*Electronic Journal of Differential Equations*, Vol. 2017 (2017), No. 79, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# HARNACK'S INEQUALITY FOR $p$-LAPLACIAN EQUATIONS WITH MUCKENHOUPT WEIGHT DEGENERATING IN PART OF THE DOMAIN

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*Communicated by Ludmila S. Pulkina*

Abstract. In this article we consider quasi-linear second-order elliptic equations of divergence structure with Makenhaupt weight that degenerates over a small part of the domain. We show that the classical Harnack's inequality does not hold in this case, and prove an appropriate Harnack's inequality for the considered equation.

## 1. Introduction and statement of results

On a domain $D \subset \mathbb{R}^n$, $n \geq 2$ we consider the family of elliptic equations

$$L_\varepsilon u = \operatorname{div}\big(\omega_\varepsilon(x)|\nabla u|^{p-2}\nabla u)\big) = 0, \quad p > 1, \tag{1.1}$$

where $p$ is a constant, $\omega_\varepsilon(x)$ is nonnegative weight depending on small parameter $\varepsilon$. It is assumed that the domain $D$ is divided by the hyperplane $\Sigma = \{x : x_n = 0\}$ into the parts $D^{(1)} = D \cap \{x : x_n > 0\}$ and $D^{(2)} = D \cap \{x : x_n < 0\}$, and

$$\omega_\varepsilon(x) = \begin{cases} \varepsilon\omega(x), & x \in D^{(1)}, \\ \omega(x), & x \in D^{(2)}, \end{cases} \tag{1.2}$$

where $\varepsilon \in (0, 1]$, $\omega(x)$ is a weight satisfying Muckenhoupt's $A_p$-condition. Note that the weight $\omega(x)$, defined in the whole space $\mathbb{R}^n$ satisfies to $A_p$-condition (see [11]), if

$$\sup\Big(\frac{1}{|B|}\int_B \omega(x)\,dx\Big)\Big(\frac{1}{|B|}\int_B \omega^{-\frac{1}{p-1}}(x)\,dx\Big)^{p-1} < \infty, \quad 1 < p < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

To define a solution of (1.1) we introduce the class of functions

$$W_{\rm loc}(D, \omega) = \{u : u \in W^{1,1}_{\rm loc}(D), |\nabla u|^p\omega \in L^1_{\rm loc}(D)\},$$

where $W^{1,1}_{\rm loc}(D)$ is a classical Sobolev's space of the functions which are local summable in the domain $D$ together with all generalized partial derivatives of the first

2010 *Mathematics Subject Classification.* 35j92, 35j65, 35j70, 35j62.
*Key words and phrases.* $p$-Laplasian; Muckenhoupt weight; Harnack's inequality.

Submitted February 5, 2017. Published March 21, 2017.

order. As a solution of the equation (1.1) we take the function $u \in W_{\rm loc}(D,\omega)$ for which the integral identity

$$\int_D \omega_\varepsilon(x)|\nabla u|^{p-2}\nabla u \cdot \nabla\xi\, dx = 0 \tag{1.3}$$

is satisfied on the finite test functions $\xi \in W_{\rm loc}(D,\omega_\varepsilon)$.

The object of the work is the problem of Harnack's inequality for the nonnegative solutions of the equation (1.1). A large number of works is devoted to this problem for the degenerate equations. The most investigated the case is when the weight function $\omega(x)$ satisfies the Muckenhoupt's $A_p-$ condition and $\varepsilon = 1$.

Below $|E|$ is $n$-dimensional Lebesque measure of the measurable set $E \subset \mathbb{R}^n$,

$$d\mu = \omega\, dx, \quad \omega(E) = \int_E \omega(x)\, dx, \quad \int_E f\, d\mu = \frac{1}{\omega(E)}\int_E f\omega\, dx.$$

The important consequences of the Muckenhoupt's $A_p$ condition are the doubling conditions [5, 11]

$$\omega(B_{2r}) \le c\omega(B_r), \tag{1.4}$$

inversion of the Holder inequality [5]

$$\Big(\frac{1}{|B_r|}\int_{B_r}\omega^{1+\delta}(x)\, dx\Big)^{1/(1+\delta)} \le C\frac{1}{|B_r|}\int_{B_r}\omega(x)\, dx, \tag{1.5}$$

Friedrichs inequality [6, 7]

$$\begin{gathered} \int_\Omega |\varphi|^p\, d\mu \le c(n,\nu,p)r^p\int_\Omega |\nabla\varphi|^p\, d\mu, \\ \varphi \in C^\infty(\overline{\Omega}), \quad \varphi|_E = 0, \quad |E| \ge \nu|\Omega|,\ \nu > 0, \end{gathered} \tag{1.6}$$

where $\Omega \subset B_r$ is Lipschitz domain, and Sobolev's inequality [6, 7]

$$\Big(⨍_{B_r}|\varphi|^{pk}\, d\mu\Big)^{1/k} \le c(n,p)r^p\int_{B_r}|\nabla\varphi|^p\, d\mu, \quad \varphi \in C_0^\infty(B_r), \quad k = \frac{n}{n-1}. \tag{1.7}$$

In [6, 7] is shown that if $\omega \in A_p$ and $\varepsilon = 1$ then solution of the equation (1.1) is of Holder property in $D$ and for all nonnegative in $B_{4R} \subset D$ solutions it holds the Harnach inequality

$$\inf_{B_R} u \ge \text{const}\cdot\sup_{B_R} u. \tag{1.8}$$

For the considered weight $\omega_\varepsilon$ the doubling condition (1.4) with a constant independent on $\varepsilon$, does not hold. This implies that in the center of the balls on the hyperplane $\Sigma$ the classical Harnack inequality (1.8) does not hold, in which the constant is independent on $\varepsilon$. This statement is set in the first section of §2.

