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IMPROVED OSCILLATION CONDITIONS FOR THIRD-ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS

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Abstract. In this article, we study the oscillatory behavior of the third-order neutral type difference equation

$$\Delta(a_n(\Delta^2(x_n + p_n x_{n-k}))^\alpha) + q_n f(x_{n-l}) = 0$$

where $\alpha > 0$, $a_n > 0$, $q_n \geq 0$ and $0 \leq p_n \leq p < \infty$. By using generalized Ricatti type transformation we present some new criteria which ensure that every solution is oscillatory. Also we provide examples that illustrate the importance of our results.

1. Introduction

This article concerns the oscillatory behavior of solutions of the third-order neutral type difference equation

$$\Delta(a_n(\Delta^2(x_n + p_n x_{n-k}))^\alpha) + q_n f(x_{n-l}) = 0, \tag{1.1}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, n_0 is a nonnegative integer, subject to the following conditions:

- (H1) $\{a_n\}$ is a positive real sequence with $\sum_{n=n_0}^{\infty} 1/a_n^{1/\alpha} = \infty$;
- (H2) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, and $0 \leq p_n \leq p < \infty$;
- (H3) $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ and $f(u)/u^\alpha \geq M > 0$ for all $u \neq 0$;
- (H4) α is a ratio of odd positive integers, and k and l are nonnegative integers.

Let $\theta = \max\{k, l\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfies equation (1.1) for all $n \geq \mathbb{N}(n_0)$. A nontrivial solution of equation (1.1) is said to be oscillatory if the terms of the sequence $\{x_n\}$ are neither eventually all positive nor eventually all negative, and nonoscillatory otherwise.

The problem of determining oscillation criteria for neutral type difference equations have been receiving great attention in the last few decades since these type of equations arise in the study of economics, mathematical biology, and many other areas of mathematics, see for example [1, 4, 5, 8].

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In [12], the authors considered the third-order neutral difference equation

$$\Delta(c_n(\Delta(d_n\Delta(x_n+p_nx_{n-\tau})))) + q_nf(x_{n-\sigma}) = 0,\ n\in\mathbb{N}(n_0), \tag{1.2}$$

and studied the oscillatory and asymptotic behavior of solutions of (1.2) subject to the conditions

$$\Delta c_n \ge 0, \quad \sum_{n=n_0}^{\infty}\frac{1}{c_n} = \sum_{n=n_0}^{\infty}\frac{1}{d_n} = \infty, \quad 0\le p_n<1. \tag{1.3}$$

In [10], the authors considered the equation

$$\Delta(c_n(\Delta(d_n\Delta(x_n+p_nx_{n-\tau})))^{\alpha}) + q_nf(x_{n-\sigma}) = 0, \quad n\in\mathbb{N}(n_0), \tag{1.4}$$

and established conditions for the oscillation and asymptotic behavior of all solutions under condition (1.3) without assuming $\Delta c_n \ge 0$.

In [16, 15], the authors considered the equation

$$\Delta(c_n(\Delta^2(x_n+p_nx_{n-\delta}))^{\alpha}) + q_nx^{\alpha}_{n+1-\tau} = 0, \quad n\in\mathbb{N}(n_0), \tag{1.5}$$

and established sufficient conditions for the oscillation and asymptotic behavior of all solutions under condition (1.3).

In [14], the authors considered equation (1.5), and established sufficient conditions for the oscillation and asymptotic behavior of all solutions under the condition

$$\sum_{n=n_0}^{\infty}\frac{1}{c_n^{1/\alpha}} = \infty, \quad 0\le p_n\le p<\infty.$$

For further results concerning the oscillatory and asymptotic behavior of third-order difference equations, one can refer to [2, 3, 9, 5] and the references cited therein.

From a review of literature it is found that all the results established in [14, 16, 10, 12, 15] for neutral type difference equations are guarantee that every solution is either oscillatory or tends to zero monotonically, and to the best of our knowledge there are no results in the literature which ensure that all solutions are just oscillatory for the third order neutral type difference equations. Therefore the purpose of this paper is to present some new oscillation criteria for equation (1.1) which ensure that all solutions are oscillatory. Thus, the results obtained in this paper improve those in [10, 12, 14, 15, 16].

