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EXISTENCE OF STANDING WAVES FOR SCHRÖDINGER EQUATIONS INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. We study a class of fractional Schrödinger equations of the form

$$\varepsilon^{2\alpha}(-\Delta)^{\alpha}u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where ε is a positive parameter, $0 < \alpha < 1$, $2\alpha < N$, $(-\Delta)^{\alpha}$ is the fractional Laplacian, $V : \mathbb{R}^N \to \mathbb{R}$ is a potential which may be bounded or unbounded and the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is superlinear and behaves like $|u|^{p-2}u$ at infinity for some $2 < p < 2^*_{\alpha} := 2N/(N-2\alpha)$. Here we use a variational approach based on the Caffarelli and Silvestre's extension developed in [3] to obtain a nontrivial solution for ε sufficiently small.

1. Introduction

In this work we are concerned with the existence of standing waves for a nonlinear differential equation directed by the fractional Laplacian. We focus on the so-called *fractional Schrödinger equation*

$$i\varepsilon\frac{\partial\Psi}{\partial t} = \varepsilon^{2\alpha}(-\Delta)^{\alpha}\Psi + (V(x)+E)\Psi - f(x,\Psi), \ \ (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $\varepsilon > 0$ is a fixed parameter, E is a real constant, V and f are continuous functions, $0 < \alpha < 1$ and $(-\Delta)^{\alpha}$ denotes the fractional Laplacian, defined for all function belongs to the Schwartz space, by

$$\widehat{(-\Delta)^{\alpha}}u(\xi) = |\xi|^{2\alpha}\widehat{u}(\xi), \tag{1.2}$$

where $\widehat{u}$ denotes the Fourier transform of u. It is worth mentioning that (1.1) comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths (see [13] for details). When $\alpha = 1$ the Lévy dynamics becomes the Brownian dynamics, and (1.1) reduces to the classical Schrödinger equation

$$i\varepsilon\frac{\partial\Psi}{\partial t} = -\varepsilon^{2\alpha}\Delta\Psi + (V(x)+E)\Psi - f(x,\Psi), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

This equation has been widely investigated by many authors in the last decades (see, for instance [14, 17] and references therein). Standing waves solutions to

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(1.1) are solutions of the form $\Psi(x,t) = u(x)\exp(-iEt)$, where u solves the elliptic equation

$$\varepsilon^{2\alpha}(-\Delta)^\alpha u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Recently, several papers have been performed for classical elliptic equations involving the fractional Laplacian. In the sequel, we will list some of them related with the existence of solutions to (1.3) that may be found in the literature. Using the Nehari variational principle, Cheng [6] proved the existence of a nontrivial solution for the fractional Schrödinger (1.3) if $f(x,u) = |u|^{q-2}u$ with $2 < q < 2^*_\alpha$ if $N > 2\alpha$ or $2 < q < \infty$ if $N \le 2\alpha$, where $2^*_\alpha := 2N/(N-2\alpha)$ is the critical Sobolev exponent. Ground states are found by imposing a coercivity assumption on $V(x)$,

$$\lim_{|x|\to+\infty} V(x) = +\infty. \tag{1.4}$$

Applying the method of [12], Secchi [15] proved the existence of a ground state under less restrictive assumptions on $f(x,u)$. It is worthwhile to remark that in [6] and [15] the hypothesis (1.4) is assumed on $V(x)$ in order to overcome the problem of lack of compactness, typical of elliptic problems defined in unbounded domains. In [10], Dipierro et al. considered the existence of radially symmetric solutions of (1.3) in the situation where $V(x)$ does not depend explicitly on the space variable x. For the first time, using rearrangement tools and following the ideas of Berestycki and Lions [1], the authors proved the existence of a nontrivial, radially symmetric solution to

$$(-\Delta)^\alpha u + u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where $2 < q < 2^*_\alpha$ if $N > 2\alpha$ or $2 < q < \infty$ if $N \le 2\alpha$. We would also like to mention that problems involving the existence and concentration of positive solution to (1.3) have been investigated by [4, 5, 8, 9] when V is positive and ε is sufficient small.