In addition to the belonging of the weighting function to the Muckenhoupt's class $A_p$ it is assumed that in the open balls $B_{R_0}$ of small enough radiuses $R_0$ with the centers on the hyperplane $\Sigma$ for almost all points $x$from the semiball $B_{R_0} \cap \{x : x_n > 0\}$ is valid

$$\omega(x) \le \gamma\omega(x'),\ \gamma = \text{const} > 0, \tag{1.9}$$

where $x'$ is a point symmetric to $x$ with respect to the hyperplane $\Sigma$. In particular to this condition satisfy the weights $|x|^\alpha$, where $-n < \alpha < n(p-1$, and $|x_n|^\alpha$, where $-1 < \alpha < p-1$. Besides any weight satisfying to the Muckenhoupt's $A_p$ condition, that is indeed even with respect to the hyperplane $\Sigma$ is suitable for this case.

The main aim of this work to formulate and prove the uniform Harnack inequality over the parameter $\varepsilon$ that corresponds to the considered equation. Since the classical Harnack's inequality (1.8) does not hold in the balls with the center on the hyperplane $\Sigma$, in the formulation of the problem takes part of such balls and is assumed that

$$B_R^- = B_R \cap \{x : -R < x_n < -R/2\}. \tag{1.10}$$

**Theorem 1.1.** *If the weight $\omega(x)$ satisfies to the Muckenhoupt's $A_p$ condition, and the conditions (1.2), (1.9) are satisfied, then for the nonnegative part in the $B_{4R} \subset D$, with the center on $\Sigma$, the solution of the equation (1.1) satisfies the inequality*

$$\inf_{B_R} u \geq c_0 \sup_{B_R^-} u, \tag{1.11}$$

*In which the positive constant $c_0 < 1$ does not depend on $u$, $R$ and $\varepsilon$.*

When $p = 2$ and $\omega \equiv 1$, Theorem 1.1 first was proved in [2]. In the present work for the case $\varepsilon = 1$ and $p = 2$ the inequality (1.11) is proved for the non-negative solutions of the equation with a particular Muckenhoupt weight $\omega$, that in general does not satisfy Muckenhoupt's $A_2$-condition.

The end of this work shows that from Theorem 1.1 it follows the Holder continuity of the solutions at the points $\Sigma \cap D$, and in consequence Holder continuity of the solutions in the domain of $D$.

Let us consider the family $\{u^\varepsilon(x)\}$ of the solutions of the equations $L_\varepsilon u^\varepsilon = 0$ bounded in $L^\infty$ uniformly over $\varepsilon$ on the compact subsets $D$.

**Theorem 1.2.** *If the weight $\omega$ satisfies the Muckenhoupt's $A_p$ condition and conditions (1.2), (1.9) take place, then there exists a constant $\alpha \in (0, 1)$ depending only on $p$, dimension of the space $n$, constant $\gamma$ from (1.9) and the weight $\omega$, such that the family $\{u^\varepsilon(x)\}$ is compact in $C^\alpha(D')$ in any subdomain of $D' \Subset D$.*

Theorem 1.2 was proved by a different method in an earlier work of the author [8][. In the case when $\omega(x) \equiv 1$ this statement is given in [7, 8]. Note also the works [3, 4], when $p = 2$ and $\varepsilon = 1$, the Holder continuity is proved for the solutions of the equations with particular Muckenhoupt's weight $\omega$, of a more general structure.

## 2. Harnack's inequality

**Absence of the classical Harnack's inequality.** Here by $B_R$ we denote the open ball of radius $R$ with the center on the hyperplane $\Sigma$ and $B_{16r}^{(1)} = B_{16r} \cap \{x_n > 0\}$, $B_{16r}^{(2)} = B_{16r} \cap \{x_n < 0\}$. Choose the points $x_0, y_0$ by the way that $B_{5r}^{x_0} \subset B_{16r}^{(1)}$, $B_{5r}^{y_0} \subset B_{16r}^{(2)}$. We assume that the points $x_0$ and $y_0$ are symmetric with respect to the hyperplane $\Sigma$. Let $\Omega = B_{16r} \backslash (\overline{B_{r/4}^{x_0}} \cup \overline{B_{r/4}^{y_0}})$. Consider the problem

$$\begin{gathered} Lu = \operatorname{div}(\omega_\varepsilon(x)|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial B_{16r} \\ u = \varepsilon^{-1}\omega^{-1}(B_r) \quad \text{on } \partial B_{r/4}^{x_0}, \quad u = \omega^{-1}(B_r) \quad \text{on } \partial B_{r/4}^{y_0}. \end{gathered} \tag{2.1}$$

Show that in the domain $\Omega$ usual Harnach's inequality with the constant not depending on $\varepsilon$ does not hold. Solution of the problem (2.1) is a minimizer of the variational problem for integral functional

$$F[v] = \int_\Omega \omega_\varepsilon(x)|\nabla v|^p\,dx \tag{2.2}$$

over the smooth functions $v$, on the closure of the domain $D$, satisfying boundary conditions (2.1). Let us continue $u$ to inside of the balls $B^{x_0}_{r/4}$ and $B^{y_0}_{r/4}$, taking $u = \varepsilon^{-1}\omega^{-1}(B_r)$ in $B^{x_0}_{r/4}$ and $u = \omega^{-1}(B_r)$ in$B^{y_0}_{r/4}$.

For the compact $K$ belonging to the open set $U$, its $p$-volume with respect to $U$ is defined as follows

$$\operatorname{cap}_p(K,U) = \inf \int_U \omega(x)|\nabla\varphi|^p\,dx,$$

where the sharp lower bound is taken over the set of functions from $C_0^\infty(U)$ being equal to unit in the neighborhood of $K$.