This article is organized as follows. In Section 2, we present the main results and in Section 3, we provide some examples to illustrate the importance of the main results.

2. Oscillation theorems

In this section, we obtain some sufficient conditions for the oscillation of all solutions of (1.1). We may deal only with the positive solutions of equation (1.1) since the proof for the negative case is similar. We also introduce a usual convention, namely, for the sequence $\{f_n\}$ and any $m\in\mathbb{N}(n_0)$ we put $\sum_{n=m}^{m-1}f_n = 0$ and $\prod_{n=m}^{m-1}f_n = 1$.

We begin with some lemmas that will be used to prove our main results. In the following, for convenience we denote

$$z_n = x_n + p_nx_{n-k}, \quad \text{and} \quad Q_n = \min\{q_n, q_{n-k}\}.$$

Lemma 2.1. *Assume that* $\alpha \geq 1$, $x_1, x_2 \in [0, \infty)$. *Then*

$$x_1^\alpha + x_2^\alpha \geq \frac{1}{2^{\alpha-1}}(x_1 + x_2)^\alpha.$$

Lemma 2.2. *Assume that* $0 < \alpha \leq 1$, $x_1, x_2 \in [0, \infty)$. *Then*

$$x_1^\alpha + x_2^\alpha \geq (x_1 + x_2)^\alpha.$$

The proof of the above lemmas can be found in [7] and [14, Lemma 2.2], respectively.

Lemma 2.3. *Let* $\{x_n\}$ *be a positive solution of equation* (1.1). *Then there are only two cases for the sequence* $\{z_n\}$*:*

(i) $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$, $\Delta(a_n(\Delta^2 z_n)^\alpha) \leq 0$;
(ii) $z_n > 0$, $\Delta z_n < 0$, $\Delta^2 z_n > 0$, $\Delta(a_n(\Delta^2 z_n)^\alpha) \leq 0$,

for all $n \geq N \in \mathbb{N}(n_0)$, *where* N *is sufficiently large.*

The proof of the above lemma is similar to that of [14, Lemma 2.1], and thus is omitted.

Lemma 2.4. *Assume that* $\{z_n\}$ *satisfies Case (i) of Lemma 2.3 for all* $n \geq N \in \mathbb{N}(n_0)$. *Then*

$$z_{n-l} \geq \frac{B(n-l, N_1)}{A(n, N)} \Delta z_n \tag{2.1}$$

where $A(n, N) = \sum_{s=N}^{n-1} \frac{1}{a_s^{1/\alpha}}$ *and* $B(n-l, N_1) = \sum_{s=N_1}^{n-l-1} \big(\sum_{t=N}^{s-1} \frac{1}{a_t^{1/\alpha}} \big)$ *for some* $N_1 > N$.

Proof. Since $\Delta(a_n(\Delta^2 z_n)^\alpha) \leq 0$, we have $a_n(\Delta^2 z_n)^\alpha$ is nonincreasing for all $n \geq N$. Then we obtain

$$\Delta z_n \geq \Delta z_n - \Delta z_N = \sum_{s=N}^{n-1} \frac{(a_s(\Delta^2 z_s)^\alpha)^{1/\alpha}}{a_s^{1/\alpha}} \geq a_n^{1/\alpha} \Delta^2 z_n A(n, N).$$

That is,

$$a_n^{-1/\alpha} \Delta z_n - \Delta^2 z_n A(n, N) \geq 0$$

which yields

$$\Delta\Big(\frac{\Delta z_n}{A(n, N)}\Big) \leq 0. \tag{2.2}$$

Since $n - l \leq n$, we have

$$\frac{\Delta z_{n-l}}{\Delta z_n} \geq \frac{A(n-l, N)}{A(n, N)}, \tag{2.3}$$

and using (2.2), we obtain

$$\begin{aligned} z_n &= z_{N_1} + \sum_{s=N_1}^{n-1} \Delta z_s \\ &\geq \sum_{s=N_1}^{n-1} \frac{\Delta z_s}{A(s, N)} A(s, N) \\ &\geq \frac{\Delta z_n}{A(n, N)} \sum_{s=N_1}^{n-1} A(s, N), \quad n \geq N_1 \geq N. \end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\frac{z_{n-l}}{\Delta z_n} = \frac{\Delta z_{n-l}}{\Delta z_n}\frac{z_{n-l}}{\Delta z_{n-l}} \geq \frac{B(n-l,N_1)}{A(n,N)}$$

