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EXISTENCE OF GLOBAL SOLUTIONS AND DECAY ESTIMATES FOR A VISCOELASTIC PETROVSKY EQUATION WITH A DELAY TERM IN THE NON-LINEAR INTERNAL FEEDBACK

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Abstract. In this article we consider a nonlinear viscoelastic Petrovsky equation in a bounded domain with a delay term in the weakly nonlinear internal feedback:

$$\begin{aligned} &|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds \\ &+ \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0. \end{aligned}$$

We prove the existence of global solutions in suitable Sobolev spaces by using the energy method combined with Faedo-Galarkin method under condition on the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study general stability estimates by using some properties of convex functions.

1. Introduction

1.1. **The model.** In this article we consider the existence and decay properties of global solutions for the initial boundary value problem of viscoelastic Petrovsky equation

$$\begin{gathered} |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds \\ + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 \quad \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) \quad \text{in } \Omega\times]0,\tau[, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, $\partial\Omega$ is a smooth boundary, $l > 0$, μ_1 and μ_2 are positive real numbers, h is a positive non-increasing function defined on $\mathbb{R}^+$, g_1 and g_2 are two functions, $\tau > 0$ is a time delay and (u_0, u_1, f_0) are the initial data in a suitable function space. Cavalcanti et al. [10] studied the following

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nonlinear viscoelastic problem with strong damping

$$|u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds - \gamma \Delta u_t = 0, \quad x \in \Omega,\ t > 0. \tag{1.2}$$

Under the assumptions $0 < l \leq \frac{2}{n-2}$ if $n \geq 3$ or $l > 0$ if $n = 1, 2$ and h decays exponentially, they obtained the global existence of weak solutions for $\gamma \geq 0$ and the uniform exponential decay rates of the energy for $\gamma > 0$. In the case of $\gamma = 0$ when a source term competes with the dissipation induced by the viscoelastic term, Messaoudi and Tatar [22] studied the equation

$$|u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds + b|u|^{p-2}u = 0, \quad x \in \Omega,\ t > 0.$$

They used the potential well method to show that the damping induced by the viscoelastic term is enough to ensure global existence and uniform decay of solutions provided that the initial data are in some stable set. Han and Wang [15], investigated a related problem with linear damping

$$|u_t|^l u_{tt} - \Delta u - \Delta u_{tt} - \int_0^t h(t-s)\Delta u(s)\,ds + u_t = 0, \quad x \in \Omega,\ t > 0.$$

Using the Faedo-Galerkin method, they showed the global existence of weak solutions and obtained uniform exponential decay of solutions by introducing a perturbed energy functional. Recently, these results have been extended by Wu [32] to a general case where a source term and a nonlinear damping term are present.

In the presence of the source term, problem (1.2) has been discussed by many authors, and related results concerning local or global existence, asymptotic behavior and blow-up of solution have been recently established (see [4, 20, 23]).

Park and Kang [26] studied the following nonlinear viscoelastic problem with damping

$$|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + u_t = 0, \quad x \in \Omega,\ t > 0.$$

Santos et al. [27] considered the existence and uniform decay for the following nonlinear beam equation in a non-cylindrical domain:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + \alpha u_t = 0, \quad \text{in } \widehat{Q},$$

where $\widehat{Q} = \cup_{0 \leq t \leq \infty} \Omega_t \times \{t\}$. Benaissa, Benguessoum and Messaoudi [6] proved the existence of global solution, as well as, a general stability result for the equation

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x, t-\tau)) = 0, \tag{1.3}$$

for $x \in \Omega$ and $t > 0$, when h is decays at a certain rate.

In the absence of the viscoelastic term (i.e. if $h \equiv 0$), problem (1.3) has been studied by many authors. It is well known that in the further absence of a damping mechanism, the delay term causes instability of the system (see, for instance, Datko et al. [11]). On the contrary, in the absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay is estimated depending on the rate of growth of g_1 (see Alabau-Boussouira, [3], Benaissa and Guesmia [8], Haraux [14], Komornik [16], Lasiecka and Tataru [18]).

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see Shinskey [28]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological, and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research (see Abdallah et al [2], Suh and Bien [29] and Zhong [31]). To stabilise a hyperbolic system involving delay terms, additional control terms are necessary (see Nicaise and Pignotti [24], Nicaise and Pignotti [25], Xu et al. [11]). In Nicaise and Pignotti [24], the authors examined the problem (P) in the linear situation (i.e. if $g_1(s) = g_2(s) = s$ for all $s \in \mathbb{R}$) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution of (1.3) will be instable if $\mu_2 \geq \mu_1$. The main approach used in Nicaise and Pignotti [24] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay were acting in the boundary domain. We also recall the result by Xu et al. [30], where the authors proved the same result as in Nicaise and Pignotti [24] for the one space dimension by adopting the spectral analysis approach. Very recently, Benaissa and Louhibi [7] extended the result of Nicaise and Pignotti [24] to the non-linear case.

Datko et al. [11] showed that a small delay in a boundary control could turn such well-behave hyperbolic system into a wild one and therefore, delay becomes a source of instability. However, sometimes it can also improve the performance of the systems (see Suh and Bien [29]).

The main purpose of this paper is to prove global solvability and energy decay estimates of the solutions of problem (1.1) when h is of exponential decay rate and g_1, g_2 are non-linear. We would like to see the influence of frictional and viscoelastic damping on the rate of decay of solutions in the presence of non-linear degenerate delay term. Of course, the most interesting case occurs when we have delay term and simultaneous and complementary damping mechanisms.

To obtain global solutions of problem (1.1), we use the Galerkin approximation scheme (see Lions [19]) together with the energy estimate method. The technique based on the theory of non-linear semi-groups used in Nicaise and Pignotti [24] does not seem to be applicable in the non-linear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. [9], Daoulatli et al. [12], Lasiecka and Doundykov [17] and Lasiecka and Tataru [18], and used by Liu and Zuazua [21], Eller et al. [13] and Alabau-Boussouira [3].

1.2. **Statement of results.** We use the Sobolev spaces $H^4(\Omega)$, $H_0^2(\Omega)$ and the Hilbert space $L^p(\Omega)$ with their usual scalar products and norms. The prime $'$ and the subscript t will denote time differentiation and we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. The constant C denotes a general positive constant, which may be different in different estimates. Now we introduce, as in the work of in Nicaise and Pignotti [24], the new variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{1.4}$$

Therefore, problem (1.1) is equivalent to

$$\begin{gathered}
|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds \\
+ \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) = 0 \quad \text{in } \Omega\times]0,+\infty[, \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega\times]0,1[\times]0,+\infty[, \\
u(x,t) = 0, \quad \text{on } \partial\Omega \times [0,\infty[, \\
z(x,0,t) = u_t(x,t), \quad \text{on } \Omega \times [0,\infty[, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \\
z(x,\rho,0) = f_0(x,-\rho\tau), \quad \text{in } \Omega\times]0,1[.
\end{gathered} \tag{1.5}$$

To state and prove our result, we use the following assumptions:

(A1) Assume that l satisfies

$$\begin{gathered}
0 < l \le \frac{2}{n-2} \quad \text{if } n \ge 3 \\
0 < l < \infty \quad \text{if } n = 1,2;
\end{gathered}$$

(A2) $g_1 : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class C^1 and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying

$$\begin{gathered}
H(0) = 0 \text{ and } H \text{ is linear on } [0,\varepsilon] \text{ or} \\
H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon] \text{ such that} \\
|g_1(s)| \le c_2|s| \quad \text{if } |s| \ge \varepsilon \\
g_1^2(s) \le H^{-1}(sg_1(s)) \quad \text{if } |s| \le \varepsilon,
\end{gathered} \tag{1.6}$$

where H^{-1} denotes the inverse function of H and ε, c_2 are positive constants. $g_2 : \mathbb{R} \to \mathbb{R}$ is an odd no decreasing function of class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$,

$$|g_2'(s)| \le c_3, \tag{1.7}$$

$$\alpha_1 s g_2(s) \le G(s) \le \alpha_2 s g_1(s), \tag{1.8}$$

where $G(s) = \int_0^s g_2(r)dr$;

(A3) $\alpha_2\mu_2 < \alpha_1\mu_1$;

(A4) For the relaxation function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1 function such that

$$\int_0^\infty h(s)\,ds = \beta < 1, \tag{1.9}$$

and we assume that there exist a positive constant ζ satisfying

$$h'(t) \le -\zeta h(t). \tag{1.10}$$

We define the energy associated with the solution of system (1.5) by

$$\begin{aligned}
E(t) = \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\Big(1 - \int_0^t h(s)\,ds\Big)\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 \\
+ \frac{1}{2}(h \circ \Delta u)(t) + \xi \int_\Omega \int_0^1 G(z(x,\rho,t))\,d\rho\,dx,
\end{aligned} \tag{1.11}$$

where ξ is a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2},$$
$$(h\circ v)(t) = \int_0^t h(t-s)\|v(t)-v(s)\|_2^2\,ds.$$

Now we have the existence of a global solution.

Theorem 1.1. *Let $u_0 \in H^4(\Omega)\cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $f_0 \in H_0^2(\Omega, H^2(0,1))$ satisfy the compatibility condition $f(\cdot,0) = u_1$. Assume that* (A1)-(A4) *hold. Then* (1.1) *admits a weak solution*

$$u \in L^\infty([0,\infty); H^4(\Omega)\cap H_0^2(\Omega)), \quad u_t \in L^\infty([0,\infty); H_0^2(\Omega)),$$
$$u_{tt} \in L^2([0,\infty);\ H_0^1(\Omega)).$$

Also we have a uniform decay rates for the energy.