Since the weight $\omega$ satisfies the Muckenhoupt's $A_p$-condition then (see [6]) in the domains $B^{(1)}_{16r}\backslash\overline{B^{x_0}_{r/4}}$ and $B^{(2)}_{16r}\backslash\overline{B^{y_0}_{r/4}}$ is valid the classical Harnachk's inequality that we use in the form

$$\sup_{B^{x_0}_{4r}\backslash\overline{B^{x_0}_{r/2}}} u \le C \inf_{B^{x_0}_{4r}\backslash\overline{B^{x_0}_{r/2}}} u, \tag{2.3}$$

$$\sup_{B^{y_0}_{4r}\backslash\overline{B^{y_0}_{r/2}}} u \le C \inf_{B^{y_0}_{4r}\backslash\overline{B^{y_0}_{r/2}}} u. \tag{2.4}$$

By [7] there exists positive constants $c_1$, $c_2$, not depending on $r$ such that

$$c_1 r^{-p}\omega(B^{x_0}_r) \le \operatorname{cap}_p(\overline{B^{x_0}_{r/4}}, B^{x_0}_{2r}) \le c_2 r^{-p}\omega(B^{x_0}_r), \tag{2.5}$$

$$c_1 r^{-p}\omega(B^{y_0}_r) \le \operatorname{cap}_p(\overline{B^{y_0}_{r/4}}, B^{y_0}_{2r}) \le c_2 r^{-p}\omega(B^{y_0}_r). \tag{2.6}$$

**Lower estimate of the solution.** Put $w = u\cdot\varepsilon\omega(B_r)$. Then

$$Lw = 0 \quad \text{in } B_{16r}, \quad w = 1 \quad \text{in } \partial B^{x_0}_{r/4}, \quad w = \varepsilon \quad \text{in } \partial B^{y_0}_{r/4}.$$

By the maximum principle $0 < w \le 1$ in $B_{16r}$.

Let $\eta$ be a cutoff function being equal to zero outside of the ball $B^{x_0}_{2r}$, and to unit in the ball $B^{x_0}_r$, $|\nabla\eta| \le Cr^{-1}$, $0 \le \eta \le 1$. Since $\omega_\varepsilon(x) = \varepsilon\omega(x)$ in $B^{x_0}_{2r}$, then by the definition of the volume

$$\int_{B^{x_0}_{2r}} \omega_\varepsilon(x)|\nabla(w\eta)|^p\,dx \ge \varepsilon\operatorname{cap}_p(\overline{B^{x_0}_{r/4}}, B^{x_0}_{2r}). \tag{2.7}$$

Then

$$\int_{B^{x_0}_{2r}} \omega_\varepsilon(x)|\nabla(w\eta)|^p\,dx \le C\int_{B^{x_0}_{2r}} \omega_\varepsilon(x)|\nabla w|^p\,dx + C\varepsilon r^{-p}\int_{B^{x_0}_{2r}\backslash B^{x_0}_{r/4}} \omega(x)w^p\,dx.$$

Using the estimate $0 \le w \le 1$, for which $w^p \le w^{p-1}$, we obtain

$$\int_{B^{x_0}_{2r}} \omega_\varepsilon(x)|\nabla(w\eta)|^p\,dx \le C\int_{B^{x_0}_{2r}} \omega_\varepsilon(x)|\nabla w|^p\,dx + C\varepsilon r^{-p}\int_{B^{x_0}_{2r}\backslash B^{x_0}_{r/4}} \omega(x)w^{p-1}\,dx.$$

Let us estimate the last integral in the right hand side of the last inequality through the sharp upper bound of $w$. Applying Harnack's inequality (2.3) to the solution $u$ of (2.1) and doubling condition (1.4) for the weight $\omega$ we arrive to the estimation

$$\int_{B^{x_0}_{2r}} \omega_\varepsilon(x)|\nabla(w\eta)|^p\,dx \le C\int_{B^{x_0}_{2r}} \omega_\varepsilon(x)|\nabla w|^p\,dx + C\varepsilon r^{-p}\omega(B_r)(\min_{\partial B^{x_0}_r} w)^{p-1}. \tag{2.8}$$

Now we estimate the first integral in the right hand side of (2.8). Choosing in the integral identity

$$\int_\Omega \omega_\varepsilon(x)|\nabla w|^{p-2}\nabla w\cdot\nabla\varphi\,dx=0 \tag{2.9}$$

The test function as $\varphi=(1-w)\xi^p$, where $\xi\in C_0^\infty(B_{3r}^{x_0})$, $\xi=1$ in $B_{2r}^{x_0}$, $|\nabla\xi|\le Cr^{-1}$ and $0\le\xi\le 1$, we obtain

$$\begin{aligned}\int_{B_{2r}^{x_0}}\omega_\varepsilon(x)|\nabla w|^p\,dx &\le p\int_{B_{3r}^{x_0}\setminus B_{2r}^{x_0}}\omega_\varepsilon(x)|\nabla w|^{p-1}(1-w)|\nabla\xi|\xi^{p-1}\,dx\\ &\le p\int_{B_{3r}^{x_0}\setminus B_{2r}^{x_0}}\omega_\varepsilon(x)|\nabla w|^{p-1}|\nabla\xi|\,dx\\ &\le Cr^{-1}\int_{B_{3r}^{x_0}\setminus B_{2r}^{x_0}}\omega_\varepsilon(x)|\nabla w|^{p-1}\,dx.\end{aligned}$$

From this by the Holder inequality we find that

$$\begin{aligned}&\int_{B_{2r}^{x_0}}\omega_\varepsilon(x)|\nabla w|^p\,dx\\ &\le C\Big(\int_{B_{3r}^{x_0}\setminus B_{2r}^{x_0}}\omega_\varepsilon(x)w^{-a}|\nabla w|^p\,dx\Big)^{(p-1)/p}\Big(\int_{B_{3r}^{x_0}\setminus B_{2r}^{x_0}}\omega_\varepsilon(x)w^{a(p-1)}r^{-p}\,dx\Big)^{1/p},\end{aligned} \tag{2.10}$$

where $0<a<1$. To estimate the fist integral in the right hand side of (2.10) we choose in the integral identity (2.9) the test function as $\varphi=w^{1-a}\eta^p$, where $\eta\in C_0^\infty(B_{4r}^{x_0}\setminus\overline{B_{3r/2}^{x_0}})$, $\eta=1$ in $B_{3r}^{x_0}\setminus\overline{B_{2r}^{x_0}}$, $|\nabla\eta|\le Cr^{-1}$ and $0\le\eta\le 1$. A a result we obtain

$$(1-a)\int_{B_{4r}^{x_0}\setminus B_{3r/2}^{x_0}}\omega_\varepsilon(x)w^{-a}|\nabla w|^p\eta^p\,dx\le p\int_{B_{4r}^{x_0}\setminus B_{3r/2}^{x_0}}\omega_\varepsilon(x)|\nabla w|^{p-1}|\nabla\eta|\eta^{p-1}\,dx.$$