Here, motivated by the papers [4, 6, 5, 8, 9, 10, 15, 17], our main goal is to study the existence of solutions for (1.3) when $0 < \alpha < 1$, $N > 2\alpha$, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions, ε is a positive parameter, $V(x)$ is a nonnegative continuous function satisfying some conditions. More precisely we assume that V and f satisfy the following assumptions:

- (A1) the set $\mathcal{Z} = \{x \in \mathbb{R}^N : V(x) = 0\}$ is nonempty;
- (A2) there exists $A > 0$ such that the level set
$$G_A = \{x \in \mathbb{R}^N : V(x) < A\}$$
has finite Lesbegue measure;
- (A3) $f(x,s) = o(|s|)$, as $s \to 0$, uniformly in $\mathbb{R}^N$;
- (A4) there exists a constant $C > 0$ such that
$$|f(x,s)| \le C(1 + |s|^p),$$
uniformly in $\mathbb{R}^N$, for all $s \in \mathbb{R}$, for some $1 < p < 2^*_\alpha - 1$, where $2^*_\alpha := 2N/(N-2\alpha)$;
- (A5) there exists a constant $\mu \in (2, p+1]$ such that
$$0 < \mu F(x,s) \le s f(x,s),$$
for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}\backslash\{0\}$; here, as usual, $F(x,s) := \int_0^s f(x,s)ds$.

A typical potentials satisfying (A1) and (A2) is

$$V(x) = \frac{|x-x_0|^\beta}{1+\lambda|x-x_0|}, \quad \beta \ge 1,\ \lambda \ge 0.$$

Our main result can be summarized as follows.

Theorem 1.1. *Suppose that* (A1)–(A6) *hold. Then there exists* $\varepsilon_0 > 0$ *such that* (1.3) *has a nontrivial and nonnegative solution for all* $\varepsilon \in (0, \varepsilon_0]$.

Remark 1.2. Our hypotheses on the potential V are inspired in [17] and the variational approach in the proof of Theorem 1.1 is based on the Caffarelli and Silvestre's extension developed in [3]. We also point out that the results of this work complement [4, 5, 6, 9, 10, 15] in the sense that the potential $V(x)$ belongs to a different class from those treated by them. To underline the role played by the potential $V(x)$, we suggest to the reader the papers [8, 9].

This work is organized as follows. In Section 2 we gather few notation and definitions. In Section 3 we make the variational framework to study the geometric properties and the Palais-Smale sequences of the associated functional. Finally, in Section 4 we prove Theorem 1.1.

2. Notation and definitions

We recall that the homogeneous Sobolev $\dot{H}^\alpha(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\dot{H}^\alpha}^2 := \int_{\mathbb{R}^N} |2\pi\xi|^{2\alpha}|\widehat{u}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u|^2 \mathrm{d}x.$$

For our setting we also consider the space $X^{2\alpha}(\mathbb{R}^{N+1}_+)$ defined as the completion of $C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|w\|_{X^{2\alpha}}^2 := \int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha}|\nabla w|^2 \mathrm{d}x\mathrm{d}y,$$

where $\kappa_\alpha = 2^{1-2\alpha}\Gamma(1-\alpha)/\Gamma(\alpha)$ and $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\}$.

In [2], it is proved that the extension operator $E_{2\alpha} : \dot{H}^\alpha(\mathbb{R}^N) \to X^{2\alpha}(\mathbb{R}^{N+1}_+)$ is well defined. Moreover, for any $\phi \in X^{2\alpha}(\mathbb{R}^{N+1}_+)$ if we denote its trace on $\mathbb{R}^N \times \{y = 0\}$ as $\phi(x,0)$, there exist $\mathcal{S}_1, \mathcal{S}_2 > 0$ such that (see [2, Lemmas 2.2 and 2.3] for details)

$$\mathcal{S}_1\|\phi(\cdot,0)\|_{2_\alpha^*} \le \|\phi(\cdot,0)\|_{\dot{H}^\alpha(\mathbb{R}^N)} \le \mathcal{S}_2\|\phi\|_{X^{2\alpha}(\mathbb{R}^{N+1}_+)}. \tag{2.1}$$

