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EXISTENCE OF SOLUTIONS FOR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS IN SPACES OF TEMPERED SEQUENCES

JÓZEF BANAŚ, MONIKA KRAJEWSKA

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Abstract. The aim of this article is to study the existence of solutions for infinite systems of differential equations. We look for solutions in Banach tempered sequence spaces, using techniques associated with measures of noncompactness, and results from differential equations in abstract Banach spaces.

1. Introduction

The theory of differential equations in Banach spaces is nowadays almost a closed branch of mathematical analysis. Roughly speaking, after the publication of [3, 10, 11, 13, 15, 16, 19, 22] there have not appeared books or article presenting essential progress in the theory in question.

One of the most important reason of such a situation is the generality of problems raised in that theory. From this point of view we may consider the mentioned theory as closed or almost closed. Nevertheless, if we consider a particular case of differential equations in Banach spaces created by infinite systems of differential equations, the situation seems to be very far to be closed or even to be satisfactory developed. Up to now there appeared only a few papers devoted to the study of the theory of infinite systems of differential equations. The current state of that theory is presented in the recent monograph [8] (cf. also [4, 5]).

On the other hand infinite systems of differential equations find numerous applications in describing of several real world problems which can be encountered in the theory of neural nets, the theory of branching processes, the theory of dissociation of polymers and a lot of others (see for example [9, 10, 14, 18, 25]). It is also worthwhile mentioning that infinite systems of differential equations are applied to solve some problems investigated in mechanics [20, 21, 26]. Moreover, when we consider some problems of partial differential equations, we can use the process of semidiscretization to transform those problems into infinite systems of differential equations (cf. [10, 23, 24]).

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To establish some preliminary facts let us consider the infinite system of ordinary differential equations which can be written in the general form

$$x_n' = f_n(t, x_1, x_2, \dots) \tag{1.1}$$

for $t \in I = [0, T]$ and for $n = 1, 2, \dots$.

The Cauchy problem for system (1.1) can be formulated as the initial value conditions

$$x_n(0) = x_o^n, \quad \text{for } n = 1, 2, \dots . \tag{1.2}$$

Let us pay attention to be fact that any solution of (1.1)-(1.2) has the form of a function sequence

$$x(t) = (x_n(t)) = (x_1(t), x_2(t), \dots) \tag{1.3}$$

where t runs over an interval $[0, T]$ (or $[0, T_1] \subset [0, T]$). To avoid unnecessary ambiguities we will denote the interval of the definition of solution (1.3) of (1.1)-(1.2) by I i.e., $I = [0, T]$. Thus, for each fixed $t \in I$ the sequence $(x_n(t))$ presents certain sequence of real numbers. Therefore, we consider the solvability of problem (1.1)-(1.2) in some sequence space c_0, c, l_p, l_∞ (cf. [8, 10]). Details concerning the mentioned sequence spaces will be given later on.

Now we show that even in rather simple situations the mentioned classical sequence spaces are not sufficient for the location of our investigations.

Example 1.1. To show the influence of the choice of initial values in a sequence space in which are located solutions of a considered initial value problem for an infinite system of differential equations, let us consider the linear diagonal infinite system of differential equations

$$x_n' = x_n \tag{1.4}$$

with the initial conditions

$$x_n(0) = n, \quad \text{for } n = 1, 2, \dots . \tag{1.5}$$

We consider problem (1.4)–(1.5) on an interval $I = [0, T]$.

It is easily seen that the solution of (1.4)-(1.5) has the form

$$x(t) = (x_n(t)) = (ne^t) = (e^t, 2e^t, 3e^t, \dots).$$

This means that $x(t) \notin l_\infty$ for each $t \in I$. Thus the sequence space l_∞ is not suitable to consider solvability of problem (1.4)–(1.5) in this space. Obviously, such a situation appears quite naturally since the initial point $(x_n^o) = (n)$ is not a member of l_∞.

Example 1.2. Let us consider the infinite system of differential equations

$$x_n' = n \frac{\sqrt{|x_n|}}{\sqrt{|x_n|} + 1} \tag{1.6}$$

for $n = 1, 2, \dots$, together with initial conditions

$$x_n(0) = 0, \quad \text{for } n = 1, 2, \dots . \tag{1.7}$$

Let us fix arbitrarily a natural number n. Then, we can easy calculate that the solution of problem (1.6)–(1.7) has the form

$$x_n(t) = \frac{n^2 t^2}{2 + nt + 2\sqrt{1 + nt}}$$

for $t \in I$. Hence, we obtain the estimate

$$
\begin{aligned}
x_n(t) &\geq \frac{n^2t^2}{2+nt+2\sqrt{1+2nt+n^2t^2}} \\
&= \frac{n^2t^2}{2+nt+2(1+nt)} \geq \frac{n^2t^2}{4+4nt} \quad = \frac{1}{4}\Big(nt-1+\frac{1}{nt+1}\Big)
\end{aligned}
\tag{1.8}
$$

for $n = 1, 2, \dots$ and for $t \in I$.

Further, let us represent the solution of (1.6)–(1.7) in the form $x(t) = (x_n(t)) = (x_1(t), x_2(t), \dots)$. Then, from estimate (1.8) we infer that $x(t) \notin l_\infty$ for any $t > 0$. On the other hand let us notice that the right-hand sides of equations (1.6) are not bounded. Indeed, we have

$$
\frac{n\sqrt{x}}{\sqrt{x}+1} \to n, \quad \text{as } x \to \infty.
$$

The above given examples suggest that we have to enlarge the spaces under considerations to ensure that solutions of infinite systems of differential equations starting from a point in such a space remain in the space in question when t runs over some interval I. It seems that a natural way to realize the enlargement is to consider the so - called tempered sequence spaces. Those spaces can be obtained from classical sequence spaces with help of a tempering sequence. For example, if we take the classical space l_∞ and the tempering sequence $\beta_n = \frac{1}{n}$ $(n = 1, 2, \dots)$ then the new sequence space l_∞^β with $\beta = (\beta_n) = (\frac{1}{n})$ is understood as the space of all sequences (x_n) such that the sequence $(\beta_n x_n) = (\frac{1}{n}x_n)$ is bounded. The details concerning tempered sequence spaces will be described later on.

It is worthwhile noticing that such an approach enables us to study an essentially larger class of infinite systems of differential equations in comparison with the classical setting.

In this article we discuss some classes of infinite systems of differential equations having solutions in the above mentioned tempered sequence spaces. The results of the paper generalize several ones obtained up to now in classical sequence spaces (see [4, 5, 8, 10, 11, 17]).

2. Auxiliary facts concerning the theory of measures of noncompactness

This section is devoted to recall a few facts concerning the theory of measures of noncompactness, which will be needed in our further considerations. Those facts come mainly from monograph [3] (cf. also [1, 2]). To set the stage for our study we establish first the notation used in this article.

By the symbol $\mathbb{R}$ we will denote the set of real numbers, and by $\mathbb{N}$ the set of natural numbers (positive integers). We write $\mathbb{R}_+$ to denote the interval $[0, \infty)$. Further, assume that E is a Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $B(x, r)$ the closed ball in E centered at x and with radius r. We write B_r to denote the ball $B(\theta, r)$. If X is a subset of E then by $\bar{X}$, $\mathrm{Conv}X$ we will denote the closure and convex closure of X, respectively. Moreover, the symbols $X + Y$, λX, $(\lambda \in \mathbb{R})$ stand for standard algebraic operations on sets X and Y.

We denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subset of the space E, and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

In what follows we will accept the following axiomatic definition of the concept of a measure of noncompactness [3].

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness if the following conditions are satisfied:

(i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;
(ii) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$;
(iii) $\mu(\bar{X}) = \mu(X)$;
(iv) $\mu(\operatorname{Conv} X) = \mu(X)$;
(v) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;
(vi) if (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ from axiom (i) is said to be *the kernel of the measure* μ.

Further, let us observe that from axiom (vi) it follows that $\mu(X_\infty) \le \mu(X_n)$ for $n = 1, 2, \dots$. This yields that $\mu(X_\infty) = 0$. Hence we conclude that the intersection set X_∞ belongs to the kernel $\ker \mu$. This simple fact plays a very essential role in applications.

In the sequel we will also consider measures of noncompactness having some additional properties. Thus, a measure μ is referred to as *sublinear* if it satisfies the following two conditions:

(vii) $\mu(\lambda X) = |\lambda|\mu(X)$, $\lambda \in \mathbb{R}$;
(viii) $\mu(X + Y) \le \mu(X) + \mu(Y)$.

We say that a measure of noncompactness has *maximum property* if

(ix) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

The measure μ is said to be *full* if

(x) $\ker \mu = \mathfrak{N}_E$.

Finally, the measure of noncompactness μ is called *regular* if it is sublinear, full and has maximum property.

The most convenient and simultaneously important regular measure of noncompactness is the so - called *Hausdorff measure* χ defined in the following way

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}.$$

It can be shown that this measure has also some other interesting and useful properties (cf. [1, 2, 3, 6]).

The usefulness of the Hausdorff measure χ leads to the question if each regular measure of noncompactness μ is equivalent to the Hausdorff measure χ. It was shown in [7] that, in general, the answer is negative. Nevertheless, we have the following theorem [3] which shows that each regular measure of noncompactness is one - sided comparable with the Hausdorff measure.

Theorem 2.2. *If μ is a regular measure then*

$$\mu(X) \le \mu(B_1)\chi(X)$$

for any set $X \in \mathfrak{M}_E$.

In practice we use those measures of noncompactness which can be expressed with help of a convenient formula associated with the structure of a considered Banach space. It turns out that we know only a few Banach spaces in which the Hausdorff measure of noncompactness can be expressed (or, at least, estimated) in such a way [3]. By these regards we mostly apply measures of noncompactness being not regular but which are connected with sufficient conditions for relative compactness in Banach spaces under considerations [3, 8].

3. Measures of noncompactness in classical sequence spaces

Now we work in the sequence spaces c_0, c, l_p and l_∞ being the classical sequence spaces. We recall briefly the definition of these spaces.

By the space c_0 we mean the set of all real (or complex) sequences $x = (x_n)$ converging to zero and normed by the classical supremum (or maximum) norm:

$$\|x\|_{c_0} = \|(x_n)\|_{c_0} = \sup\{|x_n| : n = 1, 2, \dots\} = \max\{|x_n| : n = 1, 2, \dots\}.$$

Obviously c_0 with this norm creates the Banach space.

Next, denote by c the space of all sequences $x = (x_n)$ converging to a (finite) limit, with the norm

$$\|x\|_c = \|(x_n)\|_c = \sup\{|x_n| : n = 1, 2, \dots\}.$$

The space c with the norm $\|\cdot\|_c$ is a Banach space and c_0 is a closed subspace of c.

If we fix a number p, $p \geq 1$, then by l_p we denote the space consisting of all sequences $x = (x_n)$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. If we normed it by

$$\|x\|_{l_p} = \|(x_n)\|_{l_p} = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p}$$

it becomes a Banach space.