Theorem 1.2. *Assume that* (A1)–(A4) *hold. Then, there exist a positive constants w_1, w_2, w_3 and ε_0 such that the solution of* (1.1) *satisfies*

$$E(t) \le w_3 H_1^{-1}(w_1 t + w_2) \quad \forall t > 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds, \tag{1.12}$$
$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\varepsilon] \\ tH'(\varepsilon_0 t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on }]0,\varepsilon], \end{cases}$$

here, H_1 is strictly decreasing and convex on $(0,1]$ with $\lim_{t\to 0} H_1(t) = +\infty$.

2. Preliminaries

Let λ_1 be the first eigenvalue of the spectral Dirichlet problem

$$\Delta^2 u = \lambda_1 u, \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \eta} = 0 \quad \text{in } \Gamma,$$
$$\|\nabla u\|_2 \le \frac{1}{\sqrt{\lambda_1}}\|\Delta u\|_2. \tag{2.1}$$

Next we have a Sobolev-Poincaré inequality [1].

Lemma 2.1. *Let q be a number with*

$$2 \le q < +\infty (n=1,2) \text{ or } 2 \le q \le 2n/(n-2)(n \ge 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_q \le C_s\|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega).$$

Lemma 2.2. *For $h, \Psi \in C^1([0,+\infty[,\mathbb{R})$ we have*

$$\int_\Omega h*\Psi\Psi_t\,dx = -\frac{1}{2}h(t)\|\Psi(t)\|_2^2 + \frac{1}{2}(h'\circ\Psi)(t) - \frac{1}{2}\frac{d}{dt}\Big[(h\circ\Psi)(t) - \Big(\int_0^t h(s)\,ds\Big)\|\Psi\|_2^2\Big].$$

Remark 2.3. Let us denote by Φ^* the conjugate function of the differentiable convex function Φ, i.e.,

$$\Phi^*(s) = \sup_{t \in \mathbb{R}^+} (st - \Phi(t)).$$

Then Φ^* is the Legendre transform of Φ, which is given by (see Arnold [5, p. 61-62])

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)], \quad \text{if } s \in (0, \Phi'(r)],$$

and Φ^* satisfies the generalized Young inequality

$$AB \le \Phi^*(A) + \Phi(B), \quad \text{if } A \in (0, \Phi'(r)],\ B \in (0, r]. \tag{2.2}$$

Lemma 2.4. *Let (u, z) be a solution of the problem* (1.5). *Then, the energy functional defined by* (1.11) *satisfies*

$$\begin{aligned} E'(t) \le -\beta_1 \int_\Omega u_t g_1(u_t)\,dx - \beta_2 \int_\Omega z(x,1,t) g_2(z(x,1,t))\,dx \\ - \frac{1}{2} h(t)\|\Delta u(t)\|^2 + \frac{1}{2}(h' \circ \Delta u)(t) \le 0, \end{aligned} \tag{2.3}$$

where $\beta_1 = \mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2$ and $\beta_2 = \frac{\xi \alpha_1}{\tau} - \mu_2(1 - \alpha_1)$.

Proof. By multiplying the first equation in (1.5) by u_t, integrating over Ω and using integration by parts, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2\Big] + \mu_1 \int_\Omega u_t(x,t) g_1(u_t(x,t))\,dx \\ &+ \mu_2 \int_\Omega u_t(x,t) g_2(z(x,1,t))\,dx \\ &= \int_\Omega \int_0^t h(t-s)\Delta u(s) \Delta u_t(t)\,ds\,dx. \end{aligned} \tag{2.4}$$

By applying the Lemma 2.2, the term on the right-hand side of (2.4) can be rewritten as

$$\begin{aligned} &\int_\Omega \int_0^t h(t-s)\Delta u(s)\Delta u_t(t)\,ds\,dx + \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 \\ &= \frac{1}{2}\frac{d}{dt}\Big[\int_0^t h(s)\,ds \|\Delta u(t)\|_2^2 - (h \circ \Delta u)(t)\Big] + \frac{1}{2}(h' \circ \Delta u)(t). \end{aligned}$$

Consequently, (2.4) becomes

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\Big(1 - \int_0^t h(s)\,ds\Big)\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}(h \circ \Delta u)(t)\Big] \\ &= -\mu_1 \int_\Omega u_t(x,t) g_1(u_t(x,t))\,dx - \mu_2 \int_\Omega u_t(x,t) g_2(z(x,1,t))\,dx \\ &\quad - \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 + \frac{1}{2}(h' \circ \Delta u)(t). \end{aligned} \tag{2.5}$$

We multiply the second equation in (1.5) by $\xi g_2(z)$, we integrate the result over $\Omega \times (0,1)$, to obtain

$$\begin{aligned} \xi \int_\Omega \int_0^1 z_t(x,\rho,t) g_2(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau} \int_\Omega \int_0^1 z_\rho(x,\rho,t) g_2(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{\xi}{\tau} \int_\Omega \int_0^1 \frac{\partial}{\partial \rho}\Big(G(z(x,\rho,t))\Big)\,d\rho\,dx \end{aligned}$$

$$= -\frac{\xi}{\tau}\int_\Omega (G(z(x,1,t)) - G(z(x,0,t)))\,dx.$$

Hence

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx = -\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,dx. \tag{2.6}$$

By combining (2.5) and (2.6), we obtain

$$\begin{aligned} E'(t) &= -\frac{1}{2}h(t)\|\Delta u(t)\|_2^2 + \frac{1}{2}(h'\circ\Delta u)(t) - \mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ &\quad - \mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,dx, \end{aligned}$$

and by recalling (1.8), we obtain

$$\begin{aligned} E'(t) &\le -(\mu_1 - \frac{\xi\alpha_2}{\tau})\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 \\ &\quad + \frac{1}{2}(h'\circ\Delta u)(t) - \mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,dx \\ &\quad - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx. \end{aligned} \tag{2.7}$$

From the definition of G and by using remark 2.3, we obtain

$$G^*(s) = sg_2^{-1}(s) - G(g_2^{-1}(s)), \quad \forall s \ge 0.$$

Hence

$$\begin{aligned} G^*(g_2(z(x,1,t))) &= z(x,1,t)g_2(z(x,1,t)) - G(z(x,1,t)) \\ &\le (1-\alpha_1)z(x,1,t)g_2(z(x,1,t)). \end{aligned}$$

By using (1.8) and (2.2) with $A = g_2(z(x,1,t))$ and $B = u_t(x,t)$, from (2.7) we obtain

$$\begin{aligned} E'(t) &\le -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau}\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 + \frac{1}{2}(h'\circ\Delta u)(t) \\ &\quad + \mu_2\int_\Omega (G(u_t(x,t)) + G^*(g_2(z(x,1,t)))\,dx - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx \\ &\le -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\ &\quad - \Big(\frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\ &\quad - \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 + \frac{1}{2}(h'\circ\Delta u)(t) \le 0. \end{aligned}$$

This completes the proof. □

3. Proofs of main resutls

3.1. **Proof of Theorem 1.1.** Throughout this section we assume $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $f_0 \in H_0^2(\Omega, H^2(0,1))$. We will use the Faedo-Galerkin method to prove the existence of a global solution. Let $T > 0$ be fixed and let

$\{w^k\}, k \in \mathbb{N}$ be a basis of $H_0^2(\Omega)$, V_k the space generated by $w^1, w^2, \dots, w^k$. Now, we define, for $1 \le j \le k$, the sequence $\phi^j(x,\rho)$ as follows:

$$\phi^j(x,0) = w^j.$$

Then, we may extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi^j)_j$ forms a base of $L^2(\Omega, H^2(0,1))$ and denote Z_k the space generated by $\{\phi^k\}$. We construct approximate solutions (u^k, z^k), $k = 1,2,3,\dots$, in the form

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t) w^j(x), \quad z^k(t) = \sum_{j=1}^{k} d^{jk}(t)\phi^j,$$

where c^{jk} and $d^{jk}(j = 1,2,\dots,k)$ are determined by the ordinary differential equations

$$\begin{gathered} (|u_t^k(t)|^l u_{tt}^k(t), w^j) + (\Delta_x u^k(t), \Delta_x w^j) + (\nabla_x u_{tt}^k, \nabla_x w^j) \\ - \int_0^t h(t-s)(\Delta u^k(s), \Delta w^j)\,ds + \mu_1(g_1(u_t^k), w^j) + \mu_2(g_2(z^k(.,1)), w^j) = 0, \quad (3.1) \\ z^k(x,0,t) = u_t^k(x,t), \end{gathered}$$

$$u^k(0) = u_0^k = \sum_{j=1}^{k} (u_0, w^j) w^j \to u_0, \quad \text{in } H^4(\Omega) \cap H_0^2(\Omega) \text{ as } k \to +\infty, \tag{3.2}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^{k} (u_1, w^j) w^j \to u_1, \quad \text{in } H_0^2(\Omega) \text{ as } k \to +\infty, \tag{3.3}$$

and

$$(\tau z_t^k + z_\rho^k, \phi^j) = 0, \quad 1 \le j \le k, \tag{3.4}$$

$$z^k(\rho,0) = z_0^k = \sum_{j=1}^{k} (f_0, \phi^j)\phi^j \to f_0 \quad \text{in } H_0^2(\Omega, H^2(0,1)) \text{ as } k \to +\infty. \tag{3.5}$$

Since $0 < l \le \frac{2}{n-2}$ if $n \ge 3$, by the Sobolev embedding, we have

$$H_0^2(\Omega) \hookrightarrow L^{2(l+1)}(\Omega)$$

and the same occurs for $n = 1,2$ where $l > 0$. Noting that $\frac{l}{2(l+1)} + \frac{1}{2(l+1)} + \frac{1}{2} = 1$, from the generalized Hölder inequality, the nonlinear term $(|u_t^k(t)|^l u_{tt}^k(t), w_j)$ in (3.1) makes sense. The standard theory of ODE guarantees that the system (3.1)-(3.5) has an unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (3.1) are locally Lipschitz continuous. Note that $u^k(t)$ is of class $\mathcal{C}^2$.