Given $u \in \dot{H}^\alpha(\mathbb{R}^N)$ we say that $w = E_{2\alpha}(u)$ is the α-harmonic extension of u to the upper half-space $\mathbb{R}^{N+1}_+$, if w is a solution of the problem

$$\begin{gathered} -\operatorname{div}(y^{1-2\alpha}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+, \\ w = u \quad \text{in } \mathbb{R}^N \times \{0\}. \end{gathered} \tag{2.2}$$

Furthermore, in [3] it is proved that

$$\lim_{y\to 0^+} y^{1-2\alpha}w_y(x,y) = -\frac{1}{\kappa_\alpha}(-\Delta)^\alpha u(x). \tag{2.3}$$

We would like to recall that using the change of variable $v(x) = u(\varepsilon x)$, Equation (1.3) is equivalent to the problem

$$(-\Delta)^\alpha u + V(\varepsilon x)u = f(\varepsilon x, u) \quad \text{in } \mathbb{R}^N. \tag{2.4}$$

Thus it is sufficient to consider (2.4) instead (1.3). Furthermore, from (2.2) and (2.3), we may consider the problem

$$\begin{gathered}
-\operatorname{div}(y^{1-2\alpha}\nabla w)=0 \quad \text{in } \mathbb{R}^{N+1}_+,\\
-\kappa_\alpha \frac{\partial w}{\partial \nu}=-V(\varepsilon x)u+f(\varepsilon x,u) \quad \text{in } \mathbb{R}^N\times\{0\},
\end{gathered}\tag{2.5}$$

where $\frac{\partial w}{\partial \nu}=\lim_{y\to 0^+} y^{1-2\alpha}w_y(x,y)$. To obtain a weak solution to (2.5), by using variational methods, we will consider the following subspace of $X^{2\alpha}(\mathbb{R}^{N+1}_+)$:

$$X_\varepsilon := \big\{w\in X^{2\alpha}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} V(\varepsilon x)w(x,0)^2 \mathrm{d}x<\infty\big\},$$

endowed with the inner product

$$\langle w,v\rangle_\varepsilon=\int_{\mathbb{R}^{N+1}_+}\kappa_\alpha\, y^{1-2\alpha}\nabla w\nabla v\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N} V(\varepsilon x)w(x,0)v(x,0)\,\mathrm{d}x$$

and the induced norm $\|w\|_\varepsilon=\langle w,w\rangle^{1/2}$ (see Lemma 3.1).

Throughout this work we say that $w\in X_\varepsilon$ is a weak solution to (2.5) if for any $\varphi\in X_\varepsilon$

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+}\kappa_\alpha\, y^{1-2\alpha}\nabla w\nabla \varphi\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N} V(\varepsilon x)w(x,0)\varphi(x,0)\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N} f(\varepsilon x,w(x,0))\varphi(x,0)\mathrm{d}x=0,
\end{aligned}$$

and consequently $u=w(x,0)\in H^\alpha(\mathbb{R}^N)$ is a weak solution to (2.4) (see [3]). Here $H^\alpha(\mathbb{R}^N)$ stands to the fractional Sobolev space

$$H^\alpha(\mathbb{R}^N):=\{u\in L^2(\mathbb{R}^N):\|(-\Delta)^\alpha u\|_2^2+\|u\|_2^2<\infty\},$$

endowed with the norm $\|u\|_{H^\alpha}=(\|(-\Delta)^\alpha u\|_2^2+\|u\|_2^2)^{1/2}$. We also recall that the imbedding $H^\alpha(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $q\in[2,2^*_\alpha]$ (see Proposition 3.6 in [11]).

Remark 2.1. We point out that if $u=w(x,0)\in H^\alpha(\mathbb{R}^N)$ is a weak solution to (2.4), then it is possible to get the pointwise expression of the fractional Laplacian as in (1.2). For details, the reader may see the paper [16], which addresses regularity results of weak solutions and viscosity solutions of the fractional Laplace equation.