Finally, by the symbol l_∞ we denote the space of all bounded sequences $x = (x_n)$ with the supremum norm

$$\|x\|_{l_\infty} = \|(x_n)\|_{l_\infty} = \sup\{|x_n| : n = 1, 2, \dots\}.$$

Now, we present the known facts concerning the measures of noncompactness in the above mentioned sequence spaces [3, 8]. In the case of sequence spaces c_0, c and l_p the situation concerning measures of noncompactness seems to be thoroughly recognized. Indeed, in the spaces c_0 and l_p we know formulas expressing the most convenient measure of noncompactness i.e., the Hausdorff measure χ (cf. Section 2). To present the mentioned formulas let us consider first the space c_0 and let us take an arbitrary nonempty and bounded subset of c_0 i.e., take a set $X \in \mathfrak{M}_{c_0}$. Then we have [3]

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_n)\in X} \Big\{ \sup\{|x_i| : i \geq n\} \Big\} \Big\}.$$

Next, if we fix arbitrarily a number p, $p \geq 1$, then for $X \in \mathfrak{M}_{l_p}$ we have [3, 8]

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup \Big\{ \Big(\sum_{k=n}^{\infty} |x_n|^p \Big)^{1/p} : x = (x_i) \in X \Big\} \Big\}.$$

In the case of the sequence space c the situation is a bit more complicated. Namely, we do not know a formula for the Hausdorff measure χ in c but we know only a good estimate χ. Indeed, for $X \in \mathfrak{M}_c$ let us define the quantity $\mu(X)$ by the formula

$$\mu(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_k)\in X} \Big\{ \sup\{|x_i - \lim_{k\to\infty} x_k| : i \geq n\} \Big\} \Big\}. \tag{3.1}$$

Then we have the estimate

$$\frac{1}{2}\mu(X) \le \chi(X) \le \mu(X) \tag{3.2}$$

and this estimate is sharp [3].

It can be shown that measure (3.1) is regular. Nevertheless, let us pay attention to the fact that the measure μ has only theoretical meaning since the use of formula (3.1) requires to know limits of sequences belonging to a set X. Therefore, to obtain a more convenient formula we can use the classical Cauchy condition associated with the limit of a sequence, since such an approach does not require the use of the limit of a sequence. Thus, for $X \in \mathfrak{M}_c$ we define the quantity

$$\mu_c(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \Big\{ \sup\{|x_n - x_m| : n, m \ge k\} \Big\} \Big\}. \tag{3.3}$$

It is worthwhile mentioning that in a few papers and monographs (see [3, 5, 8], for example) we can encounter results asserting that the measure μ_c defined by formula (3.3) is regular and equivalent to the Hausdorff measure χ in the space c. On the other hand there are no proof of that fact. Therefore, to bridge this gap we provide below the complete proof of the following theorem.

Theorem 3.1. *The quantity μ_c defined by formula* (3.3) *is a regular measure of noncompactness in the space c. Moreover, the following inequalities are satisfied*

$$\chi(X) \le \mu_c(X) \le 2\chi(X) \tag{3.4}$$

for $X \in \mathfrak{M}_c$

Proof. At the beginning let us observe that keeping in mind formula (3.3) it is not hard to show the quantity μ_c satisfies axioms (ii)–(v) and (vii)–(ix) of the definition of a regular measure of noncompactness (cf. Section 2 and Definition 2.1).

Next, fix arbitrarily a set $X \in \mathfrak{M}_c$ and choose a sequence $x = (x_i) \in X$. Take a fixed natural number k. Then, for arbitrary $n, m \ge k$ we have

$$|x_n - x_m| \le |x_n - \lim_{i\to\infty} x_i| + |x_m - \lim_{i\to\infty} x_i|.$$

Hence we derive the estimate

$$\mu_c(X) \le 2\mu(X), \tag{3.5}$$

where μ is the measure of noncompactness defined by (3.1). Linking (3.2) and (3.5) we obtain

$$\mu_c(X) \le 4\chi(X) \tag{3.6}$$

for $X \in \mathfrak{M}_c$.

Now, let us denote $r = \mu_c(X)$. Fix $\varepsilon > 0$ and find a natural number k_0 such that

$$|x_n - x_m| \le r + \varepsilon \tag{3.7}$$

for each $x = (x_i) \in X$ and $n, m \ge k_0$. Consider the set $X_{k_o} = \{(x_1, x_2, \dots, x_{k_0}) : x = (x_1, x_2, \dots, x_{k_0}, x_{k_0+1}, \dots) \in X\}$. Obviously X_{k_0} is a bounded subset of the Euclidean space $\mathbb{R}^{k_0}$. Thus there exists a finite ε-net of the set X_{k_0} formed by some k_0 - tuples $\tilde{y_1}, \tilde{y_2}, \dots, \tilde{y}_m$, where $\tilde{y}_p = (y_1^p, y_2^p, \dots, y_{k_0}^p)$ for $p = 1, 2, \dots, m$.

Next, we consider the sequence y_p $(p = 1, 2, \dots, m)$ defined as

$$y_p = (y_1^p, y_2^p, \dots, y_{k_0}^p, y_{k_0}^p, y_{k_0}^p, \dots).$$

We show that the set $\{y_1, y_2, \dots, y_m\}$ forms the $r+2\varepsilon$-net of the set X in the space c. To this end take an arbitrary sequence $x = (x_i) \in X$. Then, we can find a k_0-tuple $\tilde{y}_p = (y_1^p, y_2^p, \dots, y_{k_0}^p)$ $(1 \le p \le m)$ such that

$$|x_i - y_i^p| \le \varepsilon \tag{3.8}$$

for $i = 1, 2, \dots, k_0$. Further, for $i \ge k_0$, we obtain

$$|x_i - y_i^p| \le |x_i - x_{k_0}| + |x_{k_0} - y_i^p| = |x_i - x_{k_0}| + |x_{k_0} - y_{k_0}^p|.$$

Hence, from (3.7) and (3.8) we obtain

$$|x_i - y_i^p| \le r + \varepsilon + \varepsilon = r + 2\varepsilon. \tag{3.9}$$

Linking (3.8) and (3.9) we conclude that the set $\{y_1, y_2, \dots, y_m\}$ forms an $r+2\varepsilon$-net of the set X in the space c. Moreover, in view of the arbitrariness of ε this yields

$$\chi(X) \le r,$$

which leads to the inequality

$$\chi(X) \le \mu_c(X). \tag{3.10}$$

Combining estimates (3.6) and (3.10) we derive the following inequalities

$$\chi(X) \le \mu_c(X) \le 4\chi(X), \tag{3.11}$$

which are satisfied for $X \in \mathfrak{M}_c$.

Now, let us observe that from inequalities (3.11) we obtain that the quantity μ_c satisfies axioms (i) and (vi) of Definition 2.1. Thus, μ_c is a sublinear measure of noncompactness with maximum property in the space c. Applying (3.11) again we deduce that μ_c is a regular measure equivalent to the Hausdorff measure χ.

In what follows let us observe that the estimate on the right hand side of (3.11) i.e., estimate (3.6) can be improved. Indeed, since μ_c is a regular measure of noncompactness then, in view of Theorem 2.2 we have

$$\mu_c(X) \le \mu_c(B_1)\chi(X) \tag{3.12}$$

for an arbitrary set $X \in \mathfrak{M}_c$ (the symbol B_1 stands for the unit ball in c). On the other hand it is easy to calculate that $\mu_c(B_1) = 2$. Thus, from (3.12) we obtain

$$\mu_c(X) \le 2\chi(X). \tag{3.13}$$

Finally, combining estimates (3.10) and (3.13) we obtain desired estimate (3.4). The proof is complete. □

In the sequel we shall deal with measures of noncompactness in the space l_∞. Firstly, let us notice that in this space we do not know a formula which expresses the Hausdorff measure of noncompactness χ. Even more, we do not know formulas for regular measures in l_∞ [1, 3, 8]. Thus, in this case we can only obtain formulas for measures of noncompactness defined in an axiomatic way (cf. Definition 2.1). It is worthwhile mentioning that there are known and used some convenient formulas for measures of noncompactness in the space l_∞ [3, 8]. Unfortunately, in the literature there are no proofs of the correctness of those formulas. More precisely, there are no proofs of the fact that the formulas in question are measures on noncompactness in l_∞. Below we are going to fill this gap.

To present the above mentioned formulas let us fix a set $X \in \mathfrak{M}_{l_\infty}$. Next, we define the following three quantities:

$$\mu_1^\infty(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_i)\in X} \big\{ \sup\{|x_i| : i \geq n\}\big\}\Big\}, \tag{3.14}$$

$$\mu_2^\infty(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \big\{ \sup\{|x_n - x_m| : n, m \geq k\}\big\}\Big\}, \tag{3.15}$$

$$\mu_3^\infty(X) = \limsup_{n\to\infty} \operatorname{diam} X_n, \tag{3.16}$$

where $X_n = \{x_n : x = (x_i) \in X\}$ and $\operatorname{diam} X_n = \sup\big\{|x_n - y_n| : x = (x_i), y = (y_i) \in X\big\}$. Observe that the formula expressing the quantity μ_1^∞ coincides with the formula for the Hausdorff measure of noncompactness in the space c_0. On the other hand, formula (3.15) for the quantity μ_2^∞ coincides with formula (3.3) for the measure of noncompactness μ_c in the sequence space c.

Theorem 3.2. *The quantities $\mu_i^\infty (i = 1, 2, 3)$ are sublinear measures of noncompactness in the space l_∞. In addition, the measures μ_1^∞ and μ_2^∞ have maximum property. Moreover, for an arbitrary set $X \in \mathfrak{M}_{l_\infty}$ the following inequalities hold*

$$\chi(X) \leq \mu_2^\infty(X), \tag{3.17}$$

$$\chi(X) \leq \mu_3^\infty(X), \tag{3.18}$$

$$\mu_2^\infty(X) \leq 2\mu_1^\infty(X), \tag{3.19}$$

$$\mu_3^\infty(X) \leq 2\mu_1^\infty(X). \tag{3.20}$$

Proof. The proof of (3.17) can be conducted in the same way as the proof of (3.10). Indeed, it follows easily from the fact that c is a subspace of the space l_∞.

