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GLOBAL WELL-POSEDNESS FOR NONLINEAR NONLOCAL CAUCHY PROBLEMS ARISING IN ELASTICITY

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Abstract. In this article, we prove global well-posedness for a family of one dimensional nonlinear nonlocal Cauchy problems arising in elasticity. We consider the equation

$$u_{tt} - \delta L u_{xx} = \left(\beta * [(1-\delta)u + u^{2n+1}]\right)_{xx},$$

where L is a differential operator, β is an integral operator, and $\delta = 0$ or 1. (Here, the case $\delta = 1$ represents the additional doubly dispersive effect.) We prove the global well-posedness of the equation in energy spaces.

1. Introduction

There is a new trend on nonlinear nonlocal differential equations , because there exists a large class of problems in classical physics and continuum mechanics (classical field theories) that fall outside their traditional domain of application. The nonlocal effect is closely connected to length scales. If the external length scales (e.g. crack length, wavelength) are close to the internal scales (e.g. granular distance, lattice parameter), local theories fail and we need to rely on nonlocal theories that can account for the long-range interatomic attractions; for example, (i) the energy balance law is postulated to remain in global form, and (ii) a material point of the body is considered to be attracted by all points of the body, at all past times [6].

In this article, we study a family of nonlinear nonlocal equations arising in elasticity. Let $u = u(t,x) = \partial_x X(t,x)$ of the displacement $X(t,x)$ in one-dimensional, homogeneous, nonlinear and nonlocal elastic infinite medium. In general, $u(t,x)$ satisfies

$$u_{tt} = \text{second derivative of the stress } S(u).$$

We now introduce the scalar function β, which is the attenuation or influence function aimed to inject in the constitutive law the nonlocal effect at the field x produced by the local strain at x':

$$S(u) := \int_{\mathbb{R}} \beta(|x - x'|)\sigma(u(x',t))dx'.$$

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By decomposing the (classical) local stress $\sigma(u)$ into its linear and nonlinear parts,

$$\sigma(u) = u + g(u), \quad g(0) = 0,$$

we have the following model [5]:

$$u_{tt} = \big(\beta * (u + g(u))\big)_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1a}$$

$$u(0,x) = \phi(x), \tag{1.1b}$$

$$u_t(0,x) = \psi(x), \tag{1.1c}$$

where the subscripts denote partial derivatives and the symbol $*$ denotes convolution in the spatial variable. It is clear from (1.1a) that the well-posedness of the Cauchy problem depends crucially on the structure of the kernel β. The choice of appropriate kernel functions remains an interesting and hard open problem in the nonlocal theory of elasticity, but some well known forms of kernel functions, such as triangular or Dirac delta functions, are currently in use for engineering problems, see [5, 6, 9] for examples of frequently used kernel functions. For further motivations on the consideration of nonlinear nonlocal equations in elasticity see the papers [1, 3, 4, 5, 6, 7, 8, 10, 12]. More recently, the global well-posedness of (1.1) investigated in [5] with the kernel β of the form

$$0 < \widehat{\beta}(\xi) \le (1 + |\xi|^2)^{-r/2}, \quad r > 3 \tag{1.2}$$

and $g(x)$ is a super-linear function satisfying a certain growth condition.

In this article, we provide another set of r and the nonlinearity g that provides the global existence of solutions. More precisely, we assume the following conditions:

$$g(x) = x^{2n+1} \text{ for some } n \in \mathbb{N}, \quad \frac{6n+7}{2n+3} < r \le \frac{6n+2}{2n+1}. \tag{1.3}$$

The range of r will be computed in Section 2. We note that g is defocusing in the sense

$$G(x) = \int_0^x g(s)ds = \frac{x^{2n+2}}{2n+2} \ge 0 \tag{1.4}$$

so that all the terms in the left-hand side of (2.3) are non-negative. Before stating our result, we define the operator $\mathcal{P}$:

$$\widehat{Pu}(\xi) = |\xi|^{-1}\big(\widehat{\beta}(\xi)\big)^{-1/2}\widehat{u}(\xi). \tag{1.5}$$

Theorem 1.1. *Suppose r and g satisfy the conditions in* (1.3)*. For any initial data $\phi, \psi \in H^1(\mathbb{R})$ with additionally satisfying*

$$E_0 := \|\mathcal{P}\psi\|^2_{L^2(\mathbb{R})} + \|\phi\|^2_{L^2(\mathbb{R})} + 2\int_{\mathbb{R}} G(\phi(x))dx < \infty,$$

there exists a unique global-in-time solution $u \in C^1([0,\infty); H^1(\mathbb{R}))$ of (1.1)*.*

Remark 1.2. Let us compare our result with the global result in [5]. Although the condition (1.4) is stronger than the condition $G(u) \ge -ku^2$ in [5], our result improves the condition of r from $r > 3$ in [5] to the condition in (1.3) which is less than 3.

