2MNSL_3}{R_1}\le\frac{L_1}{2}. \quad (3.19)$$

For the nonlinear terms, we have similar results given by

$$\begin{gathered}(t\partial_t)^j(\partial_x^\alpha)U \ll \frac{C_dL_1t}{R^m}(\Phi_{\eta-m}^\eta)^{(d)}\\ W_i \ll 4SK(L_2+L_3)t(\Phi_{\eta-m}^\eta)^{(d)}.\end{gathered}$$

Therefore by again setting $\kappa = 2^{\eta-m}\max\{\sharp\Lambda, N\}$, the remaining terms of the right-hand side of (3.5) will be majorized by

$$\begin{aligned}&\Big(\frac{AM}{1-x/R_0}\Big)^{(d)}\Big\{\sum_{k=1}^{\infty}\big(\frac{t}{r_0}\big)^k+\sum_{\substack{k+|\nu|+|\mu|\ge 2\\ |\nu|+|\mu|\ge 1}}\big(\frac{t}{r_0}\big)^k\Big(\frac{C_dL_1t(\Phi_{\eta-m}^\eta)^{(d)}}{R^mR_1}\Big)^{|\nu|}\\ &\quad\times\Big(\frac{4SK(L_2+L_3)t(\Phi_{\eta-m}^\eta)^{(d)}}{R_1}\Big)^{|\beta|}\Big\}\\ &\ll AMK\Big\{\frac{4S}{r_0}t(\Phi_\eta^\eta)^{(d)}+\Big(\frac{1}{r_0}+\frac{\kappa C_dL_1}{R^mR_1}+\frac{\kappa 4SK(L_2+L_3)}{R_1}\Big)^2t^2(\Phi_{\eta-m}^\eta)^{(d)}\Big\}, \end{aligned} \quad (3.20)$$

where the last simplification is possible if
$$\frac{1}{r_0} + \frac{\kappa C_d L_1}{R^m R_1} + \frac{\kappa 4SK(L_2 + L_3)}{R_1} \le \frac{1}{cr_0}. \tag{3.21}$$
Therefore, by (3.15) and (3.20), we obtain the conditions
$$\frac{4SAMK}{r_0} \le \frac{L_1}{2}$$
$$AMK\Big(\frac{1}{r_0} + \frac{\kappa C_d L_1}{R^m R_1} + \frac{\kappa 4KS(L_2 + L_3)}{R_1}\Big)^2 \le \frac{L_1}{2^{m(\eta+2)+1}\eta^m cr}$$
Finally, similar to the previous section, we may choose constants L_1, L_2, L_3, c and R that will satisfy conditions (3.12), (3.17), (3.19), (3.21), (3.2) and (3.2) so that U and W_i satisfy the majorant system in Proposition 3.3.

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