3. Variational Setting

Our first lemma enables us to settle the variational setting.

Lemma 3.1. *Suppose condition* (A2) *holds. Then, for each* $0<\varepsilon<1$ *there exists* $C=C(\varepsilon)>0$ *such that*

$$\|w(x,0)\|_q\le C\|w\|_\varepsilon,\quad \text{for all } w\in X_\varepsilon \text{ and } 2\le q\le 2^*_\alpha.\tag{3.1}$$

Proof. First we show that there exists $\tau_1>0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+}\kappa_\alpha\, y^{1-2\alpha}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N} V(x)|w(x,0)|^2\,\mathrm{d}x\\
&\ge \tau_1\Big(\int_{\mathbb{R}^{N+1}_+}\kappa_\alpha\, y^{1-2\alpha}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y+\|w(x,0)\|_2^2\Big),
\end{aligned}\tag{3.2}$$

for every w for which the quantity in the left-hand side of (3.2) is finite. Indeed, since G_A has finite measure, there exists $C_2 = C_2(|G_A|, \alpha, N) > 0$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|w(x,0)|^2 \,\mathrm{d}x \\
&\geq \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \kappa_\alpha y^{1-2\alpha} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}y + C_2 \int_{G_A} |w(x,0)|^2 \mathrm{d}x + A \int_{\mathbb{R}^N \setminus G_A} |w(x,0)|^2 \,\mathrm{d}x \\
&\geq \min\{1/2, C_2, A\} \Big(\int_{\mathbb{R}^{N+1}_+} \kappa_\alpha y^{1-2\alpha} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}y + \|w(x,0)\|_2^2 \Big),
\end{aligned}
$$

where we used the Hölder and Sobolev inequalities, which give

$$
\int_{G_A} |w(x,0)|^2 \mathrm{d}x \leq |G_A|^{2\alpha/N} \|w(x,0)\|^2_{2^*_\alpha} \leq C_3 \int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}y.
$$

Take $w \in X_\varepsilon$ and put $v(x,y) = w(x/\varepsilon, y)$. Then from (3.2),

$$
\begin{aligned}
\|w\|_\varepsilon &= \varepsilon^{-n} \Big(\int_{\mathbb{R}^{N+1}_+} \varepsilon^2 \kappa_\alpha\, y^{1-2\alpha} |\nabla v|^2 \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|v(x,0)|^2 \,\mathrm{d}x \Big) \\
&\geq \varepsilon^{-n+2} \Big(\int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} |\nabla v|^2 \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|v(x,0)|^2 \,\mathrm{d}x \Big) \\
&\geq \varepsilon^{-n+2} \tau_1 \Big(\int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} |\nabla v|^2 \,\mathrm{d}x\mathrm{d}y + \|v(x,0)\|_2^2 \Big) \\
&= \varepsilon^2 \tau_1 \Big(\int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} |\nabla w|^2 \,\mathrm{d}x\mathrm{d}y + \|w(x,0)\|_2^2 \Big).
\end{aligned}
$$

Thus, from (2.1)

$$
\|w\|_\varepsilon \geq C_1(\varepsilon) \|w(x,0)\|^2_{H^\alpha}.
$$

This together with the Sobolev imbedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ imply the desired result. □

It follows by Lemma 3.1, (A2)–(A4), that the functional

$$
I_\varepsilon(w) = \frac{1}{2} \|w\|_\varepsilon^2 - \int_{\mathbb{R}^N} F(\varepsilon x, w(x,0)) \mathrm{d}x
$$

is well defined in X_ε and belongs to $C^1(X_\varepsilon, \mathbb{R})$, with Gâteaux derivative given by

$$
\begin{aligned}
I'_\varepsilon(w) \cdot \varphi = &\int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} \nabla w \nabla \varphi \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(\varepsilon x) w(x,0) \varphi(x,0) \,\mathrm{d}x \\
&- \int_{\mathbb{R}^N} f(\varepsilon x, w(x,0)) \varphi(x,0) \mathrm{d}x,
\end{aligned}
$$

for any $\varphi \in C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$. Thus critical points of I_ε are weak solutions to (2.5) and reciprocally. The functional I_ε satisfies the following geometric properties.