To prove (3.18) let us fix $X \in \mathfrak{M}_{l_\infty}$ and put $r = \mu_3^\infty(X)$. Next, take an arbitrary number $\varepsilon > 0$. Then, in view of definition (3.16) we can find a natural number n_0 such that $\operatorname{diam} X_n \leq r + \varepsilon$ for $n \geq n_0$. Hence we infer that for arbitrary elements $x = (x_i)$, $y = (y_i)$ of the set X we have

$$|x_n - y_n| \leq r + \varepsilon \tag{3.21}$$

for $n \geq n_0$. Further, we consider the set $\bar{X}_{n_0} = \{(x_1, x_2, \dots, x_{n_0}) : (x_i) \in X\}$. This set is a relatively compact subset of the Euclidean space $\mathbb{R}^{n_0}$. Thus, there exists a finite ε-net of the set $\bar{X}_{n_0}$ composed by n_0 - tuples $\tilde{y}_1 = (y_1^1, y_2^1, \dots, y_{n_0}^1)$, $\tilde{y_2} = (y_1^2, y_2^2, \dots, y_{n_0}^2)$, $\tilde{y}_m = (y_1^m, y_2^m, \dots, y_{n_0}^m)$. Next, fix an arbitrary element $y = (y_i) = (y_1, y_2, \dots, y_{n_0}, y_{n_0+1}, \dots)$ of the set X and consider the finite subset $Y = \{y_1, y_2, \dots, y_m\}$ of the space l_∞ such that

$$y_i = (y_1^i, y_2^i, \dots, y_{n_0}^i, y_{n_0+1}, y_{n_0+2}, \dots)$$

for $i = 1, 2, \dots, m$. We show that Y forms a finite $r + \varepsilon$-net of the set X. To this end take an arbitrary element $x = (x_i) \in X$ and consider the n_0-tuple $\tilde{x} = (x_1, x_2, \dots, x_{n_0})$. Then we can find a n_0-tuple $\tilde{y}_k \in \bar{X}_{n_0}$, $\tilde{y}_k = (y_1^k, y_2^k, \dots, y_{n_0}^k)$ such that

$$|x_i - y_i^k| \leq \varepsilon \tag{3.22}$$

for $i = 1, 2, \dots, n_0$.

Now, we take the element $y_k = (y_1^k, y_2^k, \dots, y_{n_0}^k, y_{n_0+1}, y_{n_0+2}, \dots) \in Y$. Then, in view of (3.21) and (3.22), we have

$$|x_n - y_n| \leq r + \varepsilon$$

for $n = 1, 2, \dots$. This means that $\|x - y_k\|_{l_\infty} \le r + \varepsilon$. Thus the set Y forms a finite $r + \varepsilon$-net of the set X. Hence we conclude that $\chi(X) \le r + \varepsilon$. In view of the arbitrariness of ε this implies inequality (3.18). Further, let us observe that estimates (3.19) and (3.20) are a simple consequence of the triangle inequality for absolute value.

Next, from (3.17) and (3.19) (or from (3.18) and (3.20)) we obtain the following estimate

$$\frac{1}{2}\chi(X) \le \mu_1^\infty(X). \tag{3.23}$$

Finally, taking into account inequalities (3.17), (3.18) and (3.23) we conclude that the quantities μ_i^∞ $(i = 1, 2, 3)$ satisfy axioms (i) and (vi) of Definition 2.1. The fact that there are satisfied other conditions (ii)–(v) and (vii), (viii) for all quantities μ_i^∞ $(i = 1, 2, 3)$ and condition (ix) for μ_1^∞ and μ_2^∞ is easy to prove. This completes the proof. □

4. Measures of noncompactness in spaces of tempered sequences

As we saw in introduction, classical sequence spaces are not always suitable to consider initial value problems for infinite systems of differential equations. Therefore, in order to consider those initial value problems we are frequently forced to treat the problems in question in enlarged sequence spaces. Such sequence spaces can be obtained if we consider the so - called tempered sequence spaces.

To define the mentioned spaces let us fix a real sequence $\beta = (\beta_n)$ such that β_n is positive for $n = 1, 2, \dots$ and the sequence (β_n) is nonincreasing. Such a sequence β will be called the tempering sequence. Next, consider the set X consisting of all real (or complex) sequences $x = (x_n)$ such that $\beta_n x_n \to 0$ as $n \to \infty$. It is easily seen that X forms a linear space over the field of real (or complex) numbers. We will denote this space by the symbol c_0^β.

It is easy to check that c_0^β is a Banach space under the norm

$$\|x\|_{c_0^\beta} = \|(x_n)\|_{c_0^\beta} = \sup\{\beta_n|x_n| : n = 1, 2, \dots\} = \max\{\beta_n|x_n| : n = 1, 2, \dots\}.$$

In a similar way we may consider the space c^β consisting of real (complex) sequences (x_n) such that the sequence $(\beta_n x_n)$ converges to a finite limit. Obviously c^β forms a linear space and it becomes a Banach space if we normed it by the supremum norm

$$\|x\|_{c^\beta} = \|(x_n)\|_{c^\beta} = \sup\{\beta_n|x_n| : n = 1, 2, \dots\}.$$

In the same way we can consider the tempered sequence space l_∞^β of all sequences (x_n) (real or complex) such that the sequence $(\beta_n x_n)$ is bounded. The space l_∞^β is a Banach space under the norm

$$\|x\|_{l_\infty^\beta} = \|(x_n)\|_{l_\infty^\beta} = \sup\{\beta_n|x_n| : n = 1, 2, \dots\}.$$

Let us pay attention to the fact that taking $\beta_n = 1$ for $n = 1, 2, \dots$ we obtain spaces $c_0^\beta = c_0$, $c^\beta = c$ and $l_\infty^\beta = l_\infty$. Similarly, if the sequence (β_n) is bounded from below by a positive constant m i.e., if $\beta_n \ge m > 0$ for $n = 1, 2, \dots$, then the norms in the tempered sequence spaces c_0^β, c^β and l_∞^β are equivalent to the classical supremum norm in each of the spaces c_0, c and l_∞. Thus, to obtain an essential enlargement of the spaces c_0, c and l_∞ we should to assume that the tempering sequence (β_n) converges to zero. In what follows we will impose such a requirement.

The most important fact for our further purposes is the assertion saying that the pairs of the spaces (c_0, c_0^β), (c, c^β) and $(l_\infty, l_\infty^\beta)$ are isometric. Indeed, consider for example the spaces l_∞ and l_∞^β. Next, take the mapping $J : l_\infty^\beta \to l_\infty$ defined in the following way

$$J(x) = J((x_n)) = (\beta_n x_n).$$

Then, for arbitrarily fixed $x, y \in l_\infty^\beta$ we have

$$\begin{aligned}\|J(x) - J(y)\|_{l_\infty} &= \|J((x_n)) - J((y_n))\|_{l_\infty} \\ &= \|(\beta_n x_n) - (\beta_n y_n)\|_{l_\infty} \\ &= \sup\{|\beta_n x_n - \beta_n y_n| : n = 1, 2, \dots\} \\ &= \sup\{\beta_n |x_n - y_n| : n = 1, 2, \dots\} = \|x - y\|_{l_\infty^\beta}.\end{aligned}$$

This shows that the mapping J is an isometry between the spaces l_∞^β and l_∞. Obviously, the same mapping establishes the isometry between the spaces c^β and c and the spaces c_0^β and c_0, respectively.

The above assertions enable us to define measures of noncompactness in the tempered sequence spaces c_0^β, c^β and l_∞^β. In fact, the Hausdorff measure of noncompactness $\chi(X)$ for $X \in \mathfrak{M}_{c_0^\beta}$ can be expressed in the following way (cf. Section 3):

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_i)\in X} \big\{ \sup \big\{ \beta_i |x_i| : i \geq n \big\} \big\} \Big\}. \tag{4.1}$$

Similarly, the analogue of the measure of noncompactness μ_c defined by formula (3.3) has the form

$$\mu_{c^\beta}(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \big\{ \sup\{|\beta_n x_n - \beta_m x_m| : n, m \geq k\} \big\} \Big\}, \tag{4.2}$$

where $X \in \mathfrak{M}_{c^\beta}$.

Obviously, in view of the fact that the spaces c and c^β are isometric (by the above mentioned isometry J), on the basis of Theorem 3.1 we have the estimates

$$\chi(X) \leq \mu_{c^\beta}(X) \leq 2\chi(X)$$

for each $X \in \mathfrak{M}_{c^\beta}$, where χ denotes the Hausdorff measure of noncompactness in the space c^β.

Now, let us take into account the tempered sequence space l_∞^β. Then, keeping in mind formulas (3.14) - (3.16) expressing measures of noncompactness in the space l_∞, we obtain the following formulas for the counterparts of those measures in the space l_∞^β:

$$\mu_1^\beta(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_i)\in X} \big\{ \sup \big\{ \beta_i |x_i| : i \geq n \big\} \big\} \Big\}, \tag{4.3}$$

$$\mu_2^\beta(X) = \lim_{k\to\infty} \Big\{ \sup_{(x_i)\in X} \big\{ \sup\{|\beta_n x_n - \beta_m x_m| : n, m \geq k\} \big\} \Big\}, \tag{4.4}$$

$$\mu_3^\beta(X) = \limsup_{n\to\infty} \operatorname{diam} X_n^\beta, \tag{4.5}$$

where $X \in \mathfrak{M}_{l_\infty^\beta}$. Moreover, X_n^β in (4.5) is understood in the following way

$$X_n^\beta = \{x_n \beta_n : (x_i) \in X\}.$$

Apart from this $\operatorname{diam} X_n^\beta = \sup \big\{ \beta_n |x_n - y_n| : (x_i), (y_i) \in X \big\}$.

Further, taking into account Theorem 3.2 we deduce the inequalities

$$\chi(X) \le \mu_2^\beta(X), \tag{4.6}$$

$$\chi(X) \le \mu_3^\beta(X), \tag{4.7}$$

$$\mu_2^\beta(X) \le 2\mu_1^\beta(X), \tag{4.8}$$

$$\mu_3^\beta(X) \le 2\mu_1^\beta(X), \tag{4.9}$$

where $X \in \mathfrak{M}_{l_\infty^\beta}$ and the symbol χ denotes the Hausdorff measure of noncompactness in the space l_∞^β.

In view of inequalities (4.6)–(4.9) it is easily seen that the kernel $\ker \mu_1^\beta$ consists of all sets X belonging to the family $\mathfrak{M}_{l_\infty^\beta}$ such that the sequences $(\beta_n x_n)$ tend to zero at infinity uniformly with respect to the set X i.e., for any $\varepsilon > 0$ there exists a natural number n_0 such that $\beta_n|x_n| \le \varepsilon$ for all $(x_i) \in X$ and for $n \ge n_0$.

Similarly, the kernel $\ker \mu_2^\beta$ consists of all sets $X \in \mathfrak{M}_{l_\infty^\beta}$ such that the sequences $(\beta_n x_n)$ tend to finite limits uniformly on the set X. In other words, the sequences $(\beta_n x_n)$ satisfy Cauchy condition uniformly with respect to X.

Finally, the kernel $\ker \mu_3^\beta$ consists of all sets X belonging to the family $\mathfrak{M}_{l_\infty^\beta}$ such that the thickness of the bundle formed by sequences $(\beta_n x_n)$, where $(x_i) \in X$, tends to zero at infinity.

Let us also observe that the measures of noncompactness μ_1^β, μ_2^β, μ_3^β are not regular in the space l_∞^β.

5. Results from differential equations in Banach spaces

This section has an auxiliary character and contains a few results from the theory of ordinary differential equations in Banach spaces (cf. [5, 8, 10]). To present those results let us assume that E is a Banach space with a norm $\|\cdot\|$. Let x_0 be a fixed element of E i.e., $x_0 \in E$ and let $B(x_0, r)$ denotes a ball in E. We will consider the differential equation

$$x' = f(t, x) \tag{5.1}$$

with the initial condition

$$x(0) = x_0. \tag{5.2}$$

Here, we assume that $f = f(t, x)$ is a given function such that $f : [0, T] \times B(x_0, r) \to E$. We will write $I = [0, T]$. Throughout this section we will assume that μ is a measure of noncompactness in the space E.

