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HANDLING GEOMETRIC SINGULARITIES BY THE MORTAR SPECTRAL ELEMENT METHOD FOR FOURTH-ORDER PROBLEMS

MOHAMED ABDELWAHED, NEJMEDDINE CHORFI, VICENŢIU D. RĂDULESCU

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Abstract. This article concerns the numerical analysis and the error estimate of the biharmonic problem with homogeneous boundary conditions using the mortar spectral element method in domains with corners. Since the solution of this problem can be written as a sum of a regular part and known singular functions, we propose to use the Strang and Fix algorithm for improving the order of the error.

1. Introduction

It is well known that the solutions of an elliptic equations in polygonal domains are not very regular despite the regularity of the second member and boundary data [16, 17, 18]. More precisely, the solution of an elliptic problem in such domains is the sum of a regular part and another one which is presented as a linear combination of functions which the regularity gets lower as the angle of singularity gets greater. This singular part of the solution pollutes the error estimate. Different numerical methods for the most part related to finite element method have been developed to calculate the singular part of the solution or to improve the error estimate [4, 5, 6]; this is the case of the mesh refinement method near the singular angle corners. Among these methods the Strang and Fix algorithm [19] which was extended to the mortar method for a spectral discretization [2, 15].

The high precision of the spectral methods makes them well adapted to the treatment of the singularities. In fact, the numerical analysis using this method in the Laplacian case [2, 14] confirms this expectation of sufficient precision. Furthermore, the study of the singular function approximation by polynomials near the singular corners shows that the convergence is better than what the general approximation theory lets to believe and explains the appearance of super convergence [10]. Calculations have also been made for the stokes system [1].

The Strang and Fix algorithm consists on the enlargement of the test function space and the resolution of the discrete problem in this space. This algorithm

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permits us the computing of the singular coefficient which is usually issued from the physics (case of the elastic crack) [3].

In this work we propose to study this algorithm for the homogeneous biharmonic problem. For that we place ourselves within the framework of the Mortar element method with spectral discretization [11, 13]. The analysis and the implementation of the mortar element method has been done in the work of Belhachmi et al. [7, 8, 9] for a problem of order 4. We present in this work an extension in the case of the non regular domains in order to improve the estimation of the order of the error.

An outline of this article is as follows. In section 2, we present the geometry aspects of the domain. In section 3 we present the continuous problem, then we give the singular functions and some regularity results. In section 4, we define the discrete problem. Section 5 is devoted to the numerical analysis and the error estimation of the mortar spectral element method of the Strang and Fix algorithm for the harmonic problem.

2. Geometric aspects

Let Ω an open polygonal, bounded, Lipschitzian and connected domain of $\mathbb{R}^2$, decomposed on K rectangles Ω^k, $1 \le k \le K$ such that

$$\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}^k \quad \text{and} \quad \Omega^k \cup \Omega^l = \emptyset\,,\ 1 \le k \ne l \le K.$$

We denote by $\overline{\Gamma}^{k,j}$, $1 \le j \le 4$ the sides of the sub-domain $\overline{\Omega}^k$, $1 \le k \le K$ and

$$\overline{\gamma}_{kl} = \overline{\Omega}^k \cap \overline{\Omega}^l, \quad 1 \le k \ne l \le K$$

the interface of the decomposition.

We define the skeleton of the decomposition

$$\mathcal{S} = \cup_{k=1}^{K} \cup_{j=1}^{4} \overline{\Gamma}^{k,j}.$$

We associate to each decomposition the set of vertices of the sub-domain, denoted by $\mathcal{V}$.

We choose $\mathcal{M}$ a set of integers m such that the open segment $\Gamma^{k(m),j(m)}$ are two by two disjoints and

$$\mathcal{S} = \cup_{m\in\mathcal{M}} \overline{\Gamma}^{k(m),j(m)}.$$

The sides $\Gamma^{k(m),j(m)}$, $m \in \mathcal{M}$ is called mortars and denoted by γ_m. We suppose that the intersection of a sub-domain Ω^k with the boundary $\partial\Omega$ can be reduced to a vertex (see Figure 1).

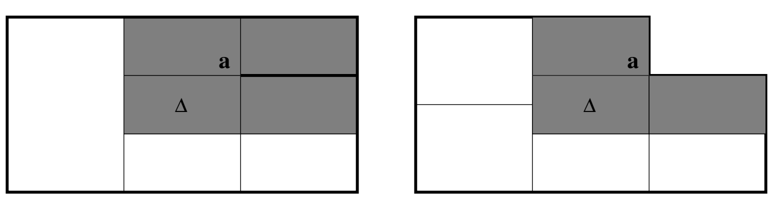

Figure 1. Domain Ω

The angles of the singular vertices are $\pi/2$, $3\pi/2$ or 2π. Thereafter we will be interested specially to the case $3\pi/2$ because of its applications in fluid mechanic (step case in Stokes flow) and to the case of 2π for its applications in mechanics (crack propagation). The local influence of the singularity allows to limit the study to one vertex. We denote **a** this vertex and ω the associated angle. To simplify the problem analysis, the sides of the sub-domains are supposed to be parallel to the axis of the scale of origin **a**. We introduce the polar coordinates (r, θ) with r the distance from a point to the vertex **a** and the line $\theta = 0$ contains a side of $\partial\Omega$.

Also we consider the following conformity assumption.