Lemma 3.2. *Suppose that* (A2)–(A5) *hold. Then*

(i) *there exist $\delta, \rho > 0$ such that $I_\varepsilon(w) \geq \delta$ if $\|w\|_\varepsilon = \rho$;*
(ii) *there exists $e \in X_\varepsilon$ such that $\|e\|_\varepsilon > \rho$ and $I_\varepsilon(e) < 0$.*

Proof. By (A3) and (A4), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}^N} |F(\varepsilon x, w(x,0))| \,\mathrm{d}x \le \frac{\epsilon}{2}\int_{\mathbb{R}^N} |w(x,0)|^2 \mathrm{d}x + \frac{C_\epsilon}{p+1}\int_{\mathbb{R}^N} |w(x,0)|^{p+1}\mathrm{d}x, \quad (3.3)$$

for all $w \in X_\varepsilon$. This and the imbedding (3.1) imply that there exist positive constants C_1 and C_2 such that

$$I_\varepsilon(w) \ge \Big(\frac{1}{2} - \frac{\epsilon C_1}{2}\Big)\rho^2 - C_2\rho^{p+1} \quad \text{if } |w\|_\varepsilon = \rho.$$

Since $p+1 > 2$, we choose $\epsilon, \rho > 0$ sufficiently small such that $\delta := \inf_{\|w\|_\varepsilon = \rho} I_\varepsilon(w) > 0$, which proves (i). To prove (ii), we consider $\varphi \in C_0^\infty(\mathbb{R}_+^{N+1}, \mathbb{R}_+)$ such that $\varphi(x,0) \not\equiv 0$. Then by (A4) and (A5) there exists a positive function $d(x) \in L^\infty(\mathbb{R}^N)$ such that

$$F(x,s) \ge d(x)|s|^\mu, \quad \text{for all } (x,s) \in \mathbb{R}^N \times \mathbb{R}. \quad (3.4)$$

Thus for $t > 0$, we get that $I_\varepsilon(t\varphi) \to -\infty$, as $t \to +\infty$. Setting $e = t\varphi$ with t large enough, the condition (ii) is satisfied. □

From Lemma 3.2, for each $\varepsilon > 0$ the minimax level

$$c_\varepsilon := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\varepsilon(\gamma(t))$$

is positive, where $\Gamma := \{\gamma \in C([0,1], X_\varepsilon); \gamma(0) = 0 \text{ and } \gamma(1) = e\}$.

Lemma 3.3. *If $1 < p < 2_\alpha^* - 1$, it holds $\lim_{\varepsilon\to 0^+} c_\varepsilon = 0$.*

Proof. Without loss of generality we may assume that $0 \in \mathcal{Z}$. Note that by (3.4), for all $w \in X_\varepsilon$ there exists $C_0 > 0$ such that

$$c_\varepsilon \le \max_{t\ge 0} I_\varepsilon(tw) \le C_0\Big(\frac{\int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha}|\nabla w|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(\varepsilon x) w(x,0)^2 \,\mathrm{d}x}{(\int_{\mathbb{R}^N} d(\varepsilon x)|w(x,0)|^\mu \,\mathrm{d}x)^{2/\mu}}\Big)^{\mu/(\mu-2)}.$$

Defining $\mathcal{M}_\varepsilon := \big\{w \in X_\varepsilon : \int_{\mathbb{R}^N} d(\varepsilon x)|w(x,0)|^\mu \,\mathrm{d}x = 1\big\}$ and

$$\bar{c}_\varepsilon := \inf_{w\in\mathcal{M}_\varepsilon} \int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha}|\nabla w|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(\varepsilon x) w(x,0)^2 \,\mathrm{d}x.$$