Applying Young's inequality to the integrand of the right-hand side, and because of the choice of the cutoff function we obtain the estimate

$$\int_{B_{3r}^{x_0}\setminus B_{2r}^{x_0}}\omega_\varepsilon(x)w^{-a}|\nabla w|^p\,dx\le Cr^{-p}\int_{B_{4r}^{x_0}\setminus B_{3r/2}^{x_0}}\omega_\varepsilon(x)w^{p-a}\,dx. \tag{2.11}$$

Thus from (2.10) and (2.11)) we have

$$\begin{aligned}&\int_{B_{2r}^{x_0}}\omega_\varepsilon(x)|\nabla w|^p\,dx\\ &\le C\Big(r^{-p}\int_{B_{4r}^{x_0}\setminus B_{3r/2}^{x_0}}\omega_\varepsilon(x)w^{p-a}\,dx\Big)^{(p-1)/p}\Big(\int_{B_{3r}^{x_0}\setminus B_{2r}^{x_0}}\omega_\varepsilon(x)w^{a(p-1)}r^{-p}\,dx\Big)^{1/p}.\end{aligned}$$

We estimate the integrals on the right side through the upper bounds $w$. Since $\omega_\varepsilon=\varepsilon\omega$ in $B_{2r}^{x_0}$, using Harnack's inequality (2.3) and doubling condition (1.4) we obtain

$$\int_{B_{2r}^{x_0}}\omega_\varepsilon(x)|\nabla w|^p\,dx\le C\varepsilon r^{-p}\omega(B_r)(\min_{\partial B_r^{x_0}}w)^{p-1}\,. \tag{2.12}$$

Comparing (2.12), (2.8), (2.7), (2.5) and using again in (2.5) the doubling condition (1.4) we obtain

$$\min_{\partial B_r^{x_0}}w\ge C,$$

From which due to the explicit form of $w$ follows,

$$\min_{\partial B_r^{x_0}} u \geq C\varepsilon^{-1}\omega^{-1}(B_r) \tag{2.13}$$

with the constant $C$ that does not depend on $\varepsilon$.

**Upper bound of the solution.** Let $\eta_1 \in C_0^\infty(B_{2r}^{x_0})$, $\eta_1 = 1$ in $B_r^{x_0}$ and $\eta_2 \in C_0^\infty(B_{2r}^{y_0})$, $\eta_2 = 1$ in $B_r^{y_0}$. Take $K_1 = \varepsilon^{-1}\omega^{-1}(B_r)$, $K_2 = \omega^{-1}(B_r)$.

Since the solution $u$ of (2.1) minimizes the functional (2.2), following to the variational principle, the choice of the cutoff functions $\eta_1$, $\eta_2$ and condition (1.2) we have

$$\int_{B_{2r}^{y_0}} \omega(x)|\nabla u|^p\,dx \leq \varepsilon \int_{B_{2r}^{x_0}} \omega(x)|\nabla(\eta_1 K_1)|^p\,dx + \int_{B_{2r}^{y_0}} \omega(x)|\nabla(\eta_2 K_2)|^p\,dx.$$

Hence, from the arbitrary choice of $\eta_1$ and $\eta_2$ it follows that

$$\int_{B_{2r}^{y_0}} \omega(x)|\nabla u|^p\,dx \leq \varepsilon K_1^p \operatorname{cap}_p(\overline{B_{r/4}^{x_0}}, B_{2r}^{x_0}) + K_2^p \operatorname{cap}_p(\overline{B_{r/4}^{y_0}}, B_{2r}^{y_0}).$$

Thus by (2.5), (2.6) and doubling condition (1.4), we obtain

$$\int_{B_{2r}^{y_0}} \omega(x)|\nabla u|^p\,dx \leq \varepsilon K_1^p \omega(B_r) r^{-p} + K_2^p \omega(B_r) r^{-p}. \tag{2.14}$$

Then by the Friedrichs inequality (1.6),

$$\int_{B_{16r}^{(2)}} \omega(x) u^p\,dx \leq C r^p \int_{B_{16r}^{(2)}} \omega(x)|\nabla u|^p\,dx$$

and from (2.14) we obtain

$$\int_{B_r^{y_0}\setminus B_{r/2}^{y_0}} \omega(x) u^p\,dx \leq \varepsilon K_1^p \omega(B_r) + K_2^p \omega(B_r).$$

Now from (2.4) and doubling condition (1.4) we have

$$\max_{\partial B_r^{y_0}} u \leq C\omega^{-1/p}(B_r)(\varepsilon^{1/p} K_1 \omega^{1/p}(B_r) + K_2 \omega^{1/p}(B_r)).$$

or considering the explicit forms of $K_1$ and $K_2$

$$\max_{\partial B_r^{y_0}} u \leq C(\varepsilon^{1/p-1}\omega^{-1}(B_r) + \omega^{-1}(B_r)).$$

or

$$\max_{\partial B_r^{y_0}} u \leq C\varepsilon^{1/p-1}\omega^{-1}(B_r) \tag{2.15}$$

with the constant $C$ that does not depend on $\varepsilon$.

If we suppose that the classical Harnack's inequality holds uniformly with respect to $\varepsilon$ in the domain $\Omega$, then

$$\min_{\partial B_r^{x_0}} u \leq C \max_{\partial B_r^{y_0}} u,$$

where $C$ does not depend on $\varepsilon$. This inequality leads to the contradiction with the estimates (2.13) and (2.15).