Further, by the symbol E_μ we will denote the so - called *kernel set* of the measure of noncompactness μ [8] which is defined in the following way

$$E_\mu = \{x \in E : \{x\} \in \ker\ \mu\}.$$

It can be shown that E_μ is a closed, convex subset of the space E. Moreover, if μ is a sublinear measure then E_μ is a linear closed subspace of E. It is worthwhile mentioning that the concept of the kernel set plays an important role in the theory of differential equations in Banach spaces.

Now, we recall a result concerning initial value problem (5.1)–(5.2) which is not very general but is useful for our purposes (cf. [3]).

Theorem 5.1. *Suppose the function f is uniformly continuous on $I \times B(x_0, r)$ and $\|f(t,x)\| \le A$, where $AT \le r$. Further, let μ be a sublinear measure of noncompactness in E such that $\{x_0\} \in \ker \mu$. We assume that for any nonempty set $X \subset B(x_0, r)$ and for almost all $t \in I$ the following inequality holds*

$$\mu(f(t,X)) \le p(t)\mu(X), \tag{5.3}$$

where $p(t)$ is an integrable function on the interval I. Then (5.1)–(5.2) has at least one solution $x = x(t)$ on the interval I such that $x(t) \in E_\mu$ for $t \in I$.

The below given theorem is a slightly modified version of the result contained in Theorem 5.1, which will be more convenient in our further considerations (cf. [5, 8]).

Theorem 5.2. *Assume that f is a function defined on $[0,T] \times E$ with values in E such that*

$$\|f(t,x)\| \le P + Q\|x\| \tag{5.4}$$

for each $t \in [0,T]$ and $x \in E$, where P and Q are nonnegative constants. Further, assume that f is uniformly continuous on the set $[0,T_1] \times B(x_0, r)$, where $QT_1 < 1$ and $r = \frac{(P+Q)T_1\|x_0\|}{1-QT_1}$. Moreover, we assume that f satisfies condition (5.3) with a sublinear measure of noncompactness μ such that $x_0 \in E_\mu$. Then, initial value problem (5.1)–(5.2) has a solution $x = x(t)$ on the interval $[0,T_1]$ such that $x(t) \in E_\mu$ for $t \in [0,T_1]$.

Remark 5.3. Observe that in the case when $\mu = \chi$ (the Hausdorff measure of noncompactness), the assumption on the uniform continuity of the function f can be replaced by the weaker one requiring only the continuity [16]. The same assertion is also true if μ is a regular measure of noncompactness equivalent to the Hausdorff measure [12, 16].

6. Infinite systems of differential equations in the tempered sequence space c_o^β

The considerations of this section will be located in the Banach tempered sequence space c_o^β described in Section 4. Thus, we will assume that $\beta = (\beta_n)$ is a sequence with positive terms which is nonincreasing. The space c_o^β consists of all sequences (x_n) such that the sequence $(\beta_n x_n)$ converges to zero. We will consider here only real sequences (x_n). The norm in the space c_o^β is defined by the formula

$$\|x\|_{c_0^\beta} = \|(x_n)\|_{c_0^\beta} = \sup\{\beta_n |x_n| : n = 1, 2, \dots\}.$$

To simplify the notation we will use the symbol $\|\cdot\|$ instead of $\|\cdot\|_{c_0^\beta}$.

The object of our study in this section will be first semilinear lower diagonal infinite systems of differential equations having the form

$$x_n' = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x_1, x_2, \dots) \tag{6.1}$$

with the initial value conditions

$$x_n(0) = x_0^n, \quad \text{for } x = 1, 2, \dots . \tag{6.2}$$

We assume that for any fixed $n \in \mathbb{N}$ the sequence $(n_1, n_2, \dots, n_{k_n})$ is such that $1 \le n_1 < n_2 < \dots < n_{k_n} \le n$. Moreover, the sequence (n_1) tends to infinity when $n \to \infty$. Apart from that we assume that there exists a natural number K such that $k_n \le K$ for all $n = 1, 2, \dots$. In other words, this means that any "linear part" of system (6.1) contains only finite number of nonzero terms and the number of those terms does not exceed K. In what follows infinite systems (6.1) satisfying the above requirement will be called infinite systems of differential equations with *linear parts of constant width*.

Apart of the requirement concerning the constant width of linear parts we will impose the following assumptions in our study of initial value problem (6.1)–(6.2):

- (i) The function $a_{nn_i} = a_{nn_i}(t)$ is continuous on a fixed interval $I = [0, T]$ for $n = 1, 2, \dots$ and for $i = 1, 2, \dots, k_n$;
- (ii) the functions $a_{nn_i}(t)$ are uniformly bounded on the interval I by a positive constant A i.e., $|a_{nn_i}(t)| \le A$ for $t \in I$ and for $n = 1, 2 \dots$ and for $i = 1, 2, \dots, k_n$;
- (iii) the sequence (x_0^n) belongs to the space c_0^β;
- (iv) for each fixed n the function $f_n(t, x_1, x_2, \dots) = f_n(t, x)$ acts from the set $I \times \mathbb{R}^\infty$ into $\mathbb{R}$. Moreover, the function $f_n : I \times c_0^\beta \to \mathbb{R}$ is continuous on $I \times c_0^\beta$;
- (v) there exists a sequence (p_n) of nonnegative terms with the property that $\beta_n p_n \to 0$ as $n \to \infty$ and such that $|f_n(t, x)| \le p_n$ for $t \in I$, $x \in c_0^\beta$ and for $n = 1, 2 \dots$.

Now, we can formulate our existence result.

Theorem 6.1. *Assume that the functions involved in system* (6.1) *having linear parts of constant width K, satisfy conditions (i)–(v). Then initial value problem* (6.1)–(6.2) *has at least one solution $x(t) = (x_n(t)) = ((x_1(t), x_2(t), \dots)$ in the sequence space c_0^β on the interval I.*

Proof. For arbitrarily fixed $n \in \mathbb{N}$ let us denote

$$g_n(t, x) = g_n(t, x_1, x_2, \dots) = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x_1, x_2, \dots),$$

where $t \in I$ and $x = (x_n) \in c_0^\beta$. Then, keeping in mind our assumptions, we obtain

$$\begin{aligned}
\beta_n |g_n(t, x_1, x_2, \dots)| &\le \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t)||x_{n_i}| + \beta_n |f_n(t, x_1, x_2, \dots)| \\
&\le \beta_n A \sum_{i=1}^{k_n} |x_{n_i}| + \beta_n p_n = A \sum_{i=1}^{k_n} \beta_n |x_{n_i}| + \beta_n p_n \\
&\le A \sum_{i=1}^{k_n} \beta_{n_i} |x_{n_i}| + \beta_n p_n \\
&\le AK \max\{\beta_{n_i} |x_{n_i}| : i = 1, 2, \dots, k_n\} + \beta_n p_n \\
&\le AK \sup\{\beta_j |x_j| : j \ge n_1\} + \beta_n p_n.
\end{aligned}$$

Hence, replacing n by j and j by i, we can write the above inequality in the form

$$\beta_j |g_j(t, x_1, x_2, \dots)| \le AK \sup\{\beta_i |x_i| : i \ge j_1\} + \beta_j p_j. \tag{6.3}$$

Next, let us notice that from estimate (6.3) it follows that the following inequality holds

$$\begin{aligned} \|g(t,x)\| &= \sup\{\beta_j|g_j(t,x_1,x_2,\dots)| : j=1,2,\dots\} \\ &\le AK\sup_{j\in\mathbb{N}}\big\{\sup\{\beta_i|x_i| : i\ge j_1\}\big\} + \sup\{\beta_j p_j : j=1,2,\dots\} \\ &= AK\|x\| + P, \end{aligned} \tag{6.4}$$

where the operator $g=g(t,x)$ is defined on the set $I\times c_0^\beta$ in the following way

$$g(t,x)=(g_1(t,x),g_2(t,x),\dots).$$

In view of estimate (6.4) we see that g transforms the set $I\times c_0^\beta$ into the space c_0^β.

Now, we show that the operator g is continuous on the set $I\times c_0^\beta$. To this end we split the operator g into two terms

$$g(t,x)=(Lx)(t)+f(t,x),$$

where the operators L and f are defined as follows:

$$(Lx)(t)=((L_1x)(t),(L_2x)(t),\dots)$$

where

$$(L_nx)(t)=\sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i}$$

$(n=1,2,\dots)$, and

$$f(t,x)=(f(t_1,x),f(t_2,x),\dots).$$

First we show that the operator f is continuous on the set $I\times c_0^\beta$. To do this fix arbitrarily a number $\varepsilon>0$ and a point $x\in c_0^\beta$. According to assumption (v) we can choose a natural number n_0 such that

$$\beta_n p_n \le \frac{\varepsilon}{2} \tag{6.5}$$

for $n\ge n_0$. Next, in view of assumption (iv) we can find a number δ_i ($i=1,2,\dots,n_0$) such that for any $y\in c_0^\beta$ such that $\|x-y\|\le\delta_i$ and for arbitrary $t\in I$ we have

$$|f_i(t,x)-f_i(t,y)|\le\frac{\varepsilon}{\beta_1}.$$

Let us take $\delta=\min\{\delta_1,\delta_2,\dots,\delta_{n_0}\}$. Then, for arbitrary $y\in c_0^\beta$ such that $\|x-y\|\le\delta$ and for $t\in I$ we have

$$|f_i(t,x)-f_i(t,y)|\le\frac{\varepsilon}{\beta_1} \tag{6.6}$$

Combining (6.5) and (6.6), for $y\in c_0^\beta$ with $\|x-y\|\le\delta$ and for $t\in I$, we obtain

$$\begin{aligned} \|f(t,x)-f(t,y)\| &= \sup\{\beta_n|f_n(t,x)-f_n(t,y)| : n=1,2,\dots\} \\ &= \max\Big\{\max\big\{\beta_n|f_n(t,x)-f_n(t,y)| : n=1,2,\dots,n_0\big\}, \\ &\qquad \sup\big\{\beta_n|f_n(t,x)-f_n(t,y)| : n>n_0\big\}\Big\} \\ &\le \max\Big\{\max\big\{\beta_1|f_n(t,x)-f_n(t,y)| : n=1,2,\dots,n_0\big\}, \\ &\qquad \sup\{\beta_n\big[|f_n(t,x)|+|f_n(t,y)|\big] : n>n_0\}\Big\} \\ &\le \max\Big\{\beta_1\Big(\frac{\varepsilon}{\beta_1}\Big), \sup\big\{2\beta_n p_n : n>n_0\big\}\Big\} = \varepsilon. \end{aligned}$$

This shows that the operator f is continuous at an arbitrary point $(t,x) \in I \times c_0^\beta$.