In the next step, we obtain a priori estimates for the solution of the system (3.1)-(3.5), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $t > 0$, using a standard compactness argument for the limiting procedure.

First estimate. Since the sequences u_0^k, u_1^k and z_0^k converge and from Lemma 2.4, we can find a positive constant C_1 independent of k such that

$$\begin{aligned} & E^k(t) - E^k(0) \\ & \le -\beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\,dx\,ds - \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\,dx\,ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_0^t h(s)\|\Delta u^k(s)\|^2\,ds + \frac{1}{2}\int_0^t (h'\circ \Delta u^k)(s)\,ds \\
&\le -\beta_1 \int_0^t\int_\Omega u_t^k g_1(u_t^k)\,dx\,ds - \beta_2\int_0^t\int_\Omega z^k(x,1,s)g_2(z^k(x,1,s))\,dx\,ds.
\end{aligned}
$$

As h is a positive non increasing function, so we obtain

$$
\begin{aligned}
&E^k(t) + \beta_1 \int_0^t\int_\Omega u_t^k g_1(u_t^k)\,dx\,ds + \beta_2\int_0^t\int_\Omega z^k(x,1,s)g_2(z^k(x,1,s))\,dx\,ds \\
&\le E^k(0) \le C_1,
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
E^k(t) &= \frac{1}{l+2}\|u_t^k\|_{l+2}^{l+2} + \frac{1}{2}\Big(1-\int_0^t h(s)\,ds\Big)\|\Delta u^k\|_2^2 \\
&\quad + \frac{1}{2}\|\nabla u_t^k\|_2^2 + \frac{1}{2}(h\circ\Delta u^k)(t) + \xi\int_\Omega\int_0^1 G(z^k(x,\rho,t))\,d\rho\,dx,
\end{aligned}
$$

and C_1 is a positive constant depending only on $\|u_0\|_{H_0^2}$ and $\|u_1\|_{H_0^1}$. Noting (1.9) and (3.6), we obtain the first estimate:

$$
\begin{aligned}
&\|u_t^k\|_{l+2}^{l+2} + \|\Delta u^k\|_2^2 + \|\nabla u_t^k\|_2^2 + (h\circ\Delta u^k)(t) + \int_\Omega\int_0^1 G(z^k(x,\rho,t))\,d\rho \\
&+ \int_0^t\int_\Omega u_t^k g_1(u_t^k)\,dx\,ds + \int_0^t\int_\Omega z^k(x,1,s)g_2(z^k(x,1,s))\,dx\,ds \le C_2,
\end{aligned} \tag{3.7}
$$

where C_2 is a positive constant depending only on $\|u_0\|_{H_0^2}$, $\|u_1\|_{H_0^1}$, l, β, ξ, τ, β_1 and β_2. These estimates imply that the solution (u^k, z^k) exists globally in $[0,+\infty)$.

Estimate (3.7) yields that

$$u^k \text{ is bounded in } L^\infty_{\rm loc}(0,\infty,H_0^2(\Omega)), \tag{3.8}$$

$$u_t^k \text{ is bounded in } L^\infty_{\rm loc}(0,\infty,H_0^1(\Omega)), \tag{3.9}$$

$$G(z^k(x,\rho,t)) \text{ is bounded in } L^\infty_{\rm loc}(0,\infty,L^1(\Omega\times(0,1))), \tag{3.10}$$

$$u_t^k(t)g_1(u_t^k(t)) \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{3.11}$$

$$z^k(x,1,t)g_2(z^k(x,1,t)) \text{ is bounded in } L^1(\Omega\times(0,T))). \tag{3.12}$$

3.2. **Second estimate.** Replacing w^j by $-\Delta_x w^j$ in (3.1), multiplying by c_t^{jk} and summing over j from 1 to k, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2\Big] - \int_\Omega |u_t^k(t)|^l u_{tt}^k(t)\Delta_x u_t^k\,dx \\
&- \int_0^t h(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla\Delta u_t^k(s)\,dx\,ds + \mu_1\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx \\
&+ \mu_2\int_\Omega \nabla_x u_t^k\nabla_x z^k(x,1,t)g_2'(z^k(x,1,t))\,dx = 0.
\end{aligned} \tag{3.13}
$$

Using the Green's formula, we have

$$
\begin{aligned}
&-\int_\Omega |u_t^k(t)|^l u_{tt}^k(t)\Delta_x u_t^k\,dx \\
&= \frac{d}{dt}\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx - (l+1)\int_\Omega |u_t^k|^l\nabla u_{tt}^k(t)\nabla_x u_t^k\,dx.
\end{aligned} \tag{3.14}
$$

Replacing ϕ^j by $-\Delta_x\phi^j$ in (3.4), multiplying by d^{jk} and summing over j from 1 to k, it follows that

$$\tau\int_\Omega \nabla_x z_t^k \nabla_x z^k\,dx + \int_\Omega \nabla_x z_\rho^k \nabla_x z^k\,dx = 0.$$

Then, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla z^k\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|\nabla_x z^k\|_2^2 = 0.$$

We integrate over $(0,1)$ to find that

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho + \frac{1}{2}\|\nabla_x z^k(x,1,t)\|_2^2 - \frac{1}{2}\|\nabla_x u_t^k(t)\|_2^2 = 0. \tag{3.15}$$

Combining (3.13)-(3.15) and using Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\Big(1-\int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k)\\
&+\tau\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho + 2\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx\Big] + \frac{1}{2}\|\nabla_x z^k(x,1,t)\|_2^2\\
&= (l+1)\int_\Omega |u_t^k|^l \nabla u_{tt}^k(t)\nabla_x u_t^k\,dx - \mu_1\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx\\
&\quad -\mu_2\int_\Omega \nabla_x u_t^k \nabla_x z^k(x,1,t) g_2'(z^k(x,1,t))\,dx + \frac{1}{2}\|\nabla_x u_t^k\|_2^2\\
&\quad -\frac{1}{2}h(t)\|\nabla\Delta u^k\|_2^2 + \frac{1}{2}(h'\circ\nabla\Delta u^k).
\end{aligned} \tag{3.16}$$

From the first estimate (3.7) and Young's inequality, we obtain

$$\begin{aligned}
(l+1)\int_\Omega |u_t^k|^l\nabla u_{tt}^k(t)\nabla_x u_t^k\,dx &\le (l+1)C_2^{l/(l+2)+1/2}\|\nabla u_{tt}^k\|_2\\
&\le \eta\|\nabla u_{tt}^k\|_2^2 + \frac{(l+1)^2 C_2^{2l/(l+2)+1}}{4\eta}, \quad \eta>0.
\end{aligned} \tag{3.17}$$

By using (1.7), (3.7) and Young's inequality, we obtain

$$\begin{aligned}
&\mu_2\int_\Omega \nabla_x u_t^k\nabla_x z^k(x,1,t)g_2'(z^k(x,1,t))\,dx\\
&\le \eta\|\nabla_x z^k(x,1,t)\|_2^2 + \frac{(\mu_2 c_3)^2}{4\eta}\|\nabla_x u_t^k\|_2^2\\
&\le \eta\|\nabla_x z^k(x,1,t)\|_2^2 + \frac{(\mu_2 c_3)^2 C_2}{4\eta}, \quad \eta>0.
\end{aligned} \tag{3.18}$$

Taking into account (3.17), (3.18) into (3.16) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\Big(1-\int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k)\\
&+\tau\int_0^1\|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho + 2\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx\Big]\\
&+\mu_1\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx + (\frac{1}{2}-\eta)\|\nabla_x z^k(x,1,t)\|_2^2\\
&\le \eta\|\nabla u_{tt}^k\|_2^2 - \frac{1}{2}h(t)\|\nabla\Delta u^k\|_2^2 + \frac{1}{2}(h'\circ\nabla\Delta u^k) + C_2(\eta).
\end{aligned} \tag{3.19}$$

Multiplying (3.1) by c_{tt}^{jk} and summing over j from 1 to k, it follows that

$$\begin{aligned} &\int_\Omega |u_t^k|^l |u_{tt}^k|^2\,dx + \|\nabla u_{tt}^k\|_2^2 \\ &= -\int_\Omega \Delta^2 u^k u_{tt}^k\,dx + \int_0^t h(t-s)\int_\Omega \Delta u^k(s)\Delta u_{tt}^k(t)\,dx\,ds \\ &- \mu_1\int_\Omega u_{tt}^k g_1(u_t^k)\,dx - \mu_2\int_\Omega u_{tt}^k g_2(z^k(x,1,t))\,dx. \end{aligned} \tag{3.20}$$