2.1. **Estimation of the minimum of thenon-negative solution.** Below $B_R \subset D$ stands for the ball with the centers on $\Sigma \cap D$, $B_R^{(i)} = B_r \cap D^{(i)}$ for the semiballs, $i = 1, 2$. Note that Sobolev's inequality (1.7) entails a corresponding inequality for the semiballs

$$\begin{gathered} \Big(\fint_{B_R^{(i)}} |\varphi|^{pk}\, d\mu_i\Big)^{1/k} \le CR^p \fint_{B_R^{(i)}} |\nabla\varphi|^p\, d\mu_i, \\ \varphi \in C_0^\infty(B_R), \quad k = \frac{n}{n-1},\ i = 1, 2. \end{gathered} \tag{2.16}$$

Let $u(x)$ be a non-negative solution of (2.1), $\tilde{u}(x)$ be even continuation of $u(x)$ from $D^{(2)}$ to $D^{(1)}$ relative to hyperplane $\Sigma$ and $B_{4R} \subset D$. Below it is assumed that

$$v(x) = \begin{cases} \min(u(x), \tilde{u}(x)), & \text{if } x \in D^{(1)} \\ u(x), & \text{if } x \in D^{(2)}. \end{cases} \tag{2.17}$$

**Lemma 2.1.** *If* (1.9) *is fulfilled then for any $q > 0$, we have*

$$\inf_{B_R} u(x) \ge C\Big(\fint_{B_{2R}} v^{-q}(x)\, d\mu\Big)^{-1/q} \tag{2.18}$$

*with the constant $C$ that does not depend on $u$ or $R$.*

*Proof.* Not loosing generality we assume that $u(x)$ is positive. Otherwise one should consider the function $u(x)+\delta$ and then pass to limit at $\delta \to 0$ in the estimate (2.18). First we show that for any $R \le \rho < r \le 2R$ and $q_0 > 0$, it holds

$$\inf_{B_\rho} u(x) \ge C\big(\frac{r-\rho}{r}\big)^a \Big(\fint_{B_r^{(1)}} v^{-q_0}(x)\, d\mu\Big)^{-1/q_0}, \tag{2.19}$$

in which $a = a(n, q_0, p) > 0$, and $C$ does not depend on $r, \rho$, $u(x)$ or $\varepsilon$. Choose in (1.3) the test function $\xi = u^\beta(x)\eta^p(x)$, where $\eta(x) \in C_0^\infty(B_{3R})$ is radially symmetric and $\beta < 1-p$. After some simple estimations using Yuong's inequality we come to the inequality

$$\fint_{B_{3R}} |\nabla u|^p u^{\beta-1}\eta^p \omega_\varepsilon\, dx \le C(p)\fint_{B_{3R}} u^{\beta+p-1}|\nabla\eta|^p \omega_\varepsilon\, dx. \tag{2.20}$$

In particular, by (2.2) and (2.9), we have

$$\begin{aligned} &\fint_{B_{3R}^{(2)}} |\nabla u|^p u^{\beta-1}\eta^p\, d\mu \\ &= \frac{1}{\omega(B_{3R}^{(2)})}\fint_{B_{3R}^{(2)}} |\nabla u|^p u^{\beta-1}\eta^p\, d\mu \\ &\le C(p,\gamma)\Big(\fint_{B_{4R}^{(1)}} u^{\beta+p-1}|\nabla\eta|^p\, d\mu + \fint_{B_{3R}^{(2)}} u^{\beta+p-1}|\nabla\eta|^p\, d\mu\Big) \end{aligned} \tag{2.21}$$

Following the Sobolev's embedding theorem (2.16) we obtain

$$\begin{aligned} &\Big(\fint_{B_{3R}^{(2)}} u^{k(\beta+p-1)}\eta^{kp}\, d\mu\Big)^{1/k} \\ &\le C(|\beta|+p-1)^p R^p \Big(\fint_{B_{3R}^{(1)}} u^{\beta+p-1}|\nabla\eta|^p\, d\mu + \fint_{B_{3R}^{(2)}} u^{\beta+p-1}|\nabla\eta|^p\, d\mu\Big), \end{aligned} \tag{2.22}$$

where $C = C(n, p, \gamma)$.

It is not possible to obtain a similar estimate in the ball $B_{4R}^{(1)}$ by this method. Take

$$G_R = B_{3R}^{(1)} \cap \{x : u(x) < \tilde{u}(x)\} \tag{2.23}$$

and assuming that $G_R \neq \emptyset$, put in (2.3) the test function

$$\xi(x) = \begin{cases} (u^\beta(x) - \tilde{u}^\beta(x))\eta^p(x) & \text{in } G_R, \\ 0, & \text{in } B_{3R} \backslash G_R, \end{cases}$$

where $\eta$ and $\beta$ have the same sense as above. This test function is valid by condition (1.9). We have

$$\begin{aligned} &|\beta| \int_{G_R} |\nabla u|^p u^{\beta-1} \eta^p \, d\mu \\ &\le |\beta| \int_{G_R} |\nabla u|^{p-1} |\nabla \tilde{u}| \tilde{u}^{\beta-1} \eta^p \, d\mu + p \int_{G_R} |\nabla u|^{p-1} |\nabla \eta| \tilde{u}^\beta \eta^{p-1} \, d\mu \\ &\quad + p \int_{G_R} |\nabla u|^{p-1} |\nabla \eta| u^\beta \eta^{p-1} \, d\mu \end{aligned}$$