We obtain that

$$c_\varepsilon \le C_0 \bar{c}_\varepsilon^{\mu/(\mu-2)}. \quad (3.5)$$

We claim that $\bar{c}_\varepsilon \to 0$ as $\varepsilon \to 0^+$. Suppose by contradiction that for some sequence $\varepsilon_n \to 0^+$, $\bar{c}_{\varepsilon_n} \ge c_0 > 0$ for all $n \in \mathbb{N}$. Since $\mu \le p+1 < 2_\alpha^*$, it is known that (see [18, Theorem 4] and [3])

$$\inf_{\substack{w\in C_0^\infty(\mathbb{R}_+^{N+1}) \\ \int_{\mathbb{R}^N} |w(x,0)|^\mu \,\mathrm{d}x = 1}} \int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha}|\nabla w|^2 \,\mathrm{d}\,x\mathrm{d}y = 0. \quad (3.6)$$

Thus we can take a sequence $(w_n) \subset C_0^\infty(\mathbb{R}_+^{N+1})$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha}|\nabla w_n|^2 \mathrm{d}x\mathrm{d}y = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |w_n(x,0)|^\mu \,\mathrm{d}x = 1. \quad (3.7)$$

Defining

$$v_n = \frac{w_n}{(\int_{\mathbb{R}^N} d(\varepsilon x)|w_n(x,0)|^\mu \,\mathrm{d}x)^{1/\mu}},$$

we have that $v_n \in \mathcal{M}_\varepsilon$. For simplicity we suppose that $d(0) = 1$. By using (3.7) we see that for each $n \in \mathbb{N}$, we can find $\varepsilon_n > 0$ such that

$$\int_{\mathbb{R}^N} d(\varepsilon x)|w_n(x,0)|^\mu \,\mathrm{d}x > \frac{1}{2}, \tag{3.8}$$

for $\varepsilon < \varepsilon_n$. Thus, for every n and every

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha}|\nabla v_n|^2 \mathrm{d}x\mathrm{d}y \le 2^{2/\mu} \int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha}|\nabla w_n|^2 \mathrm{d}x\mathrm{d}y.$$

From (3.7), we can find n_0 such that for every $\varepsilon < \varepsilon_{n_0}$,

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha}|\nabla v_{n_0}|^2 \mathrm{d}x\mathrm{d}y < \frac{c_0}{2}. \tag{3.9}$$

On the other hand, for all $n \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$c_0 \le \int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha}|\nabla v_n|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(\varepsilon x) v_n(x,0)^2 \,\mathrm{d}x. \tag{3.10}$$

Hence for $\varepsilon < \varepsilon_{n_0}$, from (3.8), (3.9) and (3.10) we have

$$\begin{aligned}\int_{\mathbb{R}^N} V(\varepsilon x)|w_{n_0}(x,0)|^2 \,\mathrm{d}x &\ge \frac{\int_{\mathbb{R}^N} V(\varepsilon x)|w_{n_0}(x,0)|^2 \,\mathrm{d}x}{2^{2/\mu}(\int_{\mathbb{R}^N} d(\varepsilon x)|w_{n_0}(x,0)|^\mu \,\mathrm{d}x)^{2/\mu}} \\ &= \frac{1}{2^{2/\mu}} \int_{\mathbb{R}^N} V(\varepsilon x)|v_{n_0}(x,0)|^2 \,\mathrm{d}x \ge \frac{c_0}{2^{\frac{2}{\mu}+1}},\end{aligned}$$

which contradicts hypothesis (A1). Therefore, $\bar{c}_\varepsilon \to 0$ as $\varepsilon \to 0^+$, which combined with (3.5) completes the proof. □

Combining Lemma 3.2 and the Ekeland's variational principle, we obtain a sequence $(w_n) \subset X_\varepsilon$ such that

$$I_\varepsilon(w_n) \to c_\varepsilon \quad \text{and} \quad \|I'_\varepsilon(w_n)\|_* \to 0 \quad \text{as } n \to +\infty. \tag{3.11}$$