Differentiating (3.4) with respect to t, we obtain

$$(\tau z_{tt}^k + z_{t\rho}^k, \phi^j) = 0,$$

Multiplying by d_t^{jk} and summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\|z_t^k\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_t^k\|_2^2 = 0,$$

Integrating over $(0,1)$ with respect to ρ, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|z_t^k\|_2^2\,d\rho + \frac{1}{2}\|z_t^k(x,1,t)\|_2^2 - \frac{1}{2}\|u_{tt}^k(x,t)\|_2^2 = 0. \tag{3.21}$$

Summing (3.20) and (3.21), we obtain

$$\begin{aligned} &\int_\Omega |u_t^k|^l|u_{tt}^k|^2\,dx + \|\nabla u_{tt}^k\|_2^2 + \frac{\tau}{2}\frac{d}{dt}\int_0^1 \|z_t^k\|_2^2\,d\rho + \frac{1}{2}\|z_t^k(x,1,t)\|_2^2 \\ &= -\int_\Omega \Delta^2 u^k u_{tt}^k\,dx + \int_0^t h(t-s)\int_\Omega \Delta u^k(s)\Delta u_{tt}^k(t)\,dx\,ds \\ &\quad + \frac{1}{2}\|u_{tt}^k(x,t)\|_2^2 - \mu_1\int_\Omega u_{tt}^k g_1(u_t^k)\,dx - \mu_2\int_\Omega u_{tt}^k g_2(z^k(x,1,t))\,dx. \end{aligned} \tag{3.22}$$

By using Young's inequality, the right hand side of (3.22) can be estimated as follows:

$$\int_\Omega \Delta^2 u^k u_{tt}^k\,dx \le \eta\|\nabla u_{tt}^k\|_2^2 + \frac{1}{4\eta}\|\nabla\Delta u^k\|_2^2, \quad \eta > 0, \tag{3.23}$$

and

$$\begin{aligned} &\int_0^t h(t-s)\int_\Omega \Delta u^k(s)\Delta u_{tt}^k(t)\,dx\,ds \\ &= -\int_0^t h(t-s)\int_\Omega \nabla\Delta u^k(s)\nabla u_{tt}^k(t)\,dx\,ds \\ &\le \eta\|\nabla u_{tt}^k\|_2^2 + \frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)|\,ds\Big)^2\,dx \\ &\le \eta\|\nabla u_{tt}^k\|_2^2 + \frac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)(|\nabla\Delta u^k(s) \\ &\quad - \nabla\Delta u^k(t)| + |\nabla\Delta u^k(t)|)\,ds\Big)^2\,dx, \end{aligned} \tag{3.24}$$

Then we use Young's inequality to obtain, for any $\eta > 0$,

$$\int_\Omega\Big(\int_0^t h(t-s)(|\nabla\Delta u^k(s) - \nabla\Delta u^k(t)| + |\nabla\Delta u^k(t)|)\,ds\Big)^2\,dx$$

$$
\begin{aligned}
&\le \int_\Omega \Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|\,ds\Big)^2\,dx\\
&\quad+\int_\Omega \Big(\int_0^t h(t-s)|\nabla\Delta u^k(t)|\,ds\Big)^2\,dx\\
&\quad+2\int_\Omega \Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|\,ds\Big)\Big(\int_0^t h(t-s)|\nabla\Delta u^k(t)|\,ds\Big)\,dx\\
&\le (1+\eta)\int_\Omega \Big(\int_0^t h(t-s)|\nabla\Delta u^k(t)|\,ds\Big)^2\,dx\\
&\quad+(1+\frac{1}{\eta})\int_\Omega \Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|\,ds\Big)^2\,dx,
\end{aligned}
$$

Using (1.9), we obtain

$$
\begin{aligned}
&\int_\Omega \Big(\int_0^t h(t-s)|\nabla\Delta u^k(s)-\nabla\Delta u^k(t)|+|\nabla\Delta u^k(t)|\,ds\Big)^2\,dx\\
&\le \beta^2(1+\eta)\|\nabla\Delta u^k(t)\|_2^2+\beta(1+\frac{1}{\eta})(h\circ\nabla\Delta u^k).
\end{aligned}
\tag{3.25}
$$

By Young's inequality, we obtain

$$
\begin{aligned}
\mu_1\int_\Omega u_{tt}^k g_1(u_t^k)\,dx &\le \eta\int_\Omega |u_{tt}^k|^2\,dx+\frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx\\
&\le \eta C_s^2\|\nabla u_{tt}^k\|_2^2+\frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx
\end{aligned}
\tag{3.26}
$$

$$
\mu_2\int_\Omega u_{tt}^k g_2(z^k(x,1,t))\,dx \le \eta C_s^2\|\nabla u_{tt}^k\|_2^2+\frac{\mu_2^2}{4\eta}\int_\Omega |g_2(z^k(x,1,t))|^2\,dx. \tag{3.27}
$$

Taking into account (3.23)–(3.27) in (3.22) yields

$$
\begin{aligned}
&\int_\Omega |u_t^k|^l|u_{tt}^k|^2\,dx+\Big(1-2\eta(1+C_s^2)-\frac{C_s^2}{2}\Big)\|\nabla u_{tt}^k\|_2^2\\
&+\frac{\tau}{2}\frac{d}{dt}\int_0^1 \|z_t^k\|_2^2\,d\rho+\frac12\|z_t^k(x,1,t)\|_2^2\\
&\le \frac{\beta^2(1+\eta)}{4\eta}\|\nabla\Delta u^k\|_2^2+\frac{\beta}{4\eta}(1+\frac{1}{\eta})(h\circ\nabla\Delta u^k)\\
&\quad+\frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx+\frac{\mu_2^2}{4\eta}\int_\Omega |g_2(z^k(x,1,t))|^2\,dx
\end{aligned}
\tag{3.28}
$$

Thus, from (3.19) and (3.28), we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big[\Big(1-\int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2+\|\Delta_x u_t^k\|_2^2+(h\circ\nabla\Delta u^k)\\
&+\tau\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho+2\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx+\tau\int_0^1 \|z_t^k\|_2^2\,d\rho\Big]\\
&+\mu_1\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx+c_2'\|\nabla_x z^k(x,1,t)\|_2^2+\int_\Omega |u_t^k|^l|u_{tt}^k|^2\,dx\\
&+\Big(1-\eta(3+2C_s^2)-\frac{C_s^2}{2}\Big)\|\nabla u_{tt}^k\|_2^2+\frac12\|z_t^k(x,1,t)\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&\le -\frac{1}{2}h(t)\|\nabla\Delta u^k\|_2^2 + \frac{1}{2}(h'\circ\nabla\Delta u^k) + \frac{\beta^2(1+\eta)}{4\eta}\|\nabla\Delta u^k\|_2^2 \\
&\quad + \frac{\beta}{4\eta}(1+\frac{1}{\eta})(h\circ\nabla\Delta u^k) + \frac{\mu_1^2}{4\eta}\int_\Omega |g_1(u_t^k)|^2\,dx \\
&\quad + \frac{\mu_2^2}{4\eta}\int_\Omega |g_2(z^k(x,1,t))|^2\,dx + C_2(\eta)\,.
\end{aligned}
$$

By choosing η small enough such that $1-\eta(3+2C_s^2)-\frac{C_s^2}{2}>0$, integrating over $(0,t)$ and using (1.10), we obtain

$$
\begin{aligned}
&\Big(1-\int_0^t h(s)\,ds\Big)\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k) + \tau\int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho \\
&+ 2\int_\Omega |u_t^k(t)|^l|\nabla_x u_t^k|^2\,dx + \tau\int_0^1 \|z_t^k\|_2^2\,d\rho + \mu_1\int_0^t\int_\Omega |\nabla_x u_t^k|^2 g_1'(u_t^k)\,dx\,ds \\
&+ c_2'\int_0^t \|\nabla_x z^k(x,1,t)\|_2^2\,ds + \int_0^t\int_\Omega |u_t^k|^l|u_{tt}^k|^2\,dx\,ds \\
&+ \Big(1-\eta(3+2C_s^2)-\frac{C_s^2}{2}\Big)\int_0^t \|\nabla u_{tt}^k\|_2^2\,ds + \frac{1}{2}\int_0^t \|z_t^k(x,1,s)\|_2^2\,ds \\
&\le \frac{\beta^2(1+\eta)}{4\eta}\int_0^t \|\nabla\Delta u^k\|_2^2\,ds + \frac{\beta}{4\eta}(1+\frac{1}{\eta})\int_0^t (h\circ\nabla\Delta u^k)\,ds \\
&\quad + \frac{\mu_1^2}{4\eta}\int_0^t\int_\Omega |g_1(u_t^k)|^2\,dx\,ds + \frac{\mu_2^2}{4\eta}\int_0^t\int_\Omega |g_2(z^k(x,1,t))|^2\,dx\,ds + C_2(\eta)T
\end{aligned}
$$