Next, we show that the operator L is continuous on the set $I \times c_0^\beta$. Similarly as before, fix arbitrarily $x \in c_0^\beta$, $t \in I$ and a number $\varepsilon > 0$. Then, for $y \in c_0^\beta$ with $\|x-y\| \le \varepsilon$ and for an arbitrary fixed natural number n, in view of imposed assumptions we obtain

$$\begin{aligned}
&\beta_n|(L_n x)(t) - (L_n y)(t))| \\
&= \beta_n\Big|\sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(t)y_{n_i}\Big| \\
&\le \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t)||x_{n_i} - y_{n_i}| \\
&\le A\sum_{i=1}^{k_n} \beta_n|x_{n_i} - y_{n_i}| \le A\sum_{i=1}^{k_n} \beta_i|x_{n_i} - y_{n_i}| \\
&\le AK\max\{\beta_i|x_{n_i} - y_{n_i}| : i = 1,2,\dots,k_n\} \\
&\le AK\sup\{\beta_j|x_j - y_j| : j \ge n_1\} \\
&\le AK\sup\{\beta_j|x_j - y_j| : j = 1,2,\dots\} = AK\|x-y\| \le AK\varepsilon.
\end{aligned}$$

Hence we deduce that the operator L is continuous on the set $I \times c_0^\beta$. Consequently, as we announced before, we conclude that the operator g is continuous on the set $I \times c_0^\beta$.

In what follows let us take a number T_1 such that $T_1 < T$ and $AKT_1 < 1$. According to assumptions of our theorem take the number $r = \frac{(P+AK)T_1\|x_0\|}{1-AKT_1}$ and consider the ball $B(x_0,r)$. Next, choose an arbitrary subset X of the ball $B(x_0,r)$. Then, for $x \in X$ and $t \in [0,T_1]$, in view of estimate (6.3), for an arbitrary fixed natural number n, we obtain:

$$\begin{aligned}
&\sup\{\beta_j|g_j(t,x_1,x_2,\dots)| : j \ge n\} \\
&\le \sup\big\{AK\sup\{\beta_i|x_i| : i \ge j_1\} : j \ge n\big\} + \sup\big\{\beta_j p_j : j \ge n\big\} \\
&\le AK\sup\big\{\sup\{\beta_i|x_i| : i \ge n_1\}, \sup\{\beta_i|x_i| : i \ge (n+1)_1\}, \\
&\quad \sup\{\beta_i|x_i| : i \ge (n+2)_1\},\dots\big\} + \sup\big\{\beta_j p_j : j \ge n\big\}.
\end{aligned}$$

This yields the estimate

$$\begin{aligned}
&\sup_{x\in X}\big\{\sup\{\beta_j|g_j(t,x_1,x_2,\dots)| : j \ge n\}\big\} \\
&\le AK\sup_{x\in X}\big\{\sup\{\sup\{\beta_i|x_i| : i \ge j_i\} : j \ge n\} + \sup\{\beta_j p_j : j \ge n\}\big\}.
\end{aligned}$$

Passing with $n \to \infty$ in the above estimate and taking into account that $j_1 \to \infty$ as $j \to \infty$, we obtain

$$\chi(g(t,X)) \le AK\chi(X),$$

where χ denotes the Hausdorff measure of noncompactness in the space c_0^β expressed with help of formula (4.1). Finally, in view of the above established facts and Theorem 5.2, we complete the proof. $\square$

The following example illustrates the result in Theorem 6.1.

Example 6.2. Consider the infinite system of differential equations

$$\begin{gathered}
x_1' = x_1 + \frac{\sqrt{|x_1|}}{\sqrt{|x_1|}+1},\\
x_2' = x_1 + x_2 + 2\frac{\sqrt{|x_2|}}{\sqrt{|x_2|}+1},\\
x_3' = x_2 + x_3 + 3\frac{\sqrt{|x_3|}}{\sqrt{|x_3|}+1},\\
\dots\\
x_n' = x_{n-1} + x_n + n\frac{\sqrt{|x_n|}}{\sqrt{|x_n|}+1},\\
\dots.
\end{gathered} \tag{6.7}$$

with initial conditions

$$x_n(0) = n \quad \text{for } n = 1, 2, \dots . \tag{6.8}$$

Observe that (6.7) is a semilinear lower diagonal infinite system of differential equations with linear parts of constant width $K = 2$. Moreover, it is easily seen that system (6.7) is a particular case of system (6.2) if we take $a_{nn_i}(t) = 1$ for $t \in I$, where we put $I = [0, T_1]$, where $T_1 > 0$ is a number chosen according to assumptions of Theorem 5.2. Additionally, $n = 1, 2 \dots$ and $i = 1, 2$ for $n \geq 2$. Hence we see that there is satisfied assumption (i) of Theorem 6.1.

Further, we have that $|a_{nn_i}(t)| \leq 1$ for $t \in I$ and $n = 1, 2, \dots$, $i = 1, 2$. This means that functions $a_{nn_i}(t)$ satisfy assumption (ii).

In what follows let us take the sequence $\beta_n = \frac{1}{n^2}$ for $n = 1, 2 \dots$. Obviously we have that $x_0 = (x_0^n) = (n) \in c_0^\beta$, where $\beta = (\beta_n) = (\frac{1}{n^2})$. Thus there is satisfied assumption (iii). From the form of system (6.7) we see that we can take

$$f_n(t, x_1, x_2, \dots) = n\frac{\sqrt{|x_n|}}{\sqrt{|x_n|}+1}$$

for $n = 1, 2 \dots$. Obviously, the function $f_n = f_n(t, x)$ is continuous on the set $I \times c_0^\beta$. Moreover, we have

$$|f_n(t, x_1, x_2, \dots)| \leq n, \quad \text{for } n = 1, 2 \dots .$$

Thus we conclude that the functions f_n satisfy assumptions (iv) and (v) with $p_n = n$ for $n = 1, 2 \dots$.

Finally, on the basis of Theorem 6.1 we deduce that there exists at least one solution $x(t) = (x_n(t))$ of initial value problem (6.7)–(6.8) defined on some interval $I = [0, T_1]$ such that for each $t \in I$ the sequence $(x_n(t))$ belongs to the space c_0^β with $\beta = (\frac{1}{n^2})$. This means that $x_n(t) = o(n^2)$ as $n \to \infty$, for any fixed $t \in [0, T_1]$.

In the sequel we will also consider the semilinear lower diagonal infinite system of differential equations of the form (6.1) i.e.,

$$x_n' = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x_1, x_2, \dots) \tag{6.9}$$

with initial value conditions

$$x_n(0) = x_n^0, \quad \text{for } n = 1, 2, \dots . \tag{6.10}$$

Now, we dispense with the assumption requiring that system (6.9) has linear parts of constant width. We replace this assumption, as well as assumption (ii), by the following hypotheses:

(ii') The sequence (n_1) tends to ∞ as $n \to \infty$;

(ii") the sequence $\left(\sum_{i=1}^{k_n} |a_{nn_i}(t)|\right)$ is uniformly bounded on the interval $I = [0, T_1]$ i.e., there exists a constant $A > 0$ such that

$$\sum_{i=1}^{k_n} |a_{nn_i}(t)| \leq A$$

for each $t \in I$ and for $n = 1, 2, \dots$.

Then we have the following result.

Theorem 6.3. *Assume that* (i), (ii'), (ii"), (iii)–(v) *of Theorem 6.1 are satisfied. Then initial value problem* (6.9)–(6.10) *has at least one solution $x(t) = (x_n(t))$ in the sequence space c_0^β defined on the interval $I = [0, T_1]$, where T_1 is a number chosen according to Theorem 5.2.*

Proof. Similarly, as in the proof of Theorem 6.1, for a fixed $n \in \mathbb{N}$ let us denote

$$g_n(t, x) = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i} + f_n(t, x),$$

$$(L_n x)(t) = \sum_{i=1}^{k_n} a_{nn_i}(t) x_{n_i},$$

where $t \in I$ and $x = (x_n) = (x_1, x_2, \dots) \in c_0^\beta$. Next, let us put

$$\begin{gathered} g(t, x) = (g_1(t, x), g_2(t, x), \dots), \\ (Lx)(t) = ((L_1 x)(t), (L_2 x)(t), \dots), \\ f(t, x) = (f_1(t, x), f_2(t, x), \dots). \end{gathered}$$

Now, in view of our assumptions, we obtain:

$$\begin{aligned} \beta_n |g_n(t, x_1, x_2, \dots)| &\leq \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t)| |x_{n_i}| + \beta_n |f_n(t, x_1, x_2, \dots)| \\ &\leq \sum_{i=1}^{k_n} |a_{nn_i}(t)| \beta_{n_i} |x_{n_i}| + \beta_n p_n \\ &\leq \sum_{i=1}^{k_n} |a_{nn_i}| \max\left\{\beta_{n_i} |x_{n_i}| : i = 1, 2, \dots, k_n\right\} + \beta_n p_n \\ &\leq A \sup\{\beta_j |x_j| : j \geq n_1\} + \beta_n p_n. \end{aligned} \tag{6.11}$$

Further, from the above estimate we obtain

$$\|g(t, x)\| = \sup\{\beta_n |g_n(t, x_1, x_2, \dots)|\} \leq A\|x\| + P, \tag{6.12}$$

where $P = \sup\{\beta_n p_n : n = 1, 2, \dots\}$. Obviously $P < \infty$ since $\beta_n p_n \to 0$ as $n \to \infty$. Next, in virtue of estimate (6.12) we have that the operator g transforms the set $I \times c_0^\beta$ into c_0^β.

In what follows observe that because of a suitable part of the proof of Theorem 6.1, we conclude that f is continuous on the set $I\times c_0^\beta$. Thus, to show the continuity of the operator g on the set $I \times c_0^\beta$ it is sufficient to show that the operator L is continuous on this set. To this end fix arbitrarily $x \in c_0^\beta, t \in I$ and a number $\varepsilon > 0$. Then, for $y \in c_0^\beta$ with $\|x-y\| \le \varepsilon$ and for a fixed $n \in \mathbb{N}$, we obtain:

$$\begin{aligned}
\beta_n|(L_nx)(t)-(L_ny)(t)| &= \beta_n\Big|\sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(t)y_{n_i}\Big| \\
&\le \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t)||x_{n_i}-y_{n_i}| \le \sum_{i=1}^{k_n} |a_{nn_i}(t)|\beta_{n_i}|x_{n_i}-y_{n_i}| \\
&\le \sum_{i=1}^{k_n} |a_{nn_i}(t)| \sup\{\beta_{n_i}|x_{n_i}-y_{n_i}| : i=1,2,\dots,k_n\} \\
&\le A\sup\{\beta_j|x_j-y_j| : j=1,2,\dots\} = A\|x-y\| \le A\varepsilon.
\end{aligned}$$

Hence we obtain that $\|(Lx)(t)-(Ly)(t)\| \le A\varepsilon$ which means that the operator L is continuous on the set $I\times c_0^\beta$. Consequently we obtain the continuity of the operator g on $I \times c_0^\beta$.