From this and using definition of $G_R$ and young's inequality we find that

$$\begin{aligned} &\int_{G_R} |\nabla u|^p u^{\beta-1} \eta^p \, d\mu \\ &\le C(p) \Big( \int_{G_R} |\nabla \tilde{u}|^p \tilde{u}^{\beta-1} \eta^p \, d\mu + \int_{G_R} \tilde{u}^{\beta+p-1} |\nabla \eta|^p \, d\mu + \int_{G_R} u^{\beta+p-1} |\nabla \eta|^p \, d\mu \Big) \end{aligned}$$

or since $\tilde{u}^{\beta-1} \le u^{\beta-1}$ on the set $G_R$ we have

$$\begin{aligned} &\int_{B_{3R}^{(1)} \backslash G_R} |\nabla \widetilde{u}|^p \widetilde{u}^{\beta-1} \eta^p \, d\mu + \int_{G_R} |\nabla u|^p u^{\beta-1} \eta^p \, d\mu \\ &\le C \Big( \int_{B_{3R}^{(1)}} |\nabla \widetilde{u}|^p \widetilde{u}^{\beta-1} \eta^p \, d\mu + \int_{B_{3R}^{(1)}} u^{\beta+p-1} |\nabla \eta|^p \, d\mu \\ &\quad + \int_{B_{3R}^{(2)}} u^{\beta+p-1} |\nabla \eta|^p \, d\mu \Big), \end{aligned} \tag{2.24}$$

where $C = C(p)$. Since the function $\tilde{u}$ is an even continuation of the function $u$ from $D^{(2)}$ to $D^{(1)}$ and the function $\eta$ is even with respect to the hyperplane $\Sigma$, by (2.9) we have

$$\int_{B_{3R}^{(1)}} |\nabla \tilde{u}|^p \tilde{u}^{\beta-1} \eta^p \, d\mu \le \gamma \int_{B_{3R}^{(2)}} |\nabla u|^p u^{\beta-1} \eta^p \, d\mu$$

and from (2.20), (1.2) we have

$$\int_{B_{3R}^{(1)}} |\nabla \widetilde{u}|^p \widetilde{u}^{\beta-1} \eta^p \, d\mu \le C(p,\gamma) \Big( \int_{B_{3R}^{(1)}} u^{\beta+p-1} |\nabla \eta|^p \, d\mu + \int_{B_{3R}^{(2)}} u^{\beta+p-1} |\nabla \eta|^p \, d\mu \Big).$$

Considering the last relation in (2.24) and using the definition of the function $v$ (see (2.17)), we obtain

$$\int_{B_{3R}^{(1)}} |\nabla v|^p v^{\beta-1} \eta^p \, d\mu \le C(p,\gamma) \Big( \int_{B_{3R}^{(1)}} v^{\beta+p-1} |\nabla \eta|^p \, d\mu + \int_{B_{3R}^{(2)}} u^{\beta+p-1} |\nabla \eta|^p \, d\mu \Big).$$

Applying here Sobolev's embedding theorem (2.16), by the doubling condition (1.4) we arrive at the estimate

$$\begin{aligned}&\Big(\fint_{B_{3R}^{(1)}} v^{k(\beta+p-1)}\eta^{pk}\,d\mu\Big)^{1/k}\\ &\le C(|\beta|+p-1)^pR^p\Big(\fint_{B_{3R}^{(1)}} v^{\beta+p-1}|\nabla\eta|^p\,d\mu+\fint_{B_{3R}^{(2)}} u^{\beta+p-1}|\nabla\eta|^p\,d\mu\Big),\end{aligned}\tag{2.25}$$

in which $C=C(n,p,\gamma)$. Thus according to (2.22), (2.25) and definition of the function $v$,

$$\begin{aligned}&\Big(\fint_{B_{3R}^{(1)}} v^{k(\beta+p-1)}\eta^{pk}\,d\mu+\fint_{B_{3R}^{(2)}} u^{k(\beta+p-1)}\eta^{pk}\,d\mu\Big)^{1/k}\\ &\le C(|\beta|+p-1)^pR^p\Big(\fint_{B_{3R}^{(1)}} v^{\beta+p-1}|\nabla\eta|^p\,d\mu+\fint_{B_{3R}^{(2)}} u^{\beta+p-1}|\nabla\eta|^p\,d\mu\Big),\end{aligned}$$

where $C=C(n,p,\gamma)$. Now from (2.7) and doubling condition (2.4) follows that

$$\begin{aligned}&\Big(\fint_{B_{3R}} v^{k(\beta+p-1)}\eta^{pk}\,d\mu\Big)^{1/k}\\ &\le C(n,p,\gamma)(|\beta|+p-1)^pR^p\fint_{B_{3R}} v^{\beta+p-1}|\nabla\eta|^p\,d\mu.\end{aligned}\tag{2.26}$$

Until now we have assumed that $G_R\neq\emptyset$. If $G_R=\emptyset$ then $v(x)=\tilde u(x)$ in $B_{3R}^{(1)}$ and (2.26) follows immediately from (2.22) and the condition (1.9). Choosing in (2.26) test function as $\eta=1$ in $B_r$, $|\nabla\eta|\le Cr(R(r-\rho))^{-1}$, by the condition (2.4) we obtain

$$\begin{aligned}&\Big(\fint_{B_\rho} v^{k(\beta+p-1)}\eta^{pk}\,d\mu\Big)^{1/k}\\ &\le C(n,p,\gamma)(|\beta|+p-1)^p(\frac{r}{r-\rho})^p\Big(\fint_{B_r} v^{\beta+p-1}\,d\mu\Big).\end{aligned}\tag{2.27}$$

Let us iterate this inequality. Let $j=0,1,\dots$. Denote $r_j=\rho+2^{-j}(r-\rho)$, $\chi_j=-q_0k^j$ and take in (2.13) $r=r_j$, $\rho=r_{j+1}$, $\beta=\chi_i+1-p$. As a result for

$$\Phi_j=\Big(\fint_{B_{r_j}^{(1)}} v^{\chi_j}\,d\mu\Big)^{1/\chi_j}$$

we obtain the following recurrence relation

$$\Phi_j\le C^{1/|\chi_j|}(2^j(1+|\chi_j|))^{p/|\chi_j|}(\frac{r}{r-\rho})^{p/|\chi_j|}\Phi_{j+1},$$

that implies estimate (2.19) (see [10]). Taking in this estimate $\rho=R$ and $r=2R$, we obtain

$$\inf_{B_R} u(x)\ge C\Big(\fint_{B_{2R}} v^{-q_0}(x)\,d\mu\Big)^{-1/q_0}.\tag{2.28}$$