Using (1.6), Jensen's inequality and the concavity of H^{-1}, we obtain

$$
\begin{aligned}
\int_\Omega |g_1(u_t^k)|^2\,dx &\le \int_{|u_t^k|\ge\varepsilon} |g_1(u_t^k)|^2\,dx + \int_{|u_t^k|\le\varepsilon} |g_1(u_t^k)|^2\,dx \\
&\le \int_{|u_t^k|\ge\varepsilon} u_t^k g_1(u_t^k)\,dx + \int_\Omega H^{-1}(u_t^k g_1(u_t^k))\,dx \\
&\le \int_{|u_t^k|\ge\varepsilon} u_t^k g_1(u_t^k)\,dx + cH^{-1}\Big(\int_\Omega u_t^k g_1(u_t^k)\,dx\Big),
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega |g_1(u_t^k)|^2\,dx &\le \int_{|u_t^k|\ge\varepsilon} u_t^k g_1(u_t^k)\,dx + c'H^*(1) + c''\int_\Omega u_t^k g_1(u_t^k)\,dx \\
&\le c'H^*(1) + c'\int_\Omega u_t^k g_1(u_t^k)\,dx \\
&\le c'H^*(1) + c(-E')
\end{aligned}
\tag{3.29}
$$

and

$$
\int_\Omega |g_2(z^k(x,1,t))|^2\,dx \le c'\int_\Omega z^k(x,1,t)g_2(z^k(x,1,t)\,dx \le c(-E')
$$

Using Gronwall' Lemma, we obtain

$$
\begin{aligned}
&\|\nabla\Delta u^k\|_2^2 + \|\Delta_x u_t^k\|_2^2 + (h\circ\nabla\Delta u^k) + \int_0^1 \|\nabla_x z^k(x,\rho,t)\|_2^2\,d\rho \\
&+ \int_0^1 \|z_t^k\|_2^2\,d\rho + \int_0^t \|\nabla u_{tt}^k(s)\|_2^2\,ds \le C_3
\end{aligned}
\tag{3.30}
$$

We observe that the estimate (3.7) and (3.30) that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ and a function u such that

$$u^m \rightharpoonup u \text{ weakly star in } L^\infty(0,T,H^4(\Omega)\cap H_0^2(\Omega)), \tag{3.31}$$

$$u_t^m \rightharpoonup u_t \text{ weakly star in } L^\infty(0,T,H_0^2(\Omega)), \tag{3.32}$$

$$g_1(u_t^m) \rightharpoonup \chi \text{ weakly star in } L^2(\Omega\times(0,T)), \tag{3.33}$$

$$u_{tt}^m \rightharpoonup u_{tt} \text{ weakly star in } L^2(0,T,H_0^1(\Omega)), \tag{3.34}$$

$$z^m \rightharpoonup z \text{ weakly star in } L^\infty(0,T,H_0^1(\Omega,L^2(0,1))), \tag{3.35}$$

$$z_t^m \rightharpoonup z_t \text{ weakly star in } L^\infty(0,T,L^2(\Omega\times(0,1))), \tag{3.36}$$

$$g_2(z^m(x,1,t)) \rightharpoonup \psi \text{ weakly star in } L^2(\Omega\times(0,T))\,. \tag{3.37}$$

From the first estimate (3.7) and Lemma 2.1, we deduce

$$\begin{aligned}\||u_t^k|^l u_t^k\|_{L^2(0,T,L^2(\Omega)} &= \int_0^T \|u_t^k\|_{2(l+1)}^{2(l+1)}\,dt \le \Big(\frac{C_s}{\sqrt{\lambda}}\Big)^{2(l+1)}\int_0^T \|\Delta u_t^k\|_2^{2(l+1)}\,dt\\ &\le \Big(\frac{C_s}{\sqrt{\lambda}}\Big)^{2(l+1)} C_3^{2(l+1)}T.\end{aligned}$$

On the other hand, from Aubin-Lions theorem, (see Lions [19]), we deduce that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ such that

$$u_t^m \to u_t \text{ strongly in } L^2(0,T,L^2(\Omega)) \tag{3.38}$$

which implies

$$u_t^m \to u_t \text{ almost everywhere in } \mathcal{A}. \tag{3.39}$$

Hence

$$|u_t^m|^l u_t^m \to |u_t|^l u_t \text{ almost everywhere in } \mathcal{A} \tag{3.40}$$

where $\mathcal{A}=\Omega\times(0,T)$. Thus, using (3.38), (3.40) and Lions Lemma, we derive

$$|u_t^m|^l u_t^m \rightharpoonup |u_t|^l u_t \text{ weakly in } L^2(0,T,L^2(\Omega)) \tag{3.41}$$

and

$$z^m \to z \text{ strongly in } L^2(0,T,L^2(\Omega))$$

which implies $z^m \to z$ almost everywhere in $\mathcal{A}$.

Lemma 3.1. *For each $T>0$, $g_1(u_t), g_2(z(x,1,t)) \in L^1(\mathcal{A})$ and $\|g_1(u')\|_{L^1(\mathcal{A})} \le K$, $\|g_2(z(x,1,t))\|_{L^1(\mathcal{A})} \le K$, where K is a constant independent of t.*

Proof. By (A2) and (3.39), we have

$$\begin{gathered} g_1(u_t^m(x,t)) \to g_1(u_t(x,t)) \quad \text{almost everywhere in } \mathcal{A},\\ 0 \le u_t^k(x,t) g_1(u_t^m(x,t)) \to u_t(x,t) g_1(u_t(x,t)) \quad \text{almost everywhere in } \mathcal{A}.\end{gathered}$$

Hence, by (3.11) and Fatou's Lemma, we have

$$\int_0^T\int_\Omega u_t(x,t) g_1(u_t(x,t))\,dx\,dt \le K_1 \quad \text{for } T>0 \tag{3.42}$$

Now, we can estimate $\int_0^T\int_\Omega |g_1(u_t(x,t))|\,dx\,dt$. By Cauchy-Schwarz inequality and using (3.29), (3.42), we have

$$\int_0^T\int_\Omega |g_1(u_t(x,t))|\,dx\,dt \le c|\mathcal{A}|^{1/2}\Big(\int_0^T\int_\Omega u_t(x,t) g_1(u_t(x,t))\,dx\,dt\Big)^{1/2}$$

$$\leq c|\mathcal{A}|^{1/2}K_1^{1/2} \equiv K.$$

Similarly, we have

$$\begin{aligned}\int_0^T\int_\Omega |g_2(z(x,1,t)))|\,dx\,dt &\leq c|\mathcal{A}|^{1/2}\Big(\int_0^T\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx\,dt\Big)^{1/2}\\ &\leq c|\mathcal{A}|^{1/2}K_1^{1/2} \equiv K.\end{aligned}$$

□

Lemma 3.2. *$g_1(u_t^k) \to g_1(u_t)$ in $L^1(\Omega\times(0,T))$ and $g_2(z^k) \to g_2(z)$ in $L^1(\Omega\times(0,T))$*

Proof. Let $E \subset \Omega\times[0,T]$ and set

$$E_1 = \Big\{(x,t)\in E : |g_1(u_t^k(x,t))| \leq \frac{1}{\sqrt{|E|}}\Big\}, \quad E_2 = E\backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s| : s\in\mathbb{R} \text{ and } |g(s)| \geq r\}$

$$\int_E |g_1(u_t^k)|\,dx\,dt \leq c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1}\int_{E_2} |u_t^k g_1(u_t^k)|\,dx\,dt.$$

By applying (3.11) we deduce that $\sup_k \int_E |g_1(u_t^k)|\,dx\,dt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that

$$g_1(u_t^k) \to g_1(u_t) \quad \text{in } L^1(\Omega\times(0,T)).$$

Similarly, we have

$$g_2(z^k) \to g_2(z) \quad \text{in } L^1(\Omega\times(0,T)).$$

This completes the proof. □

Hence

$$g_1(u_t^k) \rightharpoonup g_1(u_t) \quad \text{weak in } L^2(\Omega\times(0,T)), \tag{3.43}$$

$$g_2(z^k) \rightharpoonup g_2(z) \quad \text{weak in } L^2(\Omega\times(0,T)). \tag{3.44}$$

By multiplying (3.1) by $\theta(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

$$\begin{aligned}&-\frac{1}{l+1}\int_0^T (|u_t^k(t)|^l u_t^k(t), w^j)\theta'(t)\,dt + \int_0^T (\Delta_x u^k(t), \Delta_x w^j)\theta(t)\,dt\\ &+\int_0^T (\nabla_x u_{tt}^k, \nabla_x w^j)\theta(t)\,dt - \int_0^T\int_0^t h(t-s)(\Delta u^k(s), \Delta w^j)\theta(t)\,ds\,dt\\ &+\mu_1\int_0^T (g_1(u_t^k), w^j)\theta(t)\,dt + \mu_2\int_0^T (g_2(z^k(.,1)), w^j)\theta(t)\,dt = 0\end{aligned} \tag{3.45}$$

and multiplying (3.4) by $\theta(t) \in \mathcal{D}(0,T)$ and integrating over $(0,T)\times(0,1)$, it follows that

$$\int_0^T\int_0^1 (\tau z_t^k + z_\rho^k, \phi^j)\theta(t)\,dt\,d\rho = 0. \tag{3.46}$$

The convergence of (3.31)–(3.37), (3.41), (3.43) and (3.44) are sufficient to pass to the limit in (3.45) and (3.46) to obtain

$$-\frac{1}{l+1}\int_0^T (|u_t|^l u_t, w)\theta'(t)\,dt + \int_0^T (\Delta_x u, \Delta_x w)\theta(t)\,dt$$

$$
\begin{aligned}
&+\int_0^T (\nabla_x u_{tt}, \nabla_x w)\theta(t)\,dt - \int_0^T\int_0^t h(t-s)(\Delta u(s), \Delta w)\theta(t)\,ds\,dt\\
&+\mu_1\int_0^T (g_1(u_t), w)\theta(t)\,dt + \mu_2\int_0^T (g_2(z(.,1)), w)\theta(t)\,dt = 0,
\end{aligned}
$$

and

$$
\int_0^T\int_0^1 (\tau z_t + z_\rho, \phi)\theta(t)\,dt\,d\rho = 0,
$$

By integrating, we have

$$
\begin{aligned}
&\int_0^T \Big(|u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds\\
&+ \mu_1 g_1(u_t) + \mu_2 g_2(z(.,1)), w\Big)\theta(t)\,dt = 0,
\end{aligned}
$$

This completes the proof of Theorem 1.1.