Assumption 2.1. We denote Δ the union of sub-domains containing the vertex **a**. We suppose that the decomposition of the domain Δ is conforming (see Figure 1): If **a** is a vertex of the mortar $\Gamma^{k(m),j(m)}$ which coincides with Γ^l a side of a sub-domain Ω^l, $l \neq k(m)$ then $N_{k(m)} \leq N_l$, such that the restriction of a function to Δ is in $H^2(\Delta)$.

3. Continuous problem and singular functions

Consider the homogeneous biharmonic problem

$$\begin{gathered}
\Delta^2 u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.1}$$

For $f \in H^{-2}(\Omega)$ the problem (3.1) is equivalent to the following variational formulation: Find $u \in H_0^2(\Omega)$, such that for all $v \in H_0^2(\Omega)$,

$$a(u, v) = \langle f, v \rangle, \tag{3.2}$$

where $a(u,v) = \int_\Omega \Delta u : \Delta v \, dx$ and $\langle \cdot, \cdot \rangle$ is the duality mapping between $H^{-2}(\Omega)$ and $H_0^2(\Omega)$. Since the bilinear form $a(\cdot, \cdot)$ is continuous in $H_0^2(\Omega) \times H_0^2(\Omega)$ and coercive in $H_0^2(\Omega)$, we conclude using the Lax-Milgram theorem that for $f \in H^{-2}(\Omega)$ the problem (3.2) has a unique solution $u \in H_0^2(\Omega)$ such that

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{H^{-2}(\Omega)},$$

where C is a constant independent of Ω.

Let V a neighborhood of the singular point **a** included in the domain $\overline{\Delta}$, let $s \geq 1$ and $f \in H^{s-2}(\Omega)$ then we know that the solution of problem (3.1) is written as [16, 17]

$$u = u_R + u_S, \tag{3.3}$$

where $u_R \in H^{s+2}(\Omega) \cap H_0^2(\Omega)$ and u_S is given by

$$u_S(r,\theta) = \sum_{0<\mathrm{Real}(z_k)<s+2} \lambda_k r^{1+z_k} \varphi_k(\theta) + \sum_{0<\mathrm{Real}(\hat{z}_k)<s+2} \hat{\lambda}_k r^{1+\hat{z}_k} [\sigma_k(\theta) + \ln(r) \eta_k(\theta)] \tag{3.4}$$

with λ_k and $\hat{\lambda}_k$ are real numbers, φ_k, σ_k, η_k are functions defined on a finite dimension sub-space of $C^\infty([0,\omega]) \cap H^2([0,\omega])$ (see [17] for the explicit expression of these functions) and z_k (respectively $\hat{z}_k$) are the simple (respectively double) roots of the characteristic equation of the bilaplacian

$$\sin(\omega z)^2 = z^2 \sin(\omega^2) \tag{3.5}$$

in the band, $0 < \mathrm{Real}(z) < s+2$, except 1 if $\omega \neq \tan(\omega)$, without exception if $\omega = \tan(\omega)$ which has the unique solution $\omega_e = 1.430397\pi$ in $]0, 2\pi[$.

The study of equation (3.5) shows that z is a double root if and only if $z = 0$ or $z = \pm\sqrt{\frac{1}{\sin\omega^2} - \frac{1}{\omega^2}}$. This is given by the following necessary and sufficient condition [17]

$$\sin\big(\frac{\omega^2}{\sin\omega^2} - 1\big) = \pm\sqrt{1 - \frac{\sin\omega^2}{\omega^2}}. \tag{3.6}$$

For handling the singularities we define

$$\eta(\omega) = \inf\{\mathrm{Real}(z),\ z \text{ is a solution of (3.5)},\ z \neq \pm 1\}.$$

In the case of $\omega = 3\pi/2$ we have $\eta(\omega) = 0.54484$ and $s < 1.544$. We decompose $u = u_R + \lambda S$, such that $u_R \in H^{s+2}(\Omega)$ and

$$\|u_R\|_{H^{s+2}(\Omega)} + |\lambda| \le C\|f\|_{H^{s-2}(\Omega)},$$

where

- $$S(r,\theta) = r^{1+\eta(\omega)}\varphi(\theta) \tag{3.7}$$
  with $\varphi(\theta) = 2.093\Big(\cos(0.459\,\theta) - \cos(1.544\,\theta)\Big) + 1.093\Big(2.193\sin(0.459\,\theta) - \sin(1.544\,\theta)\Big)$,
- λ is the first singular coefficient of the singularity S.

Furthermore, if $f \in H^{s-2}(\Omega)$, with $s < 2.908$, we can again decompose the singular part as follows

$$u = \tilde{u}_R + \lambda S + \tilde{\lambda}\tilde{S}, \tag{3.8}$$

where

- $\tilde{u}_R \in H^{s+2}(\Omega)$,
- $$\tilde{S}(r,\theta) = r^{1+z_2}\psi(\theta), \tag{3.9}$$
  with z_2 is the second solution of equation (3.5) in the band $0 < \mathrm{Real}(z) < 1$ ($z_2 \simeq 0.908529$) and $\psi(\theta) = 4.302\big(\cos(0.092\theta) - \cos(1.908\theta)\big) - 1.815\big(10.869\sin(0.092\theta) - 0.524\sin(1.908\theta)\big)$,
- $\tilde{\lambda}$ is the coefficient of the second singularity $\tilde{S}$ satisfying

$$\|\tilde{u}_R\|_{H^{s+2}(\Omega)} + |\lambda| + |\tilde{\lambda}| \le C\|f\|_{H^{s-2}(\Omega)}.$$

when $\omega = 2\pi$, we have $\eta(\omega) = 0.5$ and $s < 1.5$. If f belongs to $H^{s-2}(\Omega)$, then u belongs to the space $H^{s+2}(\Omega)$. We decompose