for all $n \geq N_1$. This completes the proof. □

Lemma 2.5. *Assume that $\{z_n\}$ satisfies Case (i) of Lemma 2.3 for all $n \geq N \in \mathbb{N}(n_0)$. Then*

$$\Delta z_n \geq (a_n^{1/\alpha}\Delta^2 z_n)A(n,N),$$
$$z_n \geq (a_n^{1/\alpha}\Delta^2 z_n)E(n,N)$$

where $E(n,N) = \sum_{s=N}^{n-1} \frac{(n-1-s)}{a_s^{1/\alpha}}$.

The proof of the above lemma can be found in [14, Lemma 2.5].

Lemma 2.6. *Let $\alpha > 0$. If $f_n > 0$ and $\Delta f_n > 0$ for all $n \geq N \in \mathbb{N}(n_0)$, then*

$$\Delta f_n^\alpha \geq \alpha f_n^{\alpha-1}\Delta f_n \quad \text{if } \alpha \geq 1,$$
$$\Delta f_n^\alpha \geq \alpha f_{n+1}^{\alpha-1}\Delta f_n \quad \text{if } 0 < \alpha \leq 1$$

for all $n \geq N$.

The proof of the above lemma can be found in [14, Lemma 2.6]. Next, we state and prove our main results.

Theorem 2.7. *Consider the sequences A and B defined in Lemma 2.4. Let $\alpha \geq 1$ and $l \geq k$. Assume that there exist a positive nondecreasing real sequence $\{\rho_n\}$ and a nonnegative real sequence $\{\delta_n\}$ such that*

$$\limsup_{n\to\infty} \sum_{s=N_2}^{n-1} \Big[2^{1-\alpha} M\rho_s Q_s\Big(\frac{B(s-l,N_1)}{A(s,N)}\Big)^\alpha - G_s\Big] = \infty \tag{2.5}$$

for a sufficiently large $N \in \mathbb{N}(n_0)$, and for some $N_2 > N_1 > N$, where

$$G_n = \frac{(\Delta\rho_n)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho_n^\alpha}(a_n + p^\alpha a_{n-k}) + \Delta(\rho_n a_n\delta_n + p^\alpha\rho_n a_{n-k\delta_{n-k}}).$$

If

$$\limsup_{n\to\infty} \sum_{t=n+k}^{n+l} \Big(\sum_{s=n}^{t} \Big(\frac{1}{a_{s-k}}\sum_{i=s}^{t} Q_i\Big)^{1/\alpha}\Big) > \Big(\frac{2^{\alpha-1}(1+p^\alpha)}{M}\Big)^{1/\alpha} \tag{2.6}$$

for all $n \geq N \in \mathbb{N}(n_0)$, then every solution of equation (1.1) is oscillatory.

Proof. Assume the contrary that equation (1.1) has an eventually positive solution $\{x_n\}$, that is, there exists a $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0$, $x_{n-k} > 0$ and $x_{n-l} > 0$ for all $n \geq n_1$. From the definition of z_n, we have $z_n > 0$ for all $n \geq N \in \mathbb{N}(n_1)$, where N is chosen so that two cases of Lemma 2.3 hold for all $n \geq N$. We shall show that in each case we are led to a contradiction.

Case (i): From equation (1.1) and (H_3), we have

$$\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha\Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha) + Mq_n x_{n-l}^\alpha + Mp^\alpha q_{n-k}x_{n-k-l}^\alpha \leq 0,$$

and then using Lemma 2.1, we obtain

$$\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha\Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha) + M\frac{Q_n}{2^{\alpha-1}}z_{n-l}^\alpha \leq 0, \quad n \geq N. \tag{2.7}$$

Define

$$w_n = \rho_n\Big(\frac{a_n(\Delta^2 z_n)^\alpha}{(\Delta z_n)^\alpha} + a_n\delta_n\Big), \quad n \ge N. \tag{2.8}$$

Then $w_n > 0$ for all $n \ge N$, and from (2.8) and Lemma 2.6, we have

$$\begin{aligned}
\Delta w_n &= \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} + \rho_n\Delta(a_n\