We next consider a general class of doubly dispersive nonlinear nonlocal model [2]:

$$u_{tt} - Lu_{xx} = (\beta * g(u))_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.6a}$$

$$u(0,x) = \phi(x), \tag{1.6b}$$

$$u_t(0,x) = \psi(x), \tag{1.6c}$$

where the symbols of the operators L and β are given by

$$\widehat{L}(\xi) = (1+|\xi|^2)^{\rho/2}, \quad 0 < \widehat{\beta}(\xi) \le (1+|\xi|^2)^{-r/2}. \tag{1.7}$$

The operator L is a differential operator which provides a regularity to the linear term, while β is an integral operator for the smoothness of the nonlinear term. This equation models the bi-directional propagation of dispersive waves in a one dimensional, homogeneous, nonlinearly and non-locally elastic infinite medium. For a background of this equation, see [2] and some of the references cited therein. In [2, [Theorem 6.3], the authors proved the global existence of a solution under the conditions $\rho + r > 1$ and $\rho + 2r \ge 2$ using the representation formula of u in the Fourier variable.

In this article, we provide another set of ρ, r and the nonlinearity g that provides the global existence of solutions. We assume that ρ, r and g satisfy the conditions

$$g(x) = x^{2n+1} \text{ for some } n \in \mathbb{N}, \quad \rho + r > 1. \tag{1.8}$$

We fix the parameters as follows:

$$k \in \mathbb{N}, \quad 2k+1+\rho \ge \frac{\rho+r}{2} + 1, \quad 2k \ge \frac{r}{2}, \quad \frac{1}{2} < 2k+\rho < 2n+1.$$

Theorem 1.3. *Suppose ρ, r and g satisfy the conditions in* (1.8)*. Then for any initial data $\phi \in H^{2k+1+\rho}(\mathbb{R})$ and $\psi \in H^{2k}(\mathbb{R})$ with*

$$E_0 := \|\mathcal{P}\psi\|^2_{L^2(\mathbb{R})} + \|\sqrt{L\beta^{-1}}\phi\|^2_{L^2(\mathbb{R})} + 2\int_{\mathbb{R}} G(\phi(x))dx < \infty, \tag{1.9}$$

there exists a unique global-in-time solution

$$u \in C([0,\infty); H^{2k+1+\rho}(\mathbb{R})) \cap C^1([0,\infty); H^{2k}(\mathbb{R}))$$

such that

$$\begin{gathered} \|\mathcal{P}u_t(t)\|^2_{L^2(\mathbb{R})} + \|\sqrt{L\beta^{-1}}u(t)\|^2_{L^2(\mathbb{R})} + 2\int_{\mathbb{R}} G(u(t,x))dx = E_0, \\ \|u_t(t)\|^2_{H^{2k}(\mathbb{R})} + \|u(t)\|^2_{H^{2k+1+\rho}(\mathbb{R})} \le C(E_0,\phi,\psi,t). \end{gathered} \tag{1.10}$$

Notation. $\widehat{f}(\xi)$ is the Fourier transform of f. H^s is the energy space whose norm is given by

$$\|u\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi.$$

All constants will be denoted by C that is a generic constant depending only on the quantities specified in the context.

2. Proof of main restuls

We begin with a lemma dealing with the effect of composition by a smooth functions.

Lemma 2.1 ([11])**.** *For $g(x) = x^{2n+1}$ and $0 \le s < 2n+1$, we have*

$$\begin{gathered} \|g(u)\|_{H^s(\mathbb{R})} \le C(n, \|u\|_{L^\infty(\mathbb{R})})\|u\|_{H^1(\mathbb{R})}, \\ \|g(u)-g(v)\|_{H^s(\mathbb{R})} \le C(\|u\|_{L^\infty(\mathbb{R})}, \|v\|_{L^\infty(\mathbb{R})}, \|u\|_{H^s(\mathbb{R})}, \|v\|_{H^s(\mathbb{R})})\|u-v\|_{H^s(\mathbb{R})}. \end{gathered}$$

Proof of Theorem 1.1. Our analysis starts by rewriting (1.1) as a H^1-valued ordinary differential equations:

$$u_t = v,\ u(0,x) = \phi(x), \tag{2.1a}$$

$$v_t = K * f(u),\ v(0,x) = \psi(x), \tag{2.1b}$$

where $f(u) = u + g(u)$ and $K = \beta_{xx}$. We first note that $\widehat{K}(\xi) \in L^\infty(\mathbb{R})$ from the condition (1.3) with $r > 2$. So, $K = \beta_{xx}$ is a bounded operator in $H^1(\mathbb{R})$. Moreover, Lemma 2.1 implies that $K * f(u)$ is locally Lipschitz on $H^1(\mathbb{R})$. Therefore, the local well-posedness with initial data in $H^1(\mathbb{R})$ follows from the well-posedness of the system of ordinary differential equations. Moreover, it is shown in [5, Lemma 3.9] that there exists a global solution in $H^1(\mathbb{R})$ if and only if for any $T > 0$

$$\limsup_{t\to T^-} \|u(t)\|_{L^\infty(\mathbb{R})} < \infty.$$

Therefore, we focus on the L^∞ norm of u. To this end, we rewrite (1.1a) as

$$\mathcal{P}^2 u_{tt} + u + g(u) = 0. \tag{2.2}$$

where $\mathcal{P}$ is defined in (1.5). Multiplying (2.2) by $2u_t$ and integrating in x, we have

$$\|\mathcal{P}u_t(t)\|^2_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})} + 2\int_{\mathbb{R}} G(u(t,x))dx = E_0. \tag{2.3}$$

As shown in [5], the first term on the left-hand side of (2.3) implies that

$$\|u(t)\|_{H^{\frac{r}{2}-1}(\mathbb{R})} \le C(E_0).$$

By the Sobolev embedding, we have

$$\|u(t)\|_{L^{\frac{2}{3-r}}(\mathbb{R})} \le C(E_0) \quad \text{for } r < 3. \tag{2.4}$$

Using this, we estimate the nonlinear term $K*g(u)$. By Hausdorff-Young inequality,

$$\|K * g(u)\|_{L^\infty(\mathbb{R})} \le C\|K\|_{L^{q'}(\mathbb{R})}\|g(u)\|_{L^q(\mathbb{R})} \le C\|\widehat{K}\|_{L^q(\mathbb{R})}\|g(u)\|_{L^q(\mathbb{R})}, \quad 1 < q \le 2.$$

To bound $\|\widehat{K}\|_{L^q(\mathbb{R})} = \||\xi|^2\widehat{\beta}\|_{L^q(\mathbb{R})}$, we need

$$(r-2)q > 1. \tag{2.5}$$

On the other hand, to bound $\|g(u)\|_{L^q(\mathbb{R})}$ using (2.4), we also need

$$(2n+1)q \le \frac{2}{3-r}. \tag{2.6}$$

By solving (2.5) and (2.6) for r, we have

$$\frac{6n+7}{2n+3} < r.$$

Moreover, by choosing

$$r \le \frac{6n+2}{2n+1}$$

we have $q \le 2$ from (2.6). Combining these two inequalities for r, we obtain that

$$\frac{6n+7}{2n+3} < r \le \frac{6n+2}{2n+1} \tag{2.7}$$

and under this condition, we have

$$\|K * g(u)\|_{L^\infty(\mathbb{R})} \le C(E_0) \tag{2.8}$$

Moreover, since

$$\frac{5}{2} < \frac{6n+7}{2n+3},$$

$K \in L^2(\mathbb{R})$. Therefore,

$$\|K * u\|_{L^\infty(\mathbb{R})} \le \|K\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})} \le C(E_0). \tag{2.9}$$

We now bound $\|u\|_{L^\infty(\mathbb{R})}$ using (2.8) and (2.9). Integrating (1.1) twice in time, we have

$$u(t,x) = \phi(x) + t\psi(x) + \int_0^t (t-s)(K * f(u(s)))ds.$$

Since

$$\|K * f(u)\|_{L^\infty(\mathbb{R})} \le \|K * u\|_{L^\infty(\mathbb{R})} + \|K * g(u)\|_{L^\infty(\mathbb{R})} \le C(E_0),$$

we conclude that

$$\|u(t)\|_{L^\infty(\mathbb{R})} \le \|\phi\|_{L^\infty(\mathbb{R})} + t\|\psi\|_{L^\infty(\mathbb{R})} + C(E_0)t^2. \tag{2.10}$$

Therefore, $\|u(t)\|_{L^\infty(\mathbb{R})}$ does not blow up in finite time. This completes the proof. □

Proof of Theorem 1.3. We begin with the bounds of $\|u\|_{L^2(\mathbb{R})}$ and $\|u\|_{L^\infty(\mathbb{R})}$. To this end, we rewrite (1.6a) as

$$\mathcal{P}^2 u_{tt} + L\beta^{-1}u + g(u) = 0. \tag{2.11}$$

Multiplying (2.11) by $2u_t$ and integrating in x, we have

$$\|\mathcal{P}u_t(t)\|^2_{L^2(\mathbb{R})} + \|\sqrt{L\beta^{-1}}u(t)\|^2_{L^2(\mathbb{R})} + 2\int_{\mathbb{R}} G(u)(t,x)dx = E_0. \tag{2.12}$$

Since

$$\sqrt{\widehat{L}(\xi)}(\widehat{\beta}(\xi))^{-1/2} \ge (1+|\xi|^2)^{\frac{\rho+r}{4}} \quad \text{and} \quad \rho + r > 1$$

Equality (2.12) implies

$$\|u(t)\|_{L^2(\mathbb{R})} \le E_0, \quad \|u(t)\|_{L^\infty(\mathbb{R})} \le E_0. \tag{2.13}$$

We next obtain the energy estimates. Let $\sqrt{-\Delta} = \Lambda$. Then, we have

$$\begin{aligned} &\frac{d}{dt}\|u_t\|^2_{H^{2k}(\mathbb{R})} + \frac{d}{dt}\sum_{i=0}^k \|\Lambda^{2i}\sqrt{L}u_x\|^2_{L^2(\mathbb{R})} \\ &\le \sum_{i=0}^k \|\Lambda^{2i}\beta_{xx} * g(u)\|^2_{L^2(\mathbb{R})} + \|u_t\|^2_{H^{2k}(\mathbb{R})}. \end{aligned} \tag{2.14}$$

Since for $\rho + r \ge 1$ and $i = 0, 1, 2, \dots, k$

$$(1+|\xi|)^{2i}|\xi|^2(1+|\xi|^2)^{-\frac{r}{2}} \le C(1+|\xi|)^{2i}|\xi|(1+|\xi|^2)^{\frac{\rho}{2}} \le C(1+|\xi|)^{2k+1+\rho},$$

by Lemma 2.1 we have

$$\begin{aligned} \sum_{i=0}^k \|\Lambda^{2i}\beta_{xx} * g(u)\|^2_{L^2(\mathbb{R})} &\le C\|g(u)\|^2_{H^{2k+1+\rho}(\mathbb{R})} \\ &\le C(\|u\|_{L^\infty(\mathbb{R})}, n)\|u\|^2_{H^{2k+1+\rho}(\mathbb{R})}. \end{aligned} \tag{2.15}$$

Therefore, by (2.14), (2.15), (2.12) and (2.13),

$$\begin{aligned} &\|u_t(t)\|^2_{H^{2k}(\mathbb{R})} + \|u(t)\|^2_{H^{2k+1+\rho}(\mathbb{R})} \\ &\le C(\phi,\psi,E_0) + \int_0^t \|u_t(s)\|^2_{H^{2k}(\mathbb{R})} ds \\ &\le C(\phi,\psi,E_0) + \int_0^t [\|u_t(s)\|^2_{H^{2k}(\mathbb{R})} + \|u(s)\|^2_{H^{2k+1+\rho}(\mathbb{R})}] ds \end{aligned} \tag{2.16}$$

and thus by Gronwall's inequality, we obtain

$$\|u_t(t)\|^2_{H^{2k}(\mathbb{R})} + \|u(t)\|^2_{H^{2k+1+\rho}(\mathbb{R})} \le C(\phi,\psi,E_0)e^t. \tag{2.17}$$

To obtain a unique solution u, we iterate the equation by defining u^l as a solution of

$$u^l_{tt} - Lu^l_{xx} = \beta_{xx} * g(u^{l-1}), \tag{2.18a}$$

$$u^l(0,x) = \phi(x), \tag{2.18b}$$

$$u^l_t(0,x) = \psi(x). \tag{2.18c}$$

Let $\omega^l = u^l - u^{l-1}$. Then, ω^l satisfies

$$\omega^l_{tt} - L\omega^l_{xx} = \beta_{xx} * \left[g(u^{l-1}) - g(u^{l-2})\right], \tag{2.19a}$$

$$\omega^l(0,x) = 0, \quad \omega^l_t(0,x) = 0. \tag{2.19b}$$

By following the calculation above, we have

$$\frac{d}{dt}\|\mathcal{P}\omega^l_t(t)\|^2_{L^2(\mathbb{R})} + \frac{d}{dt}\|\sqrt{L\beta^{-1}}\omega^l(t)\|^2_{L^2(\mathbb{R})} \le C(E_0)\|\omega^{l-1}\|^2_{L^2(\mathbb{R})} + \|\omega^l_t\|^2_{L^2(\mathbb{R})} \tag{2.20}$$

and

$$\begin{aligned} &\frac{d}{dt}\|\omega^l_t\|^2_{H^{2k}(\mathbb{R})} + \frac{d}{dt}\|\omega^l(t)\|^2_{H^{2k+1+\rho}(\mathbb{R})} \\ &\le C\|g(u^{l-1}) - g(u^{l-2})\|^2_{H^{2k+1+\rho}(\mathbb{R})} + \|\omega^l_t\|^2_{H^{2k}(\mathbb{R})} \\ &\le C(\phi,\psi,E_0)\|\omega^{l-1}\|^2_{H^{2k+1+\rho}(\mathbb{R})} + \|\omega^l_t\|^2_{H^{2k}(\mathbb{R})}. \end{aligned} \tag{2.21}$$

This implies

$$\begin{aligned} &\sup_{0\le t\le T}\left[\|\omega^l_t(t)\|^2_{H^{2k}(\mathbb{R})} + \|\omega^l(t)\|^2_{H^{2k+1+\rho}(\mathbb{R})}\right] \\ &\le C(\phi,\psi,E_0)T\sup_{0\le t\le T}\left[\|\omega^{l-1}_t(t)\|^2_{H^{2k}(\mathbb{R})} + \|\omega^{l-1}(t)\|^2_{H^{2k+1+\rho}(\mathbb{R})}\right] \\ &+ T\sup_{0\le t\le T}\left[\|\omega^l_t(t)\|^2_{H^{2k}(\mathbb{R})} + \|\omega^l(t)\|^2_{H^{2k+1+\rho}(\mathbb{R})}\right]. \end{aligned} \tag{2.22}$$

We take $T > 0$ such that $T < 1/2$ and $C(\phi,\psi,E_0)T < 1/4$. Then, $\{u^l(t) : l \in \mathbb{N}\}$ is a Cauchy sequence in $E_{2k}(T)$ with

$$\|u\|_{E_{2k}(T)} = \sup_{0\le t\le T}\left[\|u_t(t)\|_{H^{2k}(\mathbb{R})} + \|u(t)\|_{H^{2k+1+\rho}(\mathbb{R})}\right].$$

Therefore, there exists a unique local in-time solution in $E_{2k}(T)$. Moreover, the solution does not blow up in finite time by the global bound (2.17). This completes the proof. □

Concluding Remark. The present article is part of a research program whose objective is the investigation of general models arising in elasticity. The well-posedness of nonlinear nonlocal elastic equations is of great scientific interest, physically relevant and presents new challenges in the analysis. In this paper, we have established the global well-posedness of (1.1) and (1.6) for large data with defocusing nonlinearity throughout our analysis. We believe that we can apply our method to other models, such as peridynamics [12]; this equation is a model proposed to describe the dynamical response of an infinite homogeneous elastic bar within the context of the peridynamic formulation of elasticity theory.

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