3.3. **Proof of Theorem 1.2.** To prove our main result, we define the functionals

$$
\psi(t) = \int_\Omega\int_0^1 e^{-2\tau\rho} G(z(x,\rho,t))\,d\rho\,dx, \tag{3.47}
$$

$$
\phi(t) = \frac{1}{l+1}\int_\Omega |u_t|^l u_t u\,dx + \int_\Omega \nabla u_t \nabla u\,dx, \tag{3.48}
$$

$$
\varphi(t) = \int_\Omega \Big(\Delta u_t - \frac{1}{l+1}|u_t|^l u_t\Big)\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx\,. \tag{3.49}
$$

Set

$$
F(t) = ME(t) + \varepsilon_1\psi(t) + \varepsilon_2\phi(t) + \varphi(t), \tag{3.50}
$$

where M, ε_1 and ε_2 are suitable positive constants to be determined later.

Lemma 3.3. *There exist two positive constants κ_0 and κ_1 depending on ε_1, ε_2 and M such that for all $t > 0$*

$$
\kappa_0 E(t) \le F(t) \le \kappa_1 E(t). \tag{3.51}
$$

Proof. Using (1.11), we have

$$
|\psi(t)| \le \frac{1}{\xi}E(t). \tag{3.52}
$$

From Young's inequality and Lemma 2.1, we deduce

$$
\begin{aligned}
&|\phi(t)|\\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\|u\|_{l+2}^{l+2} + \frac12\|\nabla u\|_2^2 + \frac12\|\nabla u_t\|_2^2\\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\|\Delta u\|_2^{l+2} + \frac{1}{2\lambda_1}\|\Delta u\|_2^2 + \frac12\|\nabla u_t\|_2^2\\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{l/2} + \frac{1}{2\lambda_1}\Big\}\|\Delta u\|_2^2\\
&\quad + \frac12\|\nabla u_t\|_2^2\,.
\end{aligned} \tag{3.53}
$$

Integrating by parts, we have

$$
\varphi(t) = -\int_\Omega \nabla u_t\int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx
$$

$$-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx,$$

we use Young's inequality applied with the conjugate exponents $\frac{l+2}{l+1}$ and $l+2$, the second term in the right hand side can be estimated as

$$\begin{aligned}
&\Big|-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx\Big|\\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\int_\Omega \Big(\int_0^t h(t-s)|u(t)-u(s)|\,ds\Big)^{l+2}dx\\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\int_\Omega \Big(\int_0^t h(s)\,ds\Big)^{l+1}\int_0^t h(t-s)|u(t)-u(s)|^{l+2}\,ds\,dx\\
&\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\beta^{l+1}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{l/2}(h\circ\Delta u)(t)
\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}
&\Big|-\int_\Omega \nabla u_t \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\Big|\\
&\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\int_\Omega \Big(\int_0^t h(t-s)|\nabla u(t)-\nabla u(s)|\,ds\Big)^2 dx\\
&\le \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{\beta}{2\lambda_1}(h\circ\Delta u)(t).
\end{aligned} \tag{3.55}$$

By combining (3.54) and (3.55), we deduce that

$$\begin{aligned}
|\varphi(t)| &\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\|\nabla u_t\|_2^2\\
&\quad + \Big\{\frac{(l+1)^{-1}}{l+2}\beta^{l+1}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{l/2} + \frac{\beta}{2\lambda_1}\Big\}(h\circ\Delta u)(t).
\end{aligned} \tag{3.56}$$

By combining (3.52), (3.53) and (3.56), we have

$$\begin{aligned}
F(t) &\le (M+\frac{\varepsilon_1}{\xi})E(t) + \frac{\varepsilon_2+1}{l+2}\|u_t\|_{l+2}^{l+2}\\
&\quad + \varepsilon_2\Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{l/2} + \frac{1}{2\lambda_1}\Big\}\|\Delta u\|_2^2\\
&\quad + \frac{\varepsilon_2+1}{2}\|\nabla u_t\|_2^2 + \Big\{\beta^{l+1}\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{l/2} + \frac{\beta}{2\lambda_1}\Big\}(h\circ\Delta u)(t)\\
&\le \kappa_1 E(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
F(t) &\ge (M-\frac{\varepsilon_1}{\xi})E(t) - \frac{\varepsilon_2+1}{l+2}\|u_t\|_{l+2}^{l+2}\\
&\quad - \varepsilon_2\Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{l/2} + \frac{1}{2\lambda_1}\Big\}\|\Delta u\|_2^2 - \frac{\varepsilon_2+1}{2}\|\nabla u_t\|_2^2\\
&\quad - \Big\{\beta^{l+1}\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{l/2} + \frac{\beta}{2\lambda_1}\Big\}(h\circ\Delta u)(t)\\
&\ge \frac{1}{l+2}\Big(M-\Big\{\frac{\varepsilon_1}{\xi}+\varepsilon_2+1\Big\}\Big)\|u_t\|_{l+2}^{l+2} + \frac{1}{2}\Big(M-\Big\{\frac{\varepsilon_1}{\xi}+\varepsilon_2+1\Big\}\Big)\|\nabla u_t\|_2^2
\end{aligned}$$

$$
\begin{aligned}
&+ (M\xi - \varepsilon_1)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx + \Big(\frac{1}{2}\Big(M - \frac{\varepsilon_1}{\xi}\Big)\Big(1 - \int_0^t h(s)\,ds\Big)\\
&- \varepsilon_2\Big\{\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{2E(0)}{1-\beta}\Big)^{l/2} + \frac{1}{2\lambda_1}\Big\}\Big)\|\Delta u\|_2^2 + \Big(\frac{1}{2}\Big(M - \frac{\varepsilon_1}{\xi}\Big)\\
&- \Big\{\beta^{l+1}\frac{(l+1)^{-1}}{l+2}\Big(\frac{C_s}{\sqrt{\lambda_1}}\Big)^{l+2}\Big(\frac{4E(0)}{1-\beta}\Big)^{l/2} + \frac{\beta}{2\lambda_1}\Big\}\Big)(h\circ\Delta u)(t)\\
&\ge \kappa_0 E(t),
\end{aligned}
$$

for M large enough. □

Lemma 3.4. *Let (u,z) be the solution to (1.5). Then*

$$
\begin{aligned}
\psi'(t) \le -2\psi(t) - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx\\
+ \frac{\alpha_2}{\tau}\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx.
\end{aligned}
\tag{3.57}
$$

Proof. By differentiating (3.47) with respect to t and using (1.4) and (1.8), we obtain

$$
\begin{aligned}
\psi'(t) &= -\frac{1}{\tau}\int_\Omega\int_0^1 e^{-2\tau\rho}\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\Big(e^{-2\tau\rho}G(z(x,\rho,t))\Big)\,dx + 2\tau e^{-2\tau\rho}G(z(x,\rho,t))\Big]\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_\Omega\Big[e^{-2\tau}G(z(x,1,t) - G(u_t(x,t))\Big]\,dx\\
&\quad - 2\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_\Omega e^{-2\tau}G(z(x,1,t)\,dx + \frac{1}{\tau}\int_\Omega G(u_t(x,t))\,dx\\
&\quad - 2\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&= -2\psi(t) + \frac{1}{\tau}\int_\Omega G(u_t(x,t))\,dx - \frac{e^{-2\tau}}{\tau}\int_\Omega G(z(x,1,t))\,dx\\
&\le -2\psi(t) + \frac{\alpha_2}{\tau}\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx\\
&\quad - \frac{\alpha_1 e^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx.
\end{aligned}
$$

The proof is complete. □

Lemma 3.5. *Let (u,z) be a solution of (1.5). Then, for any $\eta > 0$,*

$$
\begin{aligned}
\phi'(t) \le \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|_2^2 - \Big(1 - \beta - \eta - \frac{\eta C_s^2}{\lambda_1}(\mu_1 + \mu_2)\Big)\|\Delta u\|_2^2\\
+ \frac{\beta}{4\delta}(h\circ\Delta u)(t) + \frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,dx + \frac{\mu_2}{4\eta}\int_\Omega |g_2(z(x,1,t))|^2\,dx.
\end{aligned}
\tag{3.58}
$$