Now, let us choose a number T_1, $T_1 < T$ such that $AT_1 < 1$. Next, take the number $r = (P+A)T_1\|x_o\|/(1-AT_1)$ and assume that X is a nonempty subset of the ball $B(x_0, r)$. Then, arguing similarly as in the proof of Theorem 6.1 and utilizing estimate (6.11) we obtain

$$\chi(g(t,X)) \le A\chi(X),$$

where χ is the Hausdorff measure of noncompactness in the space c_0^β described by formula (4.1). Hence, applying Theorem 5.2 we complete the proof. $\square$

Next we provide an example showing the applicability of Theorem 6.3.

Example 6.4. We consider the lower diagonal infinite system of differential equations. To expose this system in a transparent way we will assume that n is an even natural number, say $n = 2k$. Then, we can present the announced system as follows:

$$\begin{gathered}
x_1' = x_1 + \frac{x_1}{1+x_1^2}, \\
x_2' = x_1 + tx_2 + 2\frac{x_1+x_2}{1+x_1^2+x_2^2}, \\
x_3' = \frac{t^2}{2!}x_3 + 3\frac{x_2+x_3}{1+x_2^2+x_3^2}, \\
x_4' = \frac{t^2}{2!}x_3 + \frac{t^3}{3!}x_4 + 4\frac{x_3+x_4}{1+x_3^2+x_4^2}, \\
\dots \qquad (6.13) \\
x_{n-1}'(=x_{2k-1}') = \frac{t^k}{k!}x_{k+1} + \dots + \frac{t^{2k-2}}{(2k-2)!}x_{2k-1} + (2k-1)\frac{x_{2k-2}+x_{2k-1}}{1+x_{2k-2}^2+x_{2k-1}^2}, \\
x_n'(=x_{2k}') = \frac{t^k}{k!}x_{k+1} + \dots + \frac{t^{2k-2}}{(2k-2)!}x_{2k-1} + \frac{t^{2k-1}}{(2k-1)!}x_{2k} + 2k\frac{x_{2k-1}+x_{2k}}{1+x_{2k-1}^2+x_{2k}^2},
\end{gathered}$$

$$\dots .$$

We also assume that the following initial conditions are satisfied

$$x_n(0) = n^2 \tag{6.14}$$

for $n = 1, 2, \dots$.

Let us observe that initial value problem (6.13)–(6.14) is a particular case of problem (6.9)–(6.10). To justify this assertion we show that the components involved in (6.13)–(6.14) satisfy assumptions of Theorem 6.3. First of all let us observe that functions $a_{nn_i}(t)$ appearing in infinite system (6.13) have the form

$$a_{nn_i}(t) = \frac{t^{n_i - 1}}{(n_i - 1)!}$$

for $n_i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n$ (if n is even) or $n_i = [\frac{n}{2}] + 2, [\frac{n}{2}] + 3, \dots, n$ (if n is odd). Obviously the functions $a_{nn_i}(t)$ are continuous on each interval of the form $[0, T]$. Thus, there is satisfied assumption (i).

Since $n_1 = \frac{n}{2} + 1$ for n even or $n_1 = [\frac{n}{2}] + 2$ for n odd, we see that assumption (ii') is satisfied. To check assumption (ii") observe that we have

$$\sum_{i=1}^{k_n} |a_{nn_i}(t)| = \sum_{i=1}^{n} |a_{nn_i}(t)| \le 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!} \le e^t$$

for $t \in [0, T]$. Hence we have that assumption (ii") is satisfied with $A = e^T$.

Further, take the tempering sequence of the form $\beta = (\beta_n) = (\frac{1}{n^3})$. Then the sequence $(x_0^n) = (n^2)$ is a member of the tempered sequence space c_0^β , so assumption (iii) is satisfied. Similarly, it is not hard to verify that the functions f_n, where

$$f_n(t, x) = f_n(t, x_1, x_2, \dots) = n \frac{x_{n-1} + x_n}{1 + x_{n-1}^2 + x_n^2}$$

$(n = 2, 3 \dots)$ are continuous on the set $I \times c_0^\beta$. Moreover, for each fixed n we obtain

$$|f_n(t, x)| \le n \frac{|x_{n-1}| + |x_n|}{1 + x_{n-1}^2 + x_n^2} \le n.$$

Thus, we can put $p_n = n$ in assumption (v). Obviously we have that $\beta_n p_n = \frac{1}{n^2} \to 0$ as $n \to \infty$. Thus assumption (v) is satisfied.

Hence, in view of Theorem 6.3, initial value problem (6.13)–(6.14) has at least one solution $x(t) = (x_n(t))$ belonging to the sequence space c_0^β and defined for $t \in I = [0, T_1]$, where T_1 satisfies the inequality $T_1 A = T_1 e^{T_1} < 1$. We can calculate that $T_1 \le 0.568 \dots$.

Remark 6.5. Observe that in Example 6.4 instead of $\beta = (\beta_n) = (1/n^3)$ we can take the tempering sequence of the form $\beta_n = 1/n^{2+\delta}$, where δ is an arbitrary positive number. Similarly, in Example 6.2 we can take the tempering sequence of the form $\beta_n = 1/n^{1+\delta}$, where $\delta > 0$ is an arbitrary number and $n = 1, 2 \dots$.

7. Infinite perturbed diagonal systems

In this section we study the existence of solutions of a perturbed diagonal infinite system of differential equations in the sequence space c^β. Consider the infinite perturbed diagonal systems of differential equations of the form

$$x_n' = a_n(t) x_n + g_n(t, x_1, x_2, \dots) \tag{7.1}$$

with the initial conditions

$$x_n(0) = x_n^0, \tag{7.2}$$

for $n = 1, 2, \dots$ and $t \in I = [0, T]$. Problem (7.1)–(7.2) will be investigated in the sequence space c^β, where $\beta = (\beta_n)$ is a tempering sequence i.e., the sequence (β_n) is nonincreasing and has positive terms.

Infinite systems of differential equations (7.1)–(7.2) contain, as particular cases, the systems considered in the theory of neural sets (cf. [10, pp. 86-87], and [18]). Let us also mention that system (7.1)–(7.2) was studied in [5]. The existence result concerning initial value problem (7.1)–(7.2) which we are going to present here, will generalize essentially results obtained in the above quoted papers [5, 18] and the monograph [10]. In our considerations we will utilize the measure of noncompactness μ_2^β in the space c^β defined by formula (4.2).

Initial value problem (7.1)–(7.2) will be studied under the following assumptions.

- (i) $x_0 = (x_n^0) \in c^\beta$;
- (ii) the mapping $g = (g_1, g_2, \dots)$ acts from the set $I \times c^\beta$ into c^β and is continuous on $I \times c^\beta$;
- (iii) There exists a sequence (p_n) with $\beta_n p_n \to 0$ as $n \to \infty$ such that
$$|g_n(t, x_1, x_2, \dots)| \le p_n$$
for $t \in I$, $x = (x_n) \in c^\beta$ and for $n = 1, 2, \dots$.
- (iv) The functions $a_n(t)$ are continuous on I and the sequence $(a_n(t))$ converges uniformly on I (to a function $a = a(t)$).

Notice that in view of the imposed assumptions the sequence $(a_n(t))$ is equibounded on I. This implies that the constant

$$A = \sup\{a_n(t) : t \in I, n = 1, 2, \dots\}$$

is finite.

Now, we can formulate our result.

Theorem 7.1. *Let assumptions (i)–(iv) be satisfied. If $AT < 1$ then initial value problem* (7.1)–(7.2) *has a solution $x(t) = (x_n(t))$ on the interval I such that $x(t) \in c^\beta$ for each $t \in I$.*

Proof. At the beginning, for $t \in I$ and $x = (x_n) \in c^\beta$ let us denote

$$f_n(t, x) = a_n(t)x_n + g_n(t, x), f(t, x) = (f_1(t, x), f_2(t, x), \dots),$$

where n is an arbitrarily fixed natural number. Further, fix arbitrary natural numbers m, n. Without loss of generality we can assume that $m < n$. Then, we obtain

$$\begin{aligned}
&|\beta_n f_n(t, x) - \beta_m f_m(t, x)| \\
&\le |\beta_n a_n(t)x_n - \beta_m a_m(t)x_m| + |\beta_n g_n(t, x) - \beta_m g_m(t, x)| \\
&\le |\beta_n a_n(t)x_n - \beta_m a_n(t)x_m| + |\beta_m a_n(t)x_m - \beta_m a_m(t)x_m| \\
&\quad + \beta_n|g_n(t, x)| + \beta_m|g_m(t, x)| \\
&\le |a_n(t)||\beta_n x_n - \beta_m x_m| + \beta_m|x_m||a_n(t) - a_m(t)| + \beta_n p_n + \beta_m p_m.
\end{aligned} \tag{7.3}$$

In view of the imposed assumptions we deduce that $(\beta_k x_k)$ is a Cauchy sequence. The same statement is also valid for the function sequence $(a_k(t))$.

Moreover, we have that $\beta_n p_n \to 0$ as $n \to \infty$. Taking into account the above established facts, from estimate (7.3) we deduce that $(\beta_n f_n(t, x))$ is a Cauchy sequence. This yields that $(f_n(t, x)) \subset c^\beta$.

Next, observe that for arbitrary $n \in \mathbb{N}, t \in I$ and for a fixed $x \in c^\beta$, we have

$$|\beta_n f_n(t,x)| \le |\beta_n a_n(t) x_n| + |\beta_n g_n(t,x)| \le |a_n(t)|\beta_n |x_n| + \beta_n p_n \le A\|x\| + P, \quad (7.4)$$

where $P = \sup\{\beta_n p_n : n = 1, 2 \dots\}$ and the symbol $\|\cdot\|$ denotes the norm in the space c^β (cf. Section 4). Obviously $P < \infty$. From estimate (7.4) we deduce the following one

$$\|f(t,x)\| \le A\|x\| + P. \quad (7.5)$$

Now, we consider the mapping $f(t,x)$ on the set $I \times B(x_0, r)$, where r is taken according to Theorem 5.2, i.e.,

$$r = \frac{(A+P)T\|x_0\|}{1 - AT}.$$

To prove the continuity of the mapping $f(t,x)$ let us fix arbitrarily $t \in I$ and $x \in B(x_0, r)$. Next, choose arbitrary $s \in I$ and $y \in B(x_0, r)$. Then, in view of the imposed assumptions, we obtain

$$\begin{aligned}
&\|f(t,x) - f(s,y)\| \\
&= \sup\Big\{|\beta_n f_n(t,x) - \beta_n f_n(s,y)| : n = 1, 2, \dots\Big\} \\
&\le \sup\Big\{\beta_n |a_n(t)x_n - a_n(s)y_n| : n = 1, 2, \dots\Big\} \\
&\quad + \sup\Big\{\beta_n |g_n(t,x) - g_n(s,y)| : n = 1, 2, \dots\Big\} \\
&\le \sup\Big\{\beta_n \Big[|a_n(t)x_n - a_n(s)x_n| + |a_n(s)x_n - a_n(s)y_n|\Big] : n = 1, 2, \dots\Big\} \\
&\quad + \sup\Big\{\beta_n |g_n(t,x) - g_n(s,y)| : n = 1, 2, \dots\Big\} \\
&\le \sup\Big\{\beta_n |x_n||a_n(t) - a_n(s)| : n = 1, 2, \dots\Big\} \\
&\quad + \sup\Big\{|a_n(s)|\beta_n |x_n - y_n| : n = 1, 2, \dots\Big\} \\
&\quad + \sup\Big\{\beta_n |g_n(t,x) - g_n(s,y)| : n = 1, 2, \dots\Big\} \\
&\le (\|x_0\| + r) \sup\Big\{|a_n(t) - a_n(s)| : n = 1, 2, \dots\Big\} \\
&\quad + A\|x - y\| + \|g(t,x) - g(s,y)\|.
\end{aligned}$$

Hence, keeping in mind the fact that the sequence $(a_n(t))$ is equicontinuous on the interval I and the mapping g is continuous at the point (t,x) we conclude that the mapping f is continuous at (t,x). In view of the arbitrariness of t and x this yields that f is continuous on the set $I \times B(x_0, r)$.