Lemma 3.4. *If $(w_n) \subset X_\varepsilon$ is a sequence satisfying* (3.11)*, then there exists a positive constant $\nu > 0$ such that*

$$\limsup_{n\to\infty} \|w_n\|_\varepsilon^2 \le \nu c_\varepsilon \quad \textit{for all } \varepsilon > 0.$$

Proof. Because of (3.11) and (A5) we have

$$\big(\frac{\mu}{2} - 1\big)\|w_n\|_\varepsilon^2 \le \mu I_\varepsilon(w_n) - I'_\varepsilon(w_n)(w_n) \le \mu c_\varepsilon + o_n(1) + o_n(1)\|w_n\|_\varepsilon,$$

which implies that (w_n) is bounded in X_ε and the lemma follows easily. In the last inequality we used $o_n(1)$ to denotes a quantity that tends to zero as $n \to \infty$. □

By Lemma 3.1, we may assume that, up to a subsequence, $w_n \rightharpoonup w_0$ weakly in X_ε, $w_n(x,0) \to w_0(x,0)$ in $L^q_{loc}(\mathbb{R}^N)$ for all $2 \le q < 2^*_\alpha$, and $w_n(x,0) \to w_0(x,0)$ almost everywhere in $\mathbb{R}^N$. Taking the limit in (3.11) we obtain that w_0 is a weak solution of (2.5). To prove that w_0 is not trivial we need the following result.

Lemma 3.5. *Let $\varepsilon > 0$ and (w_n) a sequence satisfying* (3.11)*. Then there exist positive constants $\eta = \eta(A, \mathcal{S}_2, \nu) > 0$ and $R = R(\varepsilon) > 0$ such that*

$$\limsup_{n\to\infty} \|w_n(x,0)\|^2_{H^\alpha(\mathbb{R}^N \setminus B_R)} \le \eta\, c_\varepsilon.$$

Proof. From (2.1), we have

$$\|w_n(\cdot,0)\|_{\dot{H}^\alpha(\mathbb{R}^N\setminus B_R)} \le \mathcal{S}_2\|w_n\|_{X^{2\alpha}(\mathbb{R}^{N+1}_+)}.$$

Thus, by Lemma 3.4, it is sufficient to show that for each $\varepsilon>0$ there exist positive constants $C=C(A)$ and $R=R(\varepsilon)$ such that

$$\int_{|x|>R}|w_n(x,0)|^2\mathrm{d}x \le C\|w_n\|_\varepsilon^2. \tag{3.12}$$

We denote by $B_R=\{x\in\mathbb{R}^N : |x|\le R\}$ the closed ball of radius R centered at the origin, B_R^c the complement of B_R, $G_{A,\varepsilon}=\{x\in\mathbb{R}^N : V(\varepsilon x)<A\}$ and $v_n(x,y)=w_n(x/\varepsilon,y)$. Since G_A has finite measure, by using the Hölder and Sobolev inequalities, we obtain $C_1=C(\mathcal{S}_1,\mathcal{S}_2)>0$ such that

$$\begin{aligned}
\int_{B_R^c\cap G_{A,\varepsilon}}|w_n(x,0)|^2\mathrm{d}x &= \varepsilon^{-N}\int_{B_{\varepsilon R}^c\cap G_A}|v_n(x,0)|^2\mathrm{d}x\\
&\le \varepsilon^{-N}|B_{\varepsilon R}^c\cap G_A|^{2\alpha/N}\Big(\int_{\mathbb{R}^N}|v_n(x,0)|^{2_\alpha^*}\mathrm{d}x\Big)^{2/2_\alpha^*}\\
&\le C_1\varepsilon^{-N}|B_{\varepsilon R}^c\cap G_A|^{2\alpha/N}\int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha}|\nabla v_n|^2\mathrm{d}x\mathrm{d}y\\
&= C_1\varepsilon^{-2}|B_{\varepsilon R}^c\cap G_A|^{2\alpha/N}\int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha}|\nabla w_n|^2\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

and we have that, for each $\varepsilon>0$, we can find a radius $R=R(\varepsilon)>0$ such that

$$|B_{\varepsilon R}^c\cap G_A|^{2\alpha/N}<\varepsilon^2C_1^{-1}A^{-1}.$$

Thus

$$\int_{B_R^c\cap G_{A,\varepsilon}}|w_n(x,0)|^2\mathrm{d}x \le \frac{1}{A}\|w_n\|_\varepsilon^2. \tag{3.13}$$