To prove (2.18) we take $s=2(1+\delta)\delta^{-1}$, where $\delta$ is a constant from (2.5), and apply to the integral

$$\fint_{B_{2R}} v^{-q_0}(x)\omega(x)\,dx=\fint_{B_{2R}} v^{-q_0}(x)\omega^{1/p}(x)\omega^{-1/p}(x)\omega(x)\,dx$$

triplet Holder inequality with orders $p_1 = p_2 = s$, $p_3 = (1+\delta)^{-1}$. As a result considering condition(2.5) we find

$$\begin{aligned}
&\Big(\frac{1}{\omega(B_{2R})}\int_{B_{2R}} v^{-q_0}(x)\omega(x)\,dx\Big)^{1/q_0}\\
&\le \Big(\frac{1}{\omega(B_{2R})}\Big)^{1/q_0}\Big(\int_{B_{2R}} \omega^{-1}(x)\,dx\Big)^{1/pq_0}\Big(\int_{B_{2R}} \omega^{1+\delta}(x)\,dx\Big)^{1/q_0(1+\delta)}\\
&\quad\times\Big(\int_{B_{2R}} v^{-pq_0}(x)\omega(x)\,dx\Big)^{1/pq_0}\\
&\le C\Big(\fint_{B_{2R}} v^{-pq_0}(x)\,d\mu\Big)^{1/pq_0},
\end{aligned}$$

That (2.28) leads to the estimate

$$\inf_{B_R} u(x) \ge C\Big(\fint_{B_{2R}} v^{-pq_0}(x)\,d\mu\Big)^{1/pq_0}.$$

Taking $q_0 = q/p$ we arrive to(2.18). The proof is complete. ☐

The statement of the Lemma 2.1 becomes true for the nonnegative super solutions $u(x)$ of the equation (1.1) i.e. for such nonnegative solutions $u$ that

$$\int_D \omega_\varepsilon(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\xi\,dx \ge 0,\quad \forall \xi\in C_0^\infty(D),\ \xi\ge 0.$$

2.2. **Harnack's inequality.** Below $u(x)$ stands for the nonnegative solution of (1.1) and $B_{3R}\subset D$ for the ball with the center on $\Sigma$. To prove the Harnack's inequality we need John-Nirenberg's lemma for the function $v(x)$defined in (2.17).

**Lemma 2.2.** *For an arbitrary ball $B_{2r}\subset B_{3R}$ we have*

$$\int_{B_r} |\nabla \ln v|^p\,d\mu \le Cr^{-p}\omega(B_r), \tag{2.29}$$

*in which the constant $C$ does not depend on $u$, $r$, $R$ or $\varepsilon$.*

*Proof.* As above without loss of generality we assume the solution is positive and $B_r^{(i)} = B_r\cap D^{(i)}$, $i=1,2$. Take the cutoff function $\eta\in C_0^\infty(B_{2r})$ such that $\eta\equiv 1$ in $B_r, |\nabla\eta|\le Cr^{-1}$. Assuming in (1.3) and $\xi = u^{1-p}\eta^p$ as in (2.20) we obtain

$$\int_{B_{2r}} |\nabla \ln u|^p\eta^p\omega_\varepsilon\,dx \le C(p)r^{-p}\int_{B_{2r}}\omega_\varepsilon\,dx.$$

If $B_{2r}\cap\Sigma=\emptyset$, then from (1.2) and (1.4) we arrive to (2.29). Now let $B_r^{x_0}$ be arbitrary open ball of radius $r$ with the center $x^0 = (x_1^0, x_2^0,\dots,x_n^0)$, such that $B_{2r}^{x_0}\subset B_{3R}$ and $B_{2r}^{x_0}\cap\Sigma\ne\emptyset$. To prove the statement it is sufficient to set

$$\int_{B_r^{x_0}} |\nabla \ln v|^p\,d\mu \le Cr^{-p}\omega(B_r^{x_0}) \tag{2.30}$$

with the constant $C$, not depending on $u$, $r$, $R$ and $\varepsilon$. Denote by $y_0$ the point that is symmetric to $x_0$ with respect to the hyperplane $\Sigma$ and take $d=|x_0-y_0|$. It is clear that $0<d<4r$. Consider the cylinder

$$\mathcal{C}_r = \Big\{x : \Big(\sum_{i=1}^{n-1}(x_i - x_i^0)^2\Big)^{1/2} < 2r,\ |x_n|\le d\Big\}$$

And introduce the symmetric with respect to the hyperplane $\Sigma$ set

$$Q_r = B_{2r}^{x_0} \cup B_{2r}^{y_0} \cup \mathcal{C}_r.$$

Let $Q_r^{(i)} = Q_r \cap D^{(i)}$, $i = 1, 2$. It is not difficult to see that $B_{2r}^{x_0} \subset Q_r \subset B_{3R}$ and $B_r^{x_0} \subset Q_{r/2}$.

Consider the symmetric with respect to the hyperplane $\Sigma$ cut off function $\eta \in C_0^\infty(Q_r)$, by the way that $\eta = 1$ in $Q_{r/2}$ and $|\nabla \eta| \le Cr^{-1}$. Choosing in the integral identity (1.3) the test function as $\xi = u^{1-p}\eta^p$ we obtain

$$\int_{Q_r} |\nabla \ln u|^p \eta^p \omega_\varepsilon \, dx \le C(p) r^{-p} \int_{Q_r} \omega_\varepsilon \, dx.$$

Now from (1.2) and (1.4) it follows that

$$\int_{Q_r^{(2)}} |\nabla \ln u|^p \eta^p \, d\mu \le C(p) r^{-p} \omega(B_r^{x_0}). \tag{2.31}$$

To prove a similar estimate in $Q_r^{(1)}$ first we assume that the set $G_R$ from (2.23) is not empty and choose in (1.3) the test function as

$$\xi(x) = \begin{cases} (u^{1-p}(x) - \tilde{u}^{1-p}(x))\eta^p(x) & \text{in } G_R, \\ 0 & \text{in } B_{3R} \backslash G_R, \end{cases}$$

where $\eta$ has the same sense as above. Then it is easy to see that (see (1.2))

$$\begin{aligned} &(p-1)\int_{G_R} |\nabla \ln u|^p \eta^p \, d\mu \\ &\le (p-1)\int_{G_R} |\nabla u|^{p-1} |\nabla \ln \tilde{u}| \tilde{u}^{1-p} \eta^p \, d\mu + p\int_{G_R} |\nabla u|^{p-1} |\nabla \eta| \tilde{u}^{1-p} \eta^{p-1} \, d\mu \\ &\quad + p\int_{G_R} |\nabla u|^{p-1} |\nabla \eta| u^{1-p} \eta^{p-1} \, d\mu. \end{aligned}$$