Now, let us take a nonempty subset X of the ball $B(x_0, r)$. Fix $t \in I$ and $x = (x_n) \in X$. Then, in view of (7.3), for arbitrarily fixed natural numbers m, n we obtain

$$\begin{aligned}
&|\beta_n f_n(t,x) - \beta_m f_m(t,x)| \\
&\le |a_n(t)||\beta_n x_n - \beta_m x_m| + \|x\||a_n(t) - a_m(t)| + \beta_n p_n + \beta_m p_m.
\end{aligned}$$

Hence, taking into account the imposed assumptions, we derive the estimate

$$\mu_2^\beta(f(t,X)) \le a(t)\mu_2^\beta(X), \quad (7.6)$$

where (as we mentioned above) μ_2^β is the measure of noncompactness in the space c^β defined by formula (4.2). Finally, linking estimates (7.5) and (7.6), in view of

Theorem 5.2 we conclude that problem (7.1)–(7.2) has at least one solution in the space c^β. The proof is complete. $\square$

Now we given an example illustrating our considerations.

Example 7.2. Consider the perturbed diagonal infinite system of differential equations

$$x_n' = \big(n\sin\frac{t}{n}\big)x_n + \arctan(x_n + x_{n+1}) \tag{7.7}$$

with the initial conditions of the form

$$x_n(0) = n+1 \tag{7.8}$$

for $n = 1,2,\dots$ and for $t \in I = [0,T]$, where T is a fixed positive number such that $T \le \frac{\pi}{2}$. The value of T will be estimated precisely later.

Observe that initial value problem (7.7)–(7.8) is a special case of problem (7.1) - (7.2) if we put $a_n(t) = n\sin\frac{t}{n}$, $g_n(t,x_1,x_2,\dots) = \arctan(x_n + x_{n+1})$ and if we accept the tempering sequence $\beta = (\beta_n) = (\frac{1}{n})$. We show briefly that in such a case infinite system (7.7) with initial conditions (7.8) satisfies assumptions of Theorem 7.1. To this end observe that the function sequence $(a_n(t))$ consists of functions continuous on the interval I and it is uniformly convergent on I to the function $a(t) = t$, $t \in I$. Thus the sequence $(a_n(t))$ satisfies assumption (iv).

Further, we have

$$|g_n(t,x_1,x_2,\dots)| = |\arctan(x_n + x_{n+1})| \le \frac{\pi}{2}$$

for $n = 1,2\dots$. Thus, taking $p_n = \frac{\pi}{2}$ we see that assumption (iii) is satisfied. Similarly we verify assumption (i).

To check assumption (ii) let us fix arbitrarily $x, y \in c^\beta$, $x = (x_k)$, $y = (y_k)$. Then, for a fixed $n \in \mathbb{N}$ we obtain:

$$\begin{aligned}
&\beta_n|g_n(t,x_1,x_2,\dots) - g_n(t,y_1,y_2,\dots)| \\
&= \frac{1}{n}|\arctan(x_n + x_{n+1}) - \arctan(y_n + y_{n+1})| \\
&\le \frac{1}{n}|x_n + x_{n+1} - y_n - y_{n+1}| \le \frac{1}{n}|x_n - y_n| + \frac{1}{n}|x_{n+1} - y_{n+1}| \\
&\le \frac{n+1}{n}\Big(\frac{1}{n+1}|x_n - y_n| + \frac{1}{n+1}|x_{n+1} - y_{n+1}|\Big) \\
&\le 2\Big(\frac{1}{n}|x_n - y_n| + \frac{1}{n+1}|x_{n+1} - y_{n+1}|\Big).
\end{aligned} \tag{7.9}$$

Next, in view of (7.9), for arbitrarily fixed $t, s \in I$ and $x, y \in c^\beta$, we obtain

$$\begin{aligned}
&\|g(t,x) - g(s,y)\| \\
&= \sup\Big\{\frac{1}{n}|g_n(t,x) - g_n(s,y)| : n = 1,2,\dots\Big\} \\
&\le \sup\Big\{2\Big(\frac{1}{n}|x_n - y_n| + \frac{1}{n+1}|x_{n+1} - y_{n+1}|\Big) : n = 1,2\dots\Big\} \\
&\le 2\sup\Big\{\frac{1}{n}|x_n - y_n| : n = 1,2\dots\Big\} + 2\sup\Big\{\frac{1}{n+1}|x_{n+1} - y_{n+1}| : n = 1,2\dots\Big\} \\
&\le 4\|x - y\|,
\end{aligned}$$

where the symbol $\|\cdot\|$ denotes the norm in the space c^β. Thus we showed that the mapping g is continuous on the set $I \times c^\beta$ (even Lipschitz continuous). This means that the mapping g satisfies assumption (ii).

Finally, let us observe that using standard methods of mathematical analysis, we obtain

$$\begin{aligned} A &= \sup\left\{|a_n(t)| : t \in [0,T], n = 1,2,\dots\right\} \\ &= \sup\left\{n \sin\frac{t}{n} : t \in [0,T], n = 1,2,\dots\right\} \le \frac{\pi}{2}. \end{aligned}$$

Thus, if we take $T < \frac{2}{\pi}$, then applying Theorem 7.1 we deduce that initial value problem (7.7)–(7.8) has at least one solution $x(t) = (x_n(t))$ such that $(x_n(t)) \in c^\beta$ for any $t \in [0,T]$.

8. Infinite systems of differential equations in the sequence space l_∞^β

In this section we will work in the space l_∞^β described in details in Section 4. We will assume here that the tempering sequence $\beta = (\beta_n)$ consists of positive terms and is nonincreasing. We will utilize the measure of noncompactness μ_3^β defined on the family $\mathfrak{M}_{l_\infty^\beta}$ by formula (4.5). For simplicity, that measure will be denoted by μ. Recall, that for $X \in \mathfrak{M}_{l_\infty^\beta}$ we put

$$\mu(X) = \limsup_{n\to\infty} \operatorname{diam} X_n^\beta,$$

where $X_n^\beta = \{\beta_n x_n : x = (x_i) \in X\}$. Equivalently, this formula can be written in a more convenient way

$$\mu(X) = \limsup_{n\to\infty} \operatorname{diam} \beta_n X_n, \tag{8.1}$$

where $X_n = \{x_n : x = (x_i) \in X\}$. We refer to Section 4 for the properties of the measure μ.

In what follows we will investigate the following perturbed semilinear lower diagonal infinite system of differential equations

$$x_n' = \sum_{j=k_n}^{n} a_{nj}(t)x_j + g_n(t, x_1, x_2, \dots) \tag{8.2}$$

with the initial conditions

$$x_n(0) = x_n^0 \tag{8.3}$$

for $n = 1, 2\dots$ and $t \in I = [0,T]$.
Throughout this section we will assume that the sequence (k_n) appearing in (8.2) is such that $1 \le k_n \le n$ for $n = 1,2,\dots$ and $k_n \to \infty$ as $n \to \infty$.

It is worthwhile mentioning that infinite systems of differential equations having form (8.2) were up to now considered very seldom (cf. [4, 8]).

For further purposes we denote by $f = f(t,x)$ the mapping defined on the set $I \times l_\infty^\beta$ in the following way

$$f(t,x) = (f_1(t,x), f_2(t,x), \dots),$$

where

$$f_n(t,x) = f_n(t, x_1, x_2, \dots) = \sum_{j=k_n}^{n} a_{nj}(t)x_j + g_n(t, x_1, x_2, \dots)$$

for $n = 1, 2, \dots$. Moreover, we will also define the mapping $g(t,x)$ on the set $I \times l_\infty^\beta$ by putting

$$g(t,x) = (g_1(t,x), g_2(t,x), \dots).$$

Now, we formulate assumptions under which problem (8.2)–(8.3) will be studied. Namely, we will impose the following hypotheses.

(i) $x_0 = (x_n^0) \in l_\infty^\beta$;
(ii) the mapping g acts from the set $I \times l_\infty^\beta$ into l_∞^β and is uniformly continuous on $I \times l_\infty^\beta$;
(iii) there exists a sequence (p_n) with $\beta_n p_n \to 0$ as $n \to \infty$ and such that

$$|g_n(t, x_1, x_2, \dots)| \le p_n$$

for $t \in I, x = (x_n) \in l_\infty^\beta$ and $n = 1, 2 \dots$;
(iv) The functions $a_{nj} : I \to \mathbb{R}$ ($j = k_n, k_n + 1, \dots, n$, $n = 1, 2 \dots$) are continuous and nondecreasing on I. Moreover, we assume that the function sequence $(A_n(t))$ is equicontinuous on the interval I and the sequence $(\bar{A}_n(t))$ is uniformly bounded on I, where

$$A_n(t) = \sum_{j=k_n}^{n} a_{nj}(t), \quad \bar{A}_n(t) = \sum_{j=k_n}^{n} |a_{nj}(t)|$$

for $n = 1, 2, \dots$.

Keeping in mind assumption (iv), for further purposes we can define the constant

$$A = \sup\{\bar{A}_n(t) : t \in I, n = 1, 2, \dots\}.$$

In view of assumptions (iv) we have that $A < \infty$.

Now, we can formulate that following result concerning initial value problem (8.2)–(8.3).

Theorem 8.1. *Assume that conditions* (i)–(iv) *are satisfied and $AT < 1$. Then initial value problem* (8.2)–(8.3) *has at least one solution $x(t) = (x_k(t))$ on the interval $I = [0,T]$ such that $x(t) \in l_\infty^\beta$ for $t \in I$.*

Proof. Let us take an arbitrary element $x = (x_k) \in l_\infty^\beta$. Next, fix $t \in I$ and $n \in \mathbb{N}$. Then, in view of the imposed assumptions we obtain

$$\begin{aligned}
|\beta_n f_n(t,x)| &\le \beta_n \sum_{j=k_n}^{n} |a_{nj}(t)||x_j| + \beta_n |g_n(t,x)| \\
&\le \sum_{j=k_n}^{n} |a_{nj}(t)| \beta_j |x_j| + \beta_n p_n \\
&\le \sum_{j=k_n}^{n} |a_{nj}(t)| \max\{\beta_j |x_j| : j = k_n, k_n+1, \dots, n\} + \beta_n p_n \\
&\le \bar{A}_n(t) \sup\{\beta_j |x_j| : j = 1, 2, \dots\} + \beta_n p_n \le A\|x\| + P,
\end{aligned} \tag{8.4}$$

where $\|\cdot\|$ stands for the norm in the space l_∞^β and $P = \sup\{\beta_n p_n : n = 1, 2 \dots\}$. Obviously $P < \infty$ in view of assumption (iii). Hence we infer that the mapping f transforms the set $I \times l_\infty^\beta$ into l_∞^β.