Moreover,

$$\int_{B_R^c\setminus G_{A,\varepsilon}}|w_n(x,0)|^2\mathrm{d}x \le \frac{1}{A}\int_{B_R^c\setminus G_{A,\varepsilon}}V(\varepsilon x)w_n^2(x,0)\mathrm{d}x \le \frac{1}{A}\|w_n\|_\varepsilon^2. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain that (3.12) holds. □

4. Proof of Theorem 1.1

To obtain a nonnegative solution of (2.4), we replace $f(x,s)$ by $f^+(x,s)$ where $f^+(x,s)=f(x,s)$ if $s\ge 0$ and 0 if $s<0$. Let $\eta>0$ given in Lemma 3.5. From (A3) and (A4) there exists $C>0$ such that

$$|f(x,s)|\le\frac{1}{\eta}|s|+C|s|^p \quad \text{for all } (x,s)\in\mathbb{R}^N\times\mathbb{R}.$$

Since $F(x,s)\ge 0$, from (3.11)

$$\begin{aligned}
c_\varepsilon &= \lim_{n\to\infty}\big[I_\varepsilon(w_n)-\frac{1}{2}I_\varepsilon'(w_n)(w_n)\big]\\
&= \lim_{n\to\infty}\int_{\mathbb{R}^N}\big[\frac{1}{2}f(\varepsilon x,w_n(x,0))w_n(x,0)-F(\varepsilon x,w_n(x,0))\big]\mathrm{d}x\\
&\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\big[\frac{1}{2\eta}|w_n(x,0)|^2+C|w_n(x,0)|^{p+1}\big]\,\mathrm{d}x.
\end{aligned}$$

Thus

$$\begin{aligned} c_\varepsilon \le\ & \frac{1}{2\eta}\|w_0(x,0)\|^2_{L^2(B_R)} + C\|w_0(x,0)\|^{p+1}_{L^{p+1}(B_R)} \\ & + \frac{1}{2\eta}\limsup_{n\to+\infty}\|w_n(x,0)\|^2_{L^2(\mathbb{R}^N\setminus B_R)} + C\limsup_{n\to+\infty}\|w_n(x,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^N\setminus B_R)}. \end{aligned}$$

Invoking Lemma 3.5 we get

$$c_\varepsilon \le \frac{1}{2\eta}\|w_0(x,0)\|^2_{L^2(B_R)} + C\|w_0(x,0)\|^{p+1}_{L^{p+1}(B_R)} + \frac{c_\varepsilon}{2} + C\eta^{\frac{p+1}{2}}c_\varepsilon^{\frac{p+1}{2}}.$$

Therefore,

$$\frac{1}{2\eta}\|w_0(x,0)\|^2_{L^2(B_R)} + C\|w_0(x,0)\|^{p+1}_{L^{p+1}(B_R)} \ge c_\varepsilon\Big(\frac{1}{2} - C\eta^{\frac{p+1}{2}}c_\varepsilon^{\frac{p-1}{2}}\Big). \tag{4.1}$$

From Lemma 3.3, we can find $\varepsilon_0 > 0$ such that

$$c_\varepsilon < \Big(\frac{1}{4C\eta^{\frac{p+1}{2}}}\Big)^{\frac{2}{p-1}} \quad \text{for all } \varepsilon \in (0,\varepsilon_0].$$

The last inequality and (4.1) imply that

$$\frac{1}{2\eta}\|w_0(x,0)\|^2_{L^2(B_R)} + C\|w_0(x,0)\|^{p+1}_{L^{p+1}(B_R)} \ge \frac{c_\varepsilon}{4} > 0.$$

Consequently, $w_0 \not\equiv 0$ for all $\varepsilon \in (0,\varepsilon_0]$. This completes the proof of Theorem 1.1.

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