From this considering $u(x) \le \tilde{u}(x)$ on $G_R$, by the help of Young's inequality we find

$$\int_{G_R} |\nabla \ln u|^p \eta^p \, d\mu \le C(p)\Big(\int_{G_R} |\nabla \ln \tilde{u}|^p \eta^p \, d\mu + \int_{G_R} |\nabla \eta|^p \, d\mu\Big)$$

or adding to both sides of this inequality the integral

$$\int_{Q_r^{(1)} \backslash G_R} |\nabla \ln \tilde{u}|^p \eta^p \, d\mu,$$

because of the choice of the cutoff function $\eta$ and doubling condition (1.4), we have

$$\begin{aligned} &\int_{Q_r^{(1)} \backslash G_R} |\nabla \ln \tilde{u}|^p \eta^p \, d\mu + \int_{G_R} |\nabla \ln u|^p \eta^p \, d\mu \\ &\le C(p)\Big(\int_{Q_r^{(1)}} |\nabla \ln \tilde{u}|^p \eta^p \, d\mu + r^{-p} \omega(B_r^{x_0})\Big). \end{aligned} \tag{2.32}$$

Since the function $\tilde{u}$ is an even continuation of the function $u$ from $D^{(2)}$ to $D^{(1)}$ and the cutoff function is even relative to $\Sigma$, then according to condition (1.9),

$$\int_{Q_r^{(1)}} |\nabla \ln \tilde{u}|^p \eta^p \, d\mu \le \gamma \int_{Q_r^{(2)}} |\nabla \ln u|^p \eta^p \, d\mu$$

and from (2.31) it follows that

$$\int_{Q_r^{(1)}} |\nabla \ln \tilde{u}|^p \eta^p \, d\mu \le C(p,\gamma) r^{-p} \omega(B_r^{x_0}).$$

Considering the last relation in the right-hand side of (2.32) we have

$$\int_{Q_r^{(1)} \setminus G_R} |\nabla \ln \tilde{u}|^p \eta^p \, d\mu + \int_{G_R} |\nabla \ln u|^p \eta^p \, d\mu \le C(p,\gamma) r^{-p} \omega(B_r^{x_0}). \tag{2.33}$$

Now from (2.31), (2.33) and definition of the function $v$ (see. (2.17)) we arrive to the estimate

$$\int_{Q_r} |\nabla \ln v|^p \eta^p \, d\mu \le C(p,\gamma) r^{-p} \omega(B_r^{x_0}),$$

that implies the relation (2.30) since $\eta = 1$ in $B_r^{x_0}$ and $B_r^{x_0} \subset Q_r$.

If the set $G_R$ is empty then $v(x) = \tilde{u}(x)$ in $B_{3R}^{(1)}$ and (2.30) follows from (2.31) and condition (1.9). The proof is complete. □

The statement of the Lemma 2.2 is true for the nonnegative supersolutions of the equation (1.1). The consequence of this lemma is John-Nirenberg's lemma the proof of which may be found in [7].

**Corollary 2.3.** *There exist positive constants $q$ and $C$ not depending on $u$, $R$ or $\varepsilon$, such that*

$$\Big(⨍_{B_{2R}} v^{-q}(x) \, d\mu_1\Big)^{-1/q} \ge C(n,p) \Big(⨍_{B_{2R}} v^{q}(x) \, d\mu_1\Big)^{1/q}. \tag{2.34}$$

*Proof of Theorem 1.1.* Let $u(x)$ be nonnegative solution of the equation (1.1) and $B_R^-$ be a set defined in (1.10). Using (2.18), (2.34) and doubling condition (1.4) we obtain

$$\inf_{B_R} u(x) \ge C \Big(⨍_{B_{2R}} v^{q}(x) \, d\mu\Big)^{1/q} \ge C \inf_{B_R^-} u(x).$$

Now (1.11) follows from the classical Harnack's inequality for the solutions of (1.1) in the domain $D^{(2)}$, according which $\inf_{B_R^-} u(x) \ge c(n,p) \sup_{B_R^-} u(x)$. The proof is complete. □

*Proof of Theorem 1.2.* From the results in [6, 7] is known that solution has the Holder property inside $D^{(1)}$ and $D^{(2)}$. It remains to prove the Holder property of the solutions on $\Sigma \cap D$, since the holder property inside of $D$ may be obtained by elementary gluing of the Holder property on $\Sigma \cap D$ and $D^{(1)}$, $D^{(2)}$. Let $B_{4R} \subset D$ be a ball with the center on $\Sigma$ and

$$M_{4R} = \sup_{B_{4R}} u(x), \quad m_{4R} = \inf_{B_{4R}} u(x), \quad M_R^- = \sup_{B_R^-} u(x), \quad m_R^- = \inf_{B_R^-} u(x).$$

Since the functions $M_{4R} - u(x)$ and $u(x) - m_{4R}$ are nonnegative solutions in $B_{4R}$, by the Harnack's inequality (1.11),

$$M_{4R} - M_R \ge c_0 (M_{4R} - m_R^-), \quad m_R - m_{4R} \ge c_0 (M_R^- - m_{4R}).$$

Summing these relations and using the fact that $c_0 < 1$, we obtain the scattering lemma

$$M_R - m_R \ge (1 - c_0)(M_{4R} - m_{4R}),$$

that shows the Holder continuity of the solutions on $\Sigma \cap D$. The proof is complete. □

**Acknowledgments.** The first author was supported by the Ministry of Education and Science of the Russian Federation (task No 1.3270.2017/PP) and the Russian Foundation for Basic Research (project No 15-01-00471-a).

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