$$u = u_R + \lambda S + \tilde{\lambda}\tilde{S},$$

where

$$\begin{aligned} S(r,\theta) &= r^{3/2}\big((\sin(3\theta/2) - 3\sin(\theta/2)) + (\cos(3\theta/2) - \cos(\theta/2))\big), \\ \tilde{S}(r,\theta) &= r^{5/2}\big((\cos(5\theta/2) - 5\sin(\theta/2)) + (\cos(5\theta/2) - \cos(\theta/2))\big), \end{aligned}$$

and $(\lambda, \tilde{\lambda})$ is the singular coefficient associated to the singular function $(S, \tilde{S})$. If f belongs to $H^{s-2}(\Omega)$, u_R belongs to $H^{s+2}(\Omega)$ for $s < 2,5$. We have the following stability condition:

$$\|u_R\|_{H^{s+2}(\Omega)} + |\lambda| + |\tilde{\lambda}| \le C\|f\|_{H^{s-2}(\Omega)}.$$

4. Discrete problem

Firstly, we recall the space of mortar functions. As the considered problem is posed in $H^2(\Omega)$, two matching conditions are necessary on each interface; one for the trace of the function and the other for its normal derivative.

We introduce $\delta = (N_k)_{1\le k\le K}$, a strictly positive sequence of integers. δ is called parameter of discretization and (N_k), $1 \le k \le K$, are the degrees of polynomials in each sub-domain. ($\mathbb{P}_n(\Omega)$ is the space of polynomial functions of degree less than or equal to n).

The mortar method requires the introduction of a space of functions, which we call mortar functions. These are defined on the skeleton and ensure the matching of the locally approximation functions. The space of mortar functions is then defined by

$$W^\delta = \Big\{(\varphi_0, \varphi_1); \varphi_0/_{\gamma^m} = v_\delta/_{\Gamma^{k(m),j(m)}} \text{ and } \varphi_1/_{\gamma^m} = (\frac{\partial v_\delta}{\partial n})/_{\Gamma^{k(m),j(m)}}\ \forall m \in \mathcal{M}\Big\}.$$

where v_δ is a test function.

We propose a discretization by the Galerkin method with numerical integration. In the case of the problem of order four, it is more appropriate to use a quadrature formula which takes into account the values of the function on the boundary. The following lemma defines this quadrature formula (see [11] for a proof).

Lemma 4.1. *Let $N \ge 2$ be an integer. Then there exists a unique set of points ξ_j, $1 \le j \le N-1$, a unique set of positive reals ρ_j, $1 \le j \le N-1$, ρ_+, ρ_- such that for all polynomials φ in $\mathbb{P}_{2N-1}(]-1,1[)$*

$$\int_{-1}^{1} \varphi(x)\,dx = \sum_{j=1}^{N-1} \varphi(\xi_j)\rho_j + \varphi(-1)\rho_- + \varphi(1)\rho_+\,. \tag{4.1}$$

Remark 4.2. The nodes ξ_j; $1 \le j \le N-1$, are the zeros of the derivative of the Legendre polynomial L_N. We refer to [11] for the calculation of ξ_j and ρ_j, $1 \le j \le N-1$.

Given two functions u, v continuous on $\overline{\Omega} = [-1,1] \times [-1,1]$ and vanishing on its boundary, we define the discrete scalar product

$$(u,v)_N = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} u(\xi_i,\xi_j)v(\xi_i,\xi_j)\rho_i\rho_j.$$

If T^k is the bijection from $]-1,1[^2$ in Ω_k, we define

$$(u,v)_{N_k} = \frac{|\Omega_k|}{4}\sum_{i=1}^{N_k-1}\sum_{j=1}^{N_k-1} (u\circ T^k)(\xi_i,\xi_j)(v\circ T^k)(\xi_i,\xi_j)\rho_i\rho_j.$$

Thus, we define the space of approximation X_δ as the space of functions v_δ such that

- for all k, $1 \le k \le K$, $v_\delta/_{\Omega^k} \in \mathbb{P}_{N_k}(\Omega^k)$,
- v_δ and $\frac{\partial v_\delta}{\partial n}$ vanishes on $\partial\Omega$,
- there exist a couple $(\varphi_0,\varphi_1) \in W_\delta$ such that, for all $1 \le k \le K$, $1 \le j \le 4$, and all $\psi \in \mathbb{P}_{N_k-4(\Gamma^{k,j})}$, $\int_{\Gamma^{k,j}} (v_\delta - \varphi_0)(\tau)\psi(\tau)\,d\tau = 0$ and $\int_{\Gamma^{k,j}} (\frac{\partial v_\delta}{\partial n} - \varphi_1)(\tau)\psi(\tau)\,d\tau = 0$.

Finally the discrete problem is written: For $f \in \mathcal{C}(\overline{\Omega})$, find $u_\delta \in X_\delta$ such that for all $v_\delta \in X_\delta$,

$$a_\delta(u_\delta, v_\delta) = (f, v)_\delta,$$

where $a_\delta(u_\delta, v_\delta) = \sum_{k=1}^{K} (\Delta u_\delta^k, \Delta v_\delta^k)_{N_k}$ and $(f, v_\delta)_\delta = \sum_{k=1}^{K} (f, v_\delta^k)_{N_k}$. We refer to [8] for the a priori analysis of this problem and its implementation by the mortar spectral element method.

5. Strang and Fix algorithm

The Strang and Fix algorithm [19] consists of the enlargement of the discrete space X_δ as

$$X_\delta^* = X_\delta + \mathbb{R}S,$$

where S is the first singular function. We have, then $u_\delta^* = u_\delta + \lambda S$ and $v_\delta^* = v_\delta + \mu S$ in X_δ^*,

$$\begin{aligned} a_\delta^*(u_\delta^*, v_\delta^*) = \sum_{k=1}^{K} \Big[&(\Delta u_\delta^k, \Delta v_\delta^k)_{N_k} + \lambda \int_{\Omega_k} \Delta v_\delta^k \Delta S \, dx \\ &+ \mu \int_{\Omega_k} \Delta u_\delta^k \Delta S \, dx + \lambda\mu \int_{\Omega_k} (\Delta S)^2 \, dx \Big]. \end{aligned}$$

The discrete problem becomes: Find $u_\delta^* \in X_\delta^*$ such that

$$\forall v_\delta^* \in X_\delta^*, \quad a_\delta(u_\delta^*, v_\delta^*) = \sum_{k=1}^{K} \int_{\Omega_k} f v_{\delta k}^* \, dx, \tag{5.1}$$

where $v_{\delta k}^*$ is the restriction of v_δ^* to sub-domain Ω_k.

For the analysis of this problem, we introduce the following two norms on X_δ^*,

$$\|u_\delta^*\|_{*1} = \sum_{k=1}^{K} \Big(\|u_\delta^k\|_{H^2(\Omega_k)}^2 + |\lambda|^2 \|S/_{\Omega_k}\|_{H^2(\Omega_k)}^2 \Big)^{1/2}$$

and

$$\|u_\delta^*\|_{*2} = \Big(\sum_{k=1}^{K} \|u_\delta^*\|_{H^2(\Omega_k)}^2 \Big)^{1/2}.$$

Proposition 5.1. *Let* $\eta(\delta) = \sup_{u_\delta^* \in X_\delta^*} \frac{\|u_\delta^*\|_{*2}}{\|u_\delta^*\|_{*1}}$. *Then* $\lim_{\delta \to +\infty} \eta(\delta) = 0$.

Proof. Because of the conformity of the decomposition, we consider the proof in the domain Δ. Let $N_\Delta = \min_{\Omega_k \subset \Delta}(N_k)$, $\eta(N_\Delta)$ is the sine of the angle between the space X_δ and the singular function S. Then

$$\eta(N_\Delta)^2 = 1 - \Big(\sup_{u_{N_\Delta} \in \mathbb{P}_{N_\Delta}(\Delta)} \frac{(v_{N_\Delta}, S)}{\|u_{N_\Delta}\|_{H^2(\Delta)} \|S\|_{H^2(\Delta)}} \Big)^2;$$

$(\cdot, \cdot)$ is the scalar product on $H^2(\Delta)$. If we consider $\Pi_{N_\Delta} : L^2(\Delta) \to \mathbb{P}_{N_\Delta}(\Delta)$, we conclude that

$$\eta(N_\Delta)^2 = 1 - \Big(\frac{(\Pi_{N_\Delta} S, S)}{\|\Pi_{N_\Delta} S\|_{H^2(\Delta)} \|S\|_{H^2(\Delta)}} \Big)^2.$$

Let

$$\frac{(\Pi_{N_\Delta} S, S)}{\|\Pi_{N_\Delta} S\|_{H^2(\Delta)} \|S\|_{H^2(\Delta)}} = \frac{(\Pi_{N_\Delta} S - S, S)}{\|\Pi_{N_\Delta} S\|_{H^2(\Delta)} \|S\|_{H^2(\Delta)}} + \frac{(S, S)}{\|\Pi_{N_\Delta} S\|_{H^2(\Delta)} \|S\|_{H^2(\Delta)}}.$$

We conclude that $\eta(N_\Delta)$ has the same order as $\left(\|\Pi_{N_\Delta} S - S\|_{H^2(\Delta)}\right)^{1/2}$ which is $N^{-\pi/w}$ [10]. This completes the proof. $\square$

To study problem (5.1), we begin by giving the properties of the bilinear form $a^*_\delta(\cdot,\cdot)$ in the following proposition.

Proposition 5.2. *There exist two positive functions C_1 and C_2 independent of δ such that for all u^*_δ, v^*_δ in X^*_δ,*

$$|a^*_\delta(u^*_\delta, v^*_\delta)| \le C_1 \|u^*_\delta\|_{*1} \|v^*_\delta\|_{*1}, \tag{5.2}$$

$$a^*_\delta(u^*_\delta, v^*_\delta) \ge C_2 \|u^*_\delta\|^2_{*2}. \tag{5.3}$$

Proof. Consider $\Omega_k =]a_k, b_k[\times]c_k, d_k[$. For u^*_δ and v^*_δ in X^*_δ, we have

$$\begin{aligned} a^*_\delta(u^*_\delta, v^*_\delta) = \sum_{k=1}^{K} \Big[&(\Delta u^k_\delta, \Delta v^k_\delta)_{N_k} + \lambda \int_k \Delta S\, \Delta v^k_\delta \,dx\,dy + \mu \int_{\Omega_k} \Delta u^k_\delta \Delta S \,dx\,dy \\ &+ \lambda\mu \int_{\Omega_k} (\Delta S)^2 \,dx\,dy \Big]. \end{aligned}$$

where

$$\begin{aligned} &(\Delta u^k_\delta, \Delta v^k_\delta)_{N_k} \\ &= \sum_{i=1}^{N_K-1} \sum_{j=1}^{N_K-1} \Big[\frac{\partial^2 u^k_\delta}{\partial x^2} \frac{\partial^2 v^k_\delta}{\partial x^2} + \frac{\partial^2 u^k_\delta}{\partial x^2} \frac{\partial^2 v^k_\delta}{\partial y^2} + \frac{\partial^2 u^k_\delta}{\partial y^2} \frac{\partial^2 v^k_\delta}{\partial x^2} + \frac{\partial^2 u^k_\delta}{\partial y^2} \frac{\partial^2 v^k_\delta}{\partial y^2} \Big] (\xi^k_i, \xi^k_j) \rho_i \rho_j. \end{aligned}$$

The terms $\frac{\partial^2 u^k_\delta}{\partial x^2} \frac{\partial^2 v^k_\delta}{\partial x^2}$, $\frac{\partial^2 u^k_\delta}{\partial x^2} \frac{\partial^2 v^k_\delta}{\partial y^2}$, $\frac{\partial^2 u^k_\delta}{\partial y^2} \frac{\partial^2 v^k_\delta}{\partial x^2}$ and $\frac{\partial^2 u^k_\delta}{\partial y^2} \frac{\partial^2 v^k_\delta}{\partial y^2}$ are the polynomials of degree less or equal to $2N_k - 1$ with respect to x and y respectively.

Using the exactness of the quadrature formula, the Cauchy-Schwartz inequality and $a \cdot b \le \frac{a^2+b^2}{2}$ we obtain

$$\begin{aligned} (\Delta u^k_\delta, \Delta v^k_\delta) N_k \le & \int_{a_k}^{b_k} \Big(\sum_{j=1}^{N_k-1} \frac{\partial^2 u^k_\delta}{\partial x^2}(x,\xi_j)^2 \rho_j \Big)^{1/2} \Big(\sum_{j=1}^{N_k-1} \frac{\partial^2 v^k_\delta}{\partial x^2}(x,\xi_j)^2 \rho_j \Big)^{1/2} dx \\ &+ \int_{c_k}^{d_k} \Big(\sum_{i=1}^{N_k-1} \frac{\partial^2 u^k_\delta}{\partial y^2}(\xi_i,y)^2 \rho_i \Big)^{1/2} \Big(\sum_{i=1}^{N_k-1} \frac{\partial^2 v^k_\delta}{\partial y^2}(\xi_i,y)^2 \rho_i \Big)^{1/2} dy \\ &+ \frac{1}{2} \int_{a_k}^{b_k} \Big(\sum_{j=1}^{N_k-1} \frac{\partial^2 u^k_\delta}{\partial x^2}(x,\xi_j)^2 \rho_j + \frac{\partial^2 v^k_\delta}{\partial x^2}(x,\xi_j)^2 \rho_j \Big) dx \\ &+ \frac{1}{2} \int_{c_k}^{d_k} \Big(\sum_{i=1}^{N_k-1} \frac{\partial^2 u^k_\delta}{\partial y^2}(\xi_i,y)^2 \rho_i + \frac{\partial^2 v^k_\delta}{\partial y^2}(\xi_i,y)^2 \rho_i \Big) dy. \end{aligned}$$

Using that for all $\varphi_N \in \mathbb{P}_N(\Lambda)$,

$$\|\varphi_N\|_{L^2(\Lambda)} \le (\varphi_N, \varphi_N)_N \le C \|\varphi_N\|_{L^2(\Lambda)},$$

where C is a constant independent of N [12], we deduce (5.2).

For the ellipticity proof, we have

$$a_\delta(u^*_\delta, v^*_\delta) \ge \sum_{k=1}^{K} \|\Delta u^k_\delta\|^2_{L^2(\Omega_k)} + \lambda^2 \|\Delta S\|^2_{L^2(\Omega_k)} + 2\lambda \int_{\Omega_k} \Delta u^k \Delta S \,dx$$

$$\geq \sum_{k=1}^{K} \|\Delta u^*_{\delta k}\|^2_{L^2(\Omega_k)}.$$

We distinguish the two cases $\Omega\backslash\overline{\Delta}$ and Δ.

(1) If $\Omega_k \subset \Delta$; since the functions X^*_δ and their normal derivatives vanish on $\partial\Delta$ and using the conformity hypothesis it is therefore sufficient to show that

$$\sum_{\Omega_k\subset\Delta} \|\Delta u^*_{\delta k}\|^2_{L^2(\Omega_k)} \geq C \sum_{\Omega_k\subset\Delta} |u^*_{\delta k}|_{H^2(\Omega_k)}. \tag{5.4}$$

It suffices to handle the terms of the cross product, using Green formula

$$\begin{aligned}\int_{\Omega_k} \frac{\partial^2 u^*_{\delta k}}{\partial x^2}\frac{\partial^2 u^*_{\delta k}}{\partial y^2}\,dx = \int_{\Omega_k}\Big(\frac{\partial^2 u^*_{\delta k}}{\partial x\partial y}\Big)^2 - \int_{\partial\Omega_k} \frac{\partial u^*_{\delta k}}{\partial x}\frac{\partial^2 u^*_{\delta k}}{\partial x\partial y} n^k_y\,d\tau \\ + \int_{\partial\Omega_k} \frac{\partial u^*_{\delta k}}{\partial x}\frac{\partial^2 u^*_{\delta k}}{\partial y^2} n^k_x\,d\tau.\end{aligned} \tag{5.5}$$

The sum on the mortar of Δ reveals the jumps $[\frac{\partial u^*_{\delta k}}{\partial x}\frac{\partial^2 u^*_{\delta k}}{\partial x\partial y}]$ and $[\frac{\partial u^*_{\delta k}}{\partial x}\frac{\partial^2 u^*_{\delta k}}{\partial y^2}]$ on the interfaces terms. Since S is continuous as well as its normal derivative on the interfaces, these jumps are reduced to $[\frac{\partial u^*_{\delta k}}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial x\partial y}]$ and $[\frac{\partial u^*_{\delta k}}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial y^2}]$.

These terms are then written: $[\frac{\partial u_{\delta k}}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial x\partial y}+\lambda\frac{\partial S}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial x\partial y}]$ and $[\frac{\partial u_{\delta k}}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial y^2}+\lambda\frac{\partial S}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial y^2}]$.

The integral of $[\frac{\partial u_{\delta k}}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial x\partial y}]$ and $[\frac{\partial u_{\delta k}}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial y^2}]$ vanishes (see [8]).

We also show that the integral of these terms $[\frac{\partial S}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial x\partial y}]$ and $[\frac{\partial S}{\partial x}\frac{\partial^2 u_{\delta k}}{\partial y^2}]$ vanishes since $\frac{\partial S}{\partial x} = \sum_{n\geq 0}\alpha_n L_n(x)$. Then, the sum on the sub-domain Δ in (5.5) no longer counts jump terms which gives (5.4).

(2) If $\Omega_k \subset \Omega\backslash\overline{\Delta}$, the restriction of the functions from X^*_δ to $\Omega\backslash\overline{\Delta}$ coincides with that of X_δ and we conclude (see [8])

$$\sum_{\Omega_k\subset\Omega\backslash\overline{\Delta}} \|\Delta u^*_{\delta k}\|^2_{L^2(\Omega_k)} \geq C \sum_{\Omega_k\subset\Omega\backslash\overline{\Delta}} \|u^*_{\delta k}\|^2_{H^2(\Omega_k)}. \tag{5.6}$$

Then from inequalities (5.4) and (5.5) we have that for all $v^*_\delta \in X^*_\delta$,

$$\sum_{k=1}^{K} \|\Delta u^*_{\delta k}\|^2_{L^2(\Omega_k)} \geq C\|u^*_{\delta k}\|^2_{*2}.$$

Hence we obtain the ellipticity of $a^*_\delta(\cdot,\cdot)$. □

Proposition 5.3. *For $f \in L^2(\Omega)$, the discrete problem* (5.1) *has a unique solution u^*_δ in X^*_Δ and*

$$\|u^*_\delta\|_{2*} \leq C\|f\|_{L^2(\Omega)}.$$

Remark 5.4. The norms $\|\cdot\|_{1*}$ and $\|\cdot\|_{2*}$ are equivalent with a constant depending on the discretization parameter δ. In the following we will use the norm $\|\cdot\|_{1*}$ and we will show an inf-sup condition on the bilinear form $a^*_\delta(\cdot,\cdot)$ using this norm.

Proposition 5.5. *There exists a constant α such that for all $v^*_\delta \in X^*_\delta$,*

$$\sup_{t^*_\delta\in X^*_\delta} \frac{a^*_\delta(v^*_\delta, t^*_\delta)}{\|t^*_\delta\|_{1*}} \geq \alpha\|v^*_\delta\|_{1*}. \tag{5.7}$$

Proof. Consider $t_\delta^* = v_\delta + \beta(\lambda S)$ and find a value for β which satisfies inequality (5.7),

$$\begin{aligned}
&a_\delta^*(v_\delta^*, t_\delta^*)\\
&= a_\delta^*(v_\delta + \lambda S, v_\delta + \beta(\lambda S))\\
&\geq \sum_{k=1}^{K} \Big[\int_{\Omega_k} (\Delta v_{\delta k})^2\,dx + |\lambda|(1+\beta)\int_{\Omega_k} \Delta S \Delta v_{\delta k}\,dx + \beta^2\lambda^2 \int_{\Omega_k} \Delta S^2\,dx\Big]\\
&\geq \sum_{k=1}^{K} \Big[\|\Delta v_{\delta k}\|^2_{L^2(\Omega_k)} - |\lambda|(1+\beta)\|\Delta S\|_{L^2(\Omega_k)}\|\Delta v_{\delta k}\|_{L^2(\Omega_k)} + \beta^2\lambda^2\|\Delta S\|^2_{L^2(\Omega_k)}\Big]\\
&\geq \sum_{k=1}^{K} \Big[\frac{1}{2}\|\Delta v_{\delta k}\|^2_{L^2(\Omega_k)} + |\lambda|\Big(\beta^2 - \frac{(\beta+1)^2}{2}\Big)\|\Delta S\|^2_{L^2(\Omega_k)}\Big]
\end{aligned}$$

Using Young's inequality $ab \leq \frac{a^2+b^2}{2}$ and choosing $\beta = 3$, we complete the proof. □

Using inequality (5.7) and the Strang lemma we obtain the following result.

Proposition 5.6. *The error estimate between u the solution of problem* (3.1) *and u_{δ^*} the solution of problem* (5.1) *is*

$$\begin{aligned}
&\|u - u_{\delta^*}\|_{1*}\\
&\leq C\Big[\inf_{v_\delta^* \in X_\delta^*} \Big(\|u - v_\delta^*\|_{1*} + \sup_{\omega_\delta^* \in X_\delta^*} \frac{a(v_\delta^*, \omega_\delta^*) - a_\delta^*(v_\delta^*, \omega_\delta^*)}{\|\omega_\delta^*\|_{1*}}\Big)\\
&\quad \times \sup_{\omega_\delta^* \in X_\delta^*} \frac{\sum_{k=1}^{K}\sum_{l=k+1}^{K} \big(\int_{\gamma_{kl}} \frac{\partial(\Delta u)}{\partial n}[\omega_\delta^*]\,dx - \int_{\gamma_{kl}} \Delta u[\frac{\partial \omega_\delta^*}{\partial n}]\,dx\big)}{\|\omega_\delta^*\|_{1*}}\Big],
\end{aligned} \tag{5.8}$$

where n and $[\omega]$ are respectively the normal and the jump of ω on the interfaces.

To find the order of convergence, we have to estimate each term of the inequality (5.8). Recall that the singular function S is of class $\mathcal{C}^1$, then the jump terms $(\omega_{\delta k}^* - \omega_{\delta l}^*)$ and $\big(\frac{\partial \omega_{\delta k}^*}{\partial n} - \frac{\partial \omega_{\delta l}^*}{\partial n}\big)$ through each interface γ_{kl} are reduced to $(\omega_{\delta k} - \omega_{\delta l})$ and $\big(\frac{\partial \omega_{\delta k}}{\partial n} - \frac{\partial \omega_{\delta l}}{\partial n}\big)$.

The conformity hypothesis on Δ implies that these quantities vanish. Moreover u and u_R coincide on $\Omega\backslash\overline{\Delta}$; the consistence error term is then written on each interface γ_{kl},

$$\begin{aligned}
&\int_{\gamma_{kl}} \frac{\partial(\Delta u)}{\partial n}[\omega_\delta]\,dx + \int_{\gamma_{kl}} \Delta u[\frac{\partial \omega_\delta}{\partial n}]\,dx\\
&= \int_{\gamma_{kl}} \frac{\partial(\Delta u_R)}{\partial n}(\varphi_0 - \omega_{\delta k})\,dx + \int_{\gamma_{kl}} \frac{\partial(\Delta u_R)}{\partial n}(\varphi_0 - \omega_{\delta l})\,dx\\
&\quad + \int_{\gamma_{kl}} (\Delta u_R)\big(\varphi_1 - \frac{\partial \omega_{\delta k}}{\partial n}\big)\,dx + \int_{\gamma_{kl}} (\Delta u_R)\big(\varphi_1 - \frac{\partial \omega_{\delta l}}{\partial n}\big)\,dx
\end{aligned}$$

where φ_0 and φ_1 are the mortar functions associated with $(\omega_\delta, \frac{\partial \omega_\delta}{\partial n})$. Then we obtain, [8],

$$\begin{aligned} &\sum_{k=1}^{K}\sum_{l=k+1}^{K}\int_{\gamma_{kl}}\frac{\partial(\Delta u_R)}{\partial n}[\omega_\delta]\,dx+\int_{\gamma_{kl}}\Delta u_R[\frac{\partial\omega_\delta}{\partial n}]\,dx\\ &\le c\sum_{k=1}^{K}\sum_{j=1}^{4}\Big(\inf_{\psi_{kj}\in\mathbb{P}_{N_k-4}(\Gamma^{kj})}\|\frac{\partial(\Delta u_R)}{\partial n}-\psi_{kj}\|_{(H^{3/2}(\Gamma^{kj}))'} \\ &\quad+\inf_{\psi_{kj}\in\mathbb{P}_{N_k-4}(\Gamma^{kj})}\|\Delta u_R-\psi_{kj}\|_{(H^{1/2}(\Gamma^{kj}))'}\Big). \end{aligned} \tag{5.9}$$

Following the definition of X_δ^* and (3.3), we have

$$\inf_{v_\delta^*\in X_\delta^*}\|u-v_\delta^*\|_{1*}\le C\inf_{v_\delta\in X_\delta^-}\|u_R-v_\delta\|_{1*}$$

where

$$X_\delta^-=\{v_\delta\in X_\delta; v_{\delta k}\in\mathbb{P}_{N-1}(\Omega_k)\}.$$

Finally the term

$$\sup_{\omega_\delta^*\in X_\delta^*}\frac{a(v_\delta^*,\omega_\delta^*)-a_\delta^*(v_\delta^*,\omega_\delta^*)}{\|\omega_\delta^*\|_{1*}}$$

vanishes if we choose $v_\delta^*=v_\delta\in X_\delta^-$ following the exactness of the quadrature formula (4.1).
Doing the sum of these results, we obtain

$$\begin{aligned} &\|u-u_\delta^*\|_{1*}\\ &\le C\Big[\inf_{v_\delta\in X_\delta^-}\|u-v_\delta\|_{1*}+\sum_{k=1}^{K}\sum_{j=1}^{4}\Big(\inf_{\psi_{kj}\in\mathbb{P}_{N_k-4}(\Gamma^{kj})}\|\frac{\partial\Delta u_R}{\partial n}-\psi_{kj}\|_{(H^{3/2}(\Gamma^{kj}))'}\\ &\quad+\inf_{\psi_{kj}\in\mathbb{P}_{N_k-4}(\Gamma^{kj})}\|\Delta u_R-\psi_{kj}\|_{(H^{1/2}(\Gamma^{kj}))'}\Big)\Big]. \end{aligned} \tag{5.10}$$

Suppose f in $H^{s-2}(\Omega)$ for $\eta(\omega)<s<\eta(\omega)+2$, then $u_R\in H^{s+2}(\Omega)$ and the trace (respectively the normal derivative trace) of u_R belongs to $H^{s-\frac12}(\partial\Omega_k)$ (respectively $H^{s-\frac32}(\partial\Omega_k)$); $1\le k\le K$. Taking ψ_{kj} (respectively χ_{kj}) the orthogonal projection on $\mathbb{P}_{N_k-4}(\Gamma^{kj})$, we deduce

$$\begin{gathered} \|\Delta u_R-\psi_{kj}\|_{(H^{1/2}(\Gamma^{kj}))'}\le C\,N_k^{-s}\|u_R\|_{H^{s+2}(\Omega_k)},\\ \|\frac{\partial\Delta u_R}{\partial n}-\chi_{kj}\|_{H^{-3/2}(\Gamma^{kj})}\le C\,N_k^{-s}\|u_R\|_{H^{s+2}(\Omega_k)}. \end{gathered}$$

Furthermore, we have

$$\inf_{v_\delta\in X_\delta^-}\|u-v_\delta\|_{1*}\le C\sum_{k=1}^{K}N_k^{-s}\|u_R\|_{H^{s+2}(\Omega_k)}.$$

Suppose that $f\in H^{s-2}(\Omega)$ with $s<2+\eta_1(\omega)$ where $\eta_1(\omega)$ is the second real solution of the equation (3.5), in the band $0<\mathrm{Real}(z)<s$, then from the decomposition (3.8) and Assumption 2.1, we show exactly in the same way that

$$\|u-u_\delta^*\|_{1*}$$

$$\leq C\Big[\inf_{v_\delta\in X_\delta^-}\|u_R-v_\delta\|_{1*}+\sum_{k=1}^{K}\sum_{j=1}^{4}\Big(\inf_{\psi_{kj}\in\mathbb{P}_{N_k-4}(\Gamma^{kj})}\|\frac{\partial\Delta\tilde u_R}{\partial n}-\psi_{kj}\|_{(H^{3/2}(\Gamma^{kj}))'}$$
$$+\inf_{\chi_{kj}\in\mathbb{P}_{N_k-4}(\Gamma^{kj})}\|\Delta\tilde u_R-\chi_{kj}\|_{(H^{1/2}(\Gamma^{kj}))'}\Big)\Big].$$

We note that

$$\inf_{v_\delta\in X_\delta^-}\|u_R-v_\delta\|_{1*}\leq C\Big\{\inf_{v_\delta\in X_\delta^-}\|\tilde u_R-v_\delta\|_{1*}+|\tilde\lambda|\inf_{v_\delta\in X_\delta^-}\|\tilde S-v_\delta\|_{1*}\Big\}.$$

Using the approximation result of the singular functions by polynomials [10] we have

$$\inf_{v_\delta\in X_\delta^-}\|\tilde S-v_\delta\|_{1*}\leq CN^{\varepsilon-2\eta_1(\omega)}\quad\forall\varepsilon>0.$$

Then

$$\inf_{v_\delta\in X_\delta^-}\|u_R-v_\delta\|_{1*}\leq CN^{2-s}(\|\tilde u_R\|_{H^s(\Omega)}+|\tilde\lambda|),$$

hence

$$\|u-u_\delta^*\|_{1*}\leq CN^{2-s}\|f\|_{H^{s-2}(\Omega)}\quad\text{for } s<2+\eta_1(\omega).$$

Combining these results we have the following theorem.

Theorem 5.7. *If $f\in H^{s-2}(\Omega)$ for $s>0$ and $\varepsilon>0$ then*

$$\|u-u_\delta^*\|_{1*}\leq C\Big(\sum_{k=1}^{K}N_k^{-\sigma_k}\Big)\|f\|_{H^{s-2}(\Omega)}$$

where σ_k, $1\leq k\leq K$ satisfies

$$\sigma_k=\begin{cases} s-2 & \text{if } \overline{\Omega}_k \text{ does not contain any vertices of } \Omega,\\ \inf(s-2,2\eta_1(\pi/2)-\varepsilon) & \text{if } \overline{\Omega}_k \text{ contains a vertex of } \Omega \text{ other than } \mathbf{a},\\ \inf(s-2,2\eta_1(\omega)-\varepsilon) & \text{if } \overline{\Omega}_k \text{ contains } \mathbf{a}.\end{cases}\tag{5.11}$$

Using the Aubin-Nische duality we have the following corollary.

Corollary 5.8. *Let f in $H^{s-2}(\Omega)$, for $s>0$, then, for all $\epsilon>0$,*

$$\|u-u_\delta^*\|_{L^2(\Omega)}\leq C\Big(N^{-2}(\sum_{k=1}^{K}N_k^{-\sigma_k})\Big)\|f\|_{H^{s-2}(\Omega)}$$

where σ_k satisfies (5.11) and $N=\inf_{1\leq k\leq K}N_k$.

Conclusion. We studied the biharmonic problem with homogeneous boundary conditions in a domain of $\mathbb{R}^2$ with corners. The discrete problem was studied using the mortar spectral element method. We showed that if we consider the decomposition of the solution in a regular part and a singular one, we improve the order of the error. Using the Strang and Fix algorithm, which consists on adding the singular function in the discrete space, we prove an optimal order of the error on the solution. The numerical implementation of the obtained results will be presented in a forthcoming work. The extension of this discretization to the three dimension axi-symmetric domain is presently under consideration.

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Mohamed Abdelwahed
Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia
E-mail address: `mabdelwahed@ksu.edu.sa`

Nejmeddine Chorfi
Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia
E-mail address: `nchorfi@ksu.edu.sa`

Vicenţiu Rădulescu
Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania.
Department of Mathematics, University of Craiova, 200585 Craiova, Romania
E-mail address: vicentiu.radulescu@imar.ro