Proof. Differentiating (3.48) with respect to t and using the first equation of (1.5), we obtain

$$\begin{aligned}
\phi'(t) &= \frac{1}{l+1}\int_\Omega (|u_t|^l u_t)' u\,dx + \frac{1}{l+1}\int_\Omega |u_t|^{l+2}\,dx + \int_\Omega \nabla u_{tt}\nabla u\,dx \\
&\quad + \int_\Omega \nabla u_t \nabla u_t\,dx \\
&= \int_\Omega |u_t|^l u_{tt} u\,dx + \frac{1}{l+1}\|u_t\|^{l+2} - \int_\Omega \Delta u_{tt} u\,dx + \|\nabla u_t\|_2^2 \\
&= \int_\Omega \Big(|u_t|^l u_{tt} - \Delta u_{tt}\Big) u\,dx + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|_2^2 \\
&= \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|^2 - \int_\Omega \Big(\Delta^2 u + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) \\
&\quad - \int_0^t h(t-s)\Delta^2 u(s)\,ds\Big) u\,dx \\
&= \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \|\nabla u_t\|_2^2 - \|\Delta u\|_2^2 + \int_\Omega \Delta u(t)\int_0^t h(t-s)\Delta u(s)\,ds\,dx \\
&\quad - \mu_1\int_\Omega u g_1(u_t(x,t))\,dx - \mu_2 \int_\Omega u g_2(z(x,1,t))\,dx
\end{aligned}$$

By using Young's inequality and Sobolev embedding, we can estimate the fourth term in the right side as follows:

$$\begin{aligned}
&\int_\Omega \Delta u(t)\int_0^t h(t-s)\Delta u(s)\,ds\,dx \\
&\le \int_0^t h(s)\,ds\|\Delta u(t)\|_2^2 + \int_\Omega\int_0^t h(t-s)|\Delta u(t)||\Delta u(s) - \nabla u(t)\,ds\,dx \\
&\le \int_0^t h(s)\,ds\|\Delta u(t)\|_2^2 + \eta\|\Delta u(t)\|_2^2 + \frac{\beta}{4\eta}(h\circ\Delta u)(t) \\
&\le (\beta+\eta)\|\Delta u(t)\|_2^2 + \frac{\beta}{4\eta}(h\circ\Delta u)(t)
\end{aligned}$$

Since

$$\int_\Omega u g_1(u_t)\,dx \le \frac{\eta C_s^2}{\lambda_1}\|\Delta u\|_2^2 + \frac{1}{4\eta}\int_\Omega |g_1(u_t)|^2\,dx, \tag{3.59}$$

$$\int_\Omega u g_2(z(x,1,t))\,dx \le \frac{\eta C_s^2}{\lambda_1}\|\Delta u\|_2^2 + \frac{1}{4\eta}\int_\Omega |g_2(z(x,1,t))|^2\,dx\,. \tag{3.60}$$

This completes the proof. □

Lemma 3.6. *Let* (u,z) *be a solution of* (1.5). *Then, for any* $\delta > 0$,

$$\begin{aligned}
\varphi'(t) &\le \delta(2\beta^2+1)\|\Delta u(t)\|_2^2 + \Big(\delta + \frac{\delta a_0}{l+1} - \int_0^t h(s)\,ds\Big)\|\nabla u_t\|_2^2 \\
&\quad + \beta\Big(2\delta + \frac{1}{2\delta} + \frac{\mu_1 C_s^2}{4\delta\lambda_1} + \frac{\mu_2 C_s^2}{4\delta\lambda_1}\Big)(h\circ\Delta u)(t) + \mu_1\delta\|g_1(u_t(x,t))\|_2^2 \\
&\quad - \frac{h(0)}{4\delta\lambda_1}\Big(1 + \frac{C_s^2}{(l+1)}\Big)(h'\circ\Delta u)(t) + \mu_2\delta\|g_2(z(x,1,t))\|_2^2
\end{aligned}$$

$$- \frac{1}{l+1}\int_0^t h(s)\,ds\|u_t\|_{l+2}^{l+2}.$$

Proof. By using the Liebnitz formula, and the first equation of (1.5), we have

$$\begin{aligned}
\varphi'(t) &= -\int_\Omega \Big(\int_0^t h(t-s)\Delta u(s)\,ds\Big)\Big(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))\,ds\Big)\,dx \\
&\quad + \int_\Omega \Delta u(t)\Big(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))\,ds\Big)\,dx \\
&\quad + \mu_1\int_\Omega g_1(u_t(x,t))\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \\
&\quad + \mu_2\int_\Omega g_2(z(x,1,t))\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx \\
&\quad - \int_\Omega \nabla u_t\int_0^t h'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\quad - \frac{1}{l+1}\int_\Omega |u_t|^l u_t\int_0^t h'(t-s)(u(t)-u(s))\,ds\,dx \\
&\quad - \int_0^t h(s)\,ds\|\nabla u_t(t)\|_2^2 - \frac{1}{l+1}\int_0^t h(s)\,ds\|u_t(t)\|_{l+2}^{l+2} \\
&= I_1+I_2+I_3+I_4+I_5+I_6 - \int_0^t h(s)\,ds\|\nabla u_t(t)\|_2^2 \\
&\quad - \frac{1}{l+1}\int_0^t h(s)\,ds\|u_t(t)\|_{l+2}^{l+2},
\end{aligned} \tag{3.61}$$

In what follows we will estimate $I_1,\dots,I_6$. So for $\delta>0$, we have

$$\begin{aligned}
|I_1| &\le \delta\int_\Omega \Big(\int_0^t h(t-s)|\Delta u(s)|\,ds\Big)^2 dx \\
&\quad + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t h(t-s)|\Delta u(t)-\Delta u(s)|\,ds\Big)^2 dx \\
&\le \delta\int_\Omega \Big(\int_0^t h(t-s)(|\Delta u(s)-\Delta u(t)|+|\Delta u(t)|)\,ds\Big)^2 dx \\
&\quad + \frac{1}{4\delta}\Big(\int_0^t h(s)\,ds\Big)(h\circ\Delta u)(t) \\
&\le 2\delta\Big(\int_0^t h(t)\,ds\Big)^2\|\Delta u(t)\|_2^2 + \Big(2\delta+\frac{1}{4\delta}\Big)\int_0^t h(s)\,ds\Big(h\circ\Delta u)(t) \\
&\le 2\delta\beta^2\|\Delta u(t)\|_2^2 + \beta\Big(2\delta+\frac{1}{4\delta}\Big)\Big(h\circ\Delta u)(t).
\end{aligned} \tag{3.62}$$

Similarly,

$$|I_2| \le \delta\|\Delta u(t)\|_2^2 + \frac{\beta}{4\delta}(h\circ\Delta u)(t), \tag{3.63}$$

$$|I_3| \le \delta\mu_1\|g_1(u_t(x,t))\|_2^2 + \frac{\mu_1\beta C_s^2}{4\delta\lambda_1}(h\circ\Delta u)(t), \tag{3.64}$$

$$|I_4| \le \delta\mu_2\|g_2(z(x,1,t))\|_2^2 + \frac{\mu_2\beta C_s^2}{4\delta\lambda_1}(h\circ\Delta u)(t), \tag{3.65}$$

$$
\begin{aligned}
|I_5| &\le \delta \int_\Omega |\nabla u_t|^2\,dx + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t |h'(t-s)||\nabla u(t)-\nabla u(s)|\,ds\Big)^2\,dx \\
&\le \delta\|\nabla u_t\|^2 + \frac{1}{4\delta}\int_\Omega\int_0^t -h'(s)\,ds\int_0^t -h'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx \\
&\le \delta\|\nabla u_t\|^2 - \frac{h(0)}{4\delta\lambda_1}(h'\circ\Delta u)(t),
\end{aligned} \tag{3.66}
$$

$$
\begin{aligned}
|I_6| &\le \frac{1}{l+1}\Big[\delta\int_\Omega ||u_t|^l u_t|^2\,dx + \frac{1}{4\delta}\int_\Omega\Big(\int_0^t h'(t-s)(u(t)-u(s))\,ds\Big)^2\,dx\Big] \\
&\le \frac{1}{l+1}\Big[\delta\|\ u_t\|_{2(l+1)}^{2(l+1)} - \frac{h(0)C_s^2}{4\delta\lambda_1}(h'\circ\Delta u)(t)\Big] \\
&\le \frac{\delta C_s^{2(l+1)}}{l+1}\|\nabla u_t\|_2^{2(l+1)} - \frac{h(0)C_s^2}{4\delta\lambda_1(l+1)}(h'\circ\Delta u)(t) \\
&\le \frac{\delta a_0}{l+1}\|\nabla u_t\|_2^2 - \frac{h(0)C_s^2}{4\delta\lambda_1(l+1)}(h'\circ\Delta u)(t),
\end{aligned} \tag{3.67}
$$

where $a_0 = C_s^{2(l+1)}(2E(0))^l$. □

Lemma 3.7. *Let (u,z) be a solution of (1.5) and assume that (A1)–(A4) hold. Then $F(t)$ satisfies the following estimate, along the solution and for some positive constants $m, a_6 > 0$,*

$$
F'(t) \le -mE(t) + a_6\|g_1(u_t(x,t))\|_2^2. \tag{3.68}
$$

Proof. From (2.3), (3.50), (3.57) and (3.58), we conclude that for any $t \ge t_0 > 0$,