Next we consider the mapping $f(t,x)$ on the set $I \times B(x_0, r)$, where r is taken according to Theorem 5.2 i.e., $r = \frac{(A+P)T\|x_0\|}{1-AT}$. First, we show that f is uniformly continuous on the set $I\times B(x_0,r)$. In view of assumption (ii) it is sufficient to show that the linear operator L defined by

$$(Lx)(t)=((L_1x)(t),(L_2x)(t),\dots),$$

where

$$(L_nx)(t)=\sum_{j=k_n}^{n}a_{nj}(t)x_j$$

for $n=1,2,\dots$, is continuous on the set $I\times l_\infty^\beta$. To this end fix arbitrarily $x,y\in l_\infty^\beta$, $t,s\in I$ and $n\in\mathbb{N}$. Without loss of generality we may assume that $s<t$. Then, keeping in mind our assumptions, we obtain

$$\begin{aligned}
&\beta_n|(L_nx)(t)-(L_ny)(s)|\\
&=\beta_n\Big|\sum_{j=k_n}^{n}a_{nj}(t)x_j-\sum_{j=k_n}^{n}a_{nj}(s)y_j\Big|\\
&\le\beta_n\Big|\sum_{j=k_n}^{n}a_{nj}(t)x_j-\sum_{j=k_n}^{n}a_{nj}(t)y_j\Big|+\beta_n\Big|\sum_{j=k_n}^{n}a_{nj}(t)y_j-\sum_{j=k_n}^{n}a_{nj}(s)y_j\Big|\\
&\le\beta_n\sum_{j=k_n}^{n}|a_{nj}(t)||x_j-y_j|+\beta_n\sum_{j=k_n}^{n}|a_{nj}(t)-a_{nj}(s)||y_j|\\
&\le\sum_{j=k_n}^{n}|a_{nj}(t)|\beta_j|x_j-y_j|+\sum_{j=k_n}^{n}\big(a_{nj}(t)-a_{nj}(s)\big)\beta_j|y_j|\\
&\le\bar{A}_n(t)\sup\big\{\beta_j|x_j-y_j|:j=1,2,\dots\big\}\\
&\quad+\sum_{j=k_n}^{n}\big(a_{nj}(t)-a_{nj}(s)\big)\sup\{\beta_j|y_j|:j=1,2,\dots\}\\
&\le A\|x-y\|+\Big(\sum_{j=k_n}^{n}a_{nj}(t)-\sum_{j=k_n}^{n}a_{nj}(s)\Big)\|y\|\\
&\le A\|x-y\|+\big(A_n(t)-A_n(s)\big)\|y\|.
\end{aligned}$$

Hence, we derive the estimate

$$\|(Lx)(t)-(Ly)(s)\|\le A\|x-y\|+\sup\big\{A_n(t)-A_n(s):n=1,2\dots\big\}\|y\|.$$

From this estimate and assumption (iv) we conclude that the operator L is continuous on the set $I\times l_\infty^\beta$. Obviously L is uniformly continuous as linear.

Next, let us take a nonempty subset X of the ball $B(x_0,r)$. Fix arbitrarily $x,y\in X$ and $t\in I$. Then, with help of a similar reasoning as above, for a fixed n natural, we obtain

$$\begin{aligned}
&\beta_n|f_n(t,x)-f_n(t,y)|\\
&\le\beta_n\Big|\sum_{j=k_n}^{n}a_{nj}(t)x_j-\sum_{j=k_n}^{n}a_{nj}(t)y_j\Big|+\beta_n|g_n(t,x)-g_n(t,y)|\\
&\le\beta_n\Big|\sum_{j=k_n}^{n}a_{nj}(t)(x_j-y_j)\Big|+\beta_n|g_n(t,x)|+\beta_n|g_n(t,y)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n \sum_{j=k_n}^{n} |a_{nj}(t)||x_j - y_j)| + 2\beta_n p_n \\
&\leq \sum_{j=k_n}^{n} |a_{nj}(t)|\beta_j|x_j - y_j)| + 2\beta_n p_n \\
&\leq \bar{A}_n(t) \sup \big\{\beta_j |x_j - y_j| : j = k_n, k_n + 1, \dots \big\} + 2\beta_n p_n \\
&\leq \bar{A}_n(t) \sup \big\{\beta_j \operatorname{diam} X_j : j = k_n, k_n + 1, \dots \big\} + 2\beta_n p_n.
\end{aligned}
$$

Hence, we derive the inequality

$$
\operatorname{diam} \beta_n f_n(t, X) \leq \bar{A}_n(t) \sup \big\{\beta_j \operatorname{diam} X_j : j \geq k_n \big\} + 2\beta_n p_n.
$$

From the above estimate, in view of assumptions (iii) and (iv) we have

$$
\mu(f(t, X)) \leq A\mu(X), \tag{8.5}
$$

where μ is the measure of noncompactness defined by (8.1). Finally, combining estimates (8.4) and (8.5), on the basis of Theorem 5.2, we complete the proof. $\square$

Remark 8.2. Observe that instead of the requirement imposed in assumption (iv) that the functions a_{nj} ($j = k_n, k_n + 1, \dots, n$; $n = 1, 2, \dots$) are nondecreasing on I, we can assume that those functions are nonincreasing on I.

The next example shows the applicability of the result in Theorem 8.1.

Example 8.3. Consider the semilinear lower diagonal perturbed infinite system of differential equations

$$
x_n' = \sum_{j=k_n}^{n} t^{n+j} x_j + \sin(x_n + x_{n+1} + x_{n+2}) \tag{8.6}
$$

with the initial conditions

$$
x_n(0) = n, \tag{8.7}
$$

for $n = 1, 2 \dots$ and for $t \in I = [0, T]$, where $T < 1$. Moreover, we assume that (k_n) is a nondecreasing sequence of natural numbers such that $1 \leq k_n \leq n$ and $k_n \to \infty$ as $n \to \infty$.

Observe that (8.6)–(8.7) is a special case of (8.2)–(8.3), where $a_{nj}(t) = t^{n+j}$ for $j = k_n, k_n + 1, \dots, n$ and for $n = 1, 2, \dots$. Apart from this, the function g_n has the form

$$
g_n(t, x_1, x_2, \dots) = \sin(x_n + x_{n+1} + x_{n+2})
$$

for $n = 1, 2, \dots$. It is easily seen that infinite system (8.6) with initial conditions (8.7) satisfies assumptions of Theorem 8.1 if we take the tempering sequence (β_n) of the form $\beta_n = \frac{1}{n}$ for $n = 1, 2, \dots$. Indeed, we have obviously that $(x_0^n) = (n) \in l_\infty^\beta$. This means that assumption (i) is satisfied.

Now, take an arbitrary element $x = (x_k) \in l_\infty^\beta$ and a number $t \in I$. Then, for a fixed natural number n we obtain

$$
\begin{aligned}
\beta_n |g_n(t, x_1, x_2, \dots)| &= \frac{1}{n} |\sin(x_n + x_{n+1} + x_{n+2})| \\
&\leq \frac{1}{n} \Big(|x_n| + |x_{n+1}| + |x_{n+2}|\Big) \\
&= \frac{n+2}{n} \Big(\frac{1}{n+2}|x_n| + \frac{1}{n+2}|x_{n+1}| + \frac{1}{n+2}|x_{n+2}|\Big)
\end{aligned}
$$

$$\le 3\Big(\frac{1}{n}|x_n| + \frac{1}{n+1}|x_{n+1}| + \frac{1}{n+2}|x_{n+2}|\Big) \le 3\|x\|,$$

where the symbol $\|\cdot\|$ denotes the norm in the space l_∞^β. Hence we obtain

$$\|g(t,x)\| \le 3\|x\|$$

which shows that g acts from the set $I \times l_\infty^\beta$ into l_∞^β.

Further, if we fix arbitrarily $n \in \mathbb{N}$, $x = (x_1, x_2, \dots) \in l_\infty^\beta$, $y = (y_1, y_2, \dots) \in l_\infty^\beta$ and $t, s \in I$, then we obtain

$$\begin{aligned}
&\beta_n |g_n(t, x_1, x_2, \dots) - g_n(s, y_1, y_2, \dots)| \\
&= \frac{1}{n}|\sin(x_n + x_{n+1} + x_{n+2}) - \sin(y_n + y_{n+1} + y_{n+2})| \\
&\le \frac{1}{n}\Big(|x_n - y_n| + |x_{n+1} - y_{n+1}| + |x_{n+2} - y_{n+2}|\Big) \\
&= \frac{n+2}{n}\Big(\frac{1}{n+2}|x_n - y_n| + \frac{1}{n+2}|x_{n+1} - y_{n+1}| + \frac{1}{n+2}|x_{n+2} - y_{n+2}|\Big) \\
&\le 3\Big(\frac{1}{n}|x_n - y_n| + \frac{1}{n+1}|x_{n+1} - y_{n+1}| + \frac{1}{n+2}|x_{n+2} - y_{n+2}|\Big) \\
&\le 3\|x - y\|.
\end{aligned}$$

Hence we derive the estimate

$$\|g(t,x) - g(s,y)\| \le 3\|x - y\|$$

which shows that the mapping g is uniformly continuous on the set $I \times l_\infty^\beta$. Thus the mapping g satisfies assumption (ii) of Theorem 8.1.

Now, we have

$$|g_n(t, x_1, x_2, \dots)| = |\sin(x_n + x_{n+1} + x_{n+2})| \le 1$$

which shows that there is satisfied assumption (iii) with $p_n = 1$ for, $n = 1, 2 \dots$. To show that there is satisfied assumption (iv) let us notice that we have

$$A_n(t) = \bar{A}_n(t) = t^{n+k_n}\frac{1 - t^{n-k_n+1}}{1-t}$$

for $t \in I = [0, T]$ and for $n = 1, 2, \dots$. Using the standard methods of analysis it is not hard to show that the sequence $(A_n(t))$ is equicontinuous on the interval I. Moreover, we have the estimate

$$A_n(t) = \bar{A}_n(t) \le A \le \frac{1}{1-T}$$

for any $t \in I$. Summing up we see that initial value problem (8.6)–(8.7) satisfies the assumptions in Theorem 8.1. Therefore the infinite system (8.6) with initial value conditions (8.7) has at least one solution in the space l_∞^β.

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Józef Banaś
Department of Nonlinear Analysis, Rzeszów University of Technology, al. Powstańców Warszawy 8, 35 - 959 Rzeszów, Poland
E-mail address: jbanas@prz.edu.pl

Monika Krajewska
Institute of Economics and Management, State Higher School of Technology and Economics in Jarosław, ul. Czarnieckiego 16, 37 - 500 Jarosław, Poland
E-mail address: monika.krajewska@pwste.edu.pl