$$
\begin{aligned}
F'(t) &= ME'(t) + \varepsilon_1\psi'(t) + \varepsilon_2\phi'(t) + \varphi'(t) \\
&\le -(M\beta_1 - \varepsilon_1\frac{\alpha_2}{\tau})\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx \\
&\quad - \Big(M\beta_2 - c_3\mu_2\{\delta + \frac{\varepsilon_2}{4\delta}\} - \varepsilon_1\frac{\alpha_1 e^{-2\tau}}{\tau}\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\
&\quad - 2\varepsilon_1\psi(t) - \frac{1}{l+1}(h_0-\varepsilon_2)\|u_t\|_{l+2}^{l+2} - \Big(h_0 - \varepsilon_2 - \delta\Big(1+\frac{a_0}{l+1}\Big)\Big)\|\nabla u_t\|_2^2 \\
&\quad - \Big(\frac{Mh_1}{2} + \varepsilon_2\{1-\beta-\delta-\frac{\delta C_s^2}{\lambda_1}(\mu_1+\mu_2)\} - \delta\Big(2\beta^2+1\Big)\Big)\|\Delta u\|_2^2 \\
&\quad + \Big(\frac{M}{2} - \frac{h(0)}{4\delta\lambda_1}\Big\{1+\frac{C_s^2}{l+1}\Big\}\Big)(h'\circ\Delta u)(t) \\
&\quad + \Big(\frac{\beta\varepsilon_2}{4\delta} + 2\beta\delta + \frac{\beta}{2\delta} + \frac{C_s^2\beta}{2\delta\lambda_1}\{\mu_1+\mu_2\}\Big)(h\circ\Delta u)(t) \\
&\quad + \mu_1(\delta + \frac{\varepsilon_2}{4\delta})\|g_1(u_t(x,t))\|_2^2
\end{aligned}
$$

where $h_0 = \int_0^{t_0} h(s)\,ds > 0$ and $h_1 = \min\{h(t)|$ for all $t \ge t_0\}$. We take $\varepsilon_2 < h_0$ and $\delta > 0$ sufficiently small such that

$$
a_1 = \frac{1}{l+1}(h_0-\varepsilon_2) > 0, \quad a_2 = h_0 - \varepsilon_2 - \delta\Big(1+\frac{a_0}{l+1}\Big) > 0.
$$

We choose M large enough such that

$$
a_3 = \frac{Mh_1}{2} + \varepsilon_2\{1-\beta-\delta-\frac{\delta C_s^2}{\lambda_1}(\mu_1+\mu_2)\} - \delta\Big(2\beta^2+1\Big) > 0,
$$

$$a_4=\zeta\Big(\frac{M}{2}-\frac{h(0)}{4\delta\lambda_1}\Big\{1+\frac{C_s^2}{l+1}\Big\}\Big)-\Big(\frac{\beta\varepsilon_2}{4\eta}+2\beta\delta+\frac{\beta}{2\delta}+\frac{C_s^2}{2\delta\lambda_1}\{\mu_1+\mu_2\}\Big)>0,$$
$$M\beta_1-\varepsilon_1\frac{\alpha_2}{\tau}>0,\quad M\beta_2-c_3\mu_2\{\delta+\frac{\varepsilon_2}{4\delta}\}-\varepsilon_1\frac{\alpha_1e^{-2\tau}}{\tau}>0.$$

Then

$$\begin{aligned}F'(t)&\le -a_1\|u_t\|_{l+2}^{l+2}-a_2\|\nabla u_t\|_2^2-a_3\|\Delta u\|_2^2-a_4(h\circ\Delta u)(t)\\&\quad-a_5\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho+\frac{\varepsilon_1}{\xi}E(t)+a_6\|g_1(u_t(x,t))\|_2^2,\end{aligned}$$

where $a_5=2\varepsilon_1$ and $a_6=\mu_1(\delta+\frac{\varepsilon_2}{4\eta})$. □

Proof of Theorem 1.2. As in Komornik [16], we consider the following partition of Ω,

$$\Omega_1=\{x\in\Omega:|u_t|>\varepsilon\},\quad \Omega_2=\{x\in\Omega:|u_t|\le\varepsilon\}$$

By using (1.6), we have

$$\int_{\Omega_1}|g_1(u_t)|^2\,dx\le c_2\int_{\Omega_1}u_tg_1(u_t)\,dx\le -cE'(t). \tag{3.69}$$

Case 1. H is linear on $[0,\varepsilon]$. In this case, one can easily check that there exists $c_1>0$, such that $|g_1(s)|\le c_1s$ for all $s\le\varepsilon$, and thus,

$$\int_{\Omega_2}|g_1(u_t)|^2\,dx\le c_1\int_{\Omega_2}u_tg_1(u_t)\,dx\le -cE'(t), \tag{3.70}$$
$$(F(t)+cE(t))'\le -mH_2(E(t)). \tag{3.71}$$

Case 2. $H'(0)$ and $H''>0$ on $]0,\varepsilon]$ we define

$$I(t)=\frac{1}{|\Omega_2|}\int_{\Omega_2}u_tg(u_t)\,dx,$$

and use Jensen's inequality and the concavity of H^{-1} to obtain

$$H^{-1}(I(t))\ge c\int_{\Omega_2}H^{-1}(u_tg(u_t))\,dx,$$

by using (1.6), we obtain

$$\begin{aligned}\int_{\Omega_2}|g_1(u_t)|^2\,dx&\le c\int_{\Omega_2}H^{-1}(u_tg(u_t))\,dx\\&\le cH^{-1}(I(t))\le cH^{-1}(-cE'(t)).\end{aligned}\tag{3.72}$$

A combination of (3.68), (3.69) and (3.72) yields

$$(F(t)+cE(t))'\le -mE(t)+cH^{-1}(-cE'(t)),\quad t\ge t_0. \tag{3.73}$$

By recalling that $E'\le0$, $H'>0$, and $H''>0$ on $(0,\varepsilon]$ and using (3.73), we obtain

$$\begin{aligned}&\Big(H'(\varepsilon_0E(t))\{F(t)+cE(t)\}+cE(t)\Big)'\\&=\varepsilon_0E'(t)H'(\varepsilon_0E(t))(F(t)+cE(t))+H'(\varepsilon_0E(t))(F(t)+cE(t))'+cE'(t)\\&\le -mH'(\varepsilon_0E(t))E(t)+cH'(\varepsilon_0E(t))H^{-1}(-cE'(t))+cE'(t),\end{aligned}\tag{3.74}$$

by using Remark 2.3 with H^*, the convex conjugate of H in the sense of Young, we obtain

$$\begin{aligned}
&\Big(H'(\varepsilon_0 E(t))\{F(t)+cE(t)\}+cE(t)\Big)' \\
&\le -mH'(\varepsilon_0 E(t))E(t)+cH^*(H'(\varepsilon_0 E(t))) \\
&\le -mH'(\varepsilon_0 E(t))E(t)+c\varepsilon_0 H'(\varepsilon_0 E(t))E(t) \\
&\le -cH'(\varepsilon_0 E(t))E(t) = -cH_2(E(t)).
\end{aligned} \tag{3.75}$$

Let

$$\widetilde{F}(t)=\begin{cases} F(t)+cE(t) \quad \text{if } H \text{is linear on } [0,\varepsilon], \\ H'(\varepsilon_0 E(t))\{F(t)+cE(t))\}+cE(t) \\ \quad \text{if } H'(0)>0 \text{ and } H''>0 \text{ on }]0,\varepsilon], \end{cases} \tag{3.76}$$

From (3.71) and (3.75), it follows that

$$\frac{d}{dt}\widetilde{F}(t)\le -cH_2(E(t)), \quad \forall t\ge t_0.$$

On the other hand, after choosing $M>0$ larger if needed, we can observe from Lemma 3.3 that $F(t)$ is equivalent to $E(t)$. So, $\widetilde{F}(t)$ is also equivalent to $E(t)$, for some positive constants $\widetilde{\epsilon_1}$ and $\widetilde{\epsilon_2}$

$$\widetilde{\epsilon_1}E(t)\le \widetilde{F}(t)\le \widetilde{\epsilon_2}E(t). \tag{3.77}$$

By setting $L(t)=\epsilon\widetilde{F}(t)$ for $\epsilon<1/\widetilde{\epsilon_2}$, we easily see that, by (3.77), we have $L(t)\sim E(t)$ and

$$\begin{aligned}
L'(t)&\le \epsilon\widetilde{F}(t)\le -\epsilon cH_2(E(t)) \\
&\le -\epsilon cH_2\Big(\frac{1}{\widetilde{\epsilon_2}}\widetilde{F}(t)\Big) \\
&\le -\epsilon cH_2\Big(\epsilon\ \widetilde{F}(t)\Big) \\
&\le -\epsilon cH_2(L(t)).
\end{aligned}$$

Then

$$\frac{L'(t)}{H_2(L(t))}\le -\epsilon c \tag{3.78}$$

By recalling (1.12), we deduce that $H_2(t)=-1/H_1'(t)$, hence

$$L'(t)H_1'(L(t))\ge \epsilon c, \quad \forall t\ge t_0.$$

A simple integration over (t_0,t) yields

$$H_1(L(t))\ge H_1(L(t_0))+\epsilon c(t-t_0).$$

By choosing $\epsilon>0$ sufficiently small such that $H_1(L(t_0))-\epsilon ct_0>0$, and exploiting the fact that H_1^{-1} is decreasing, we infer that

$$L(t))\le H_1^{-1}(\epsilon ct+H_1(L(t_0))-\epsilon ct_0). \tag{3.79}$$

Consequently, the equivalence of F, $\widetilde{F}$, L and E yields the estimate

$$E(t)\le w_3H_1^{-1}(w_1t+w_2).$$

This completes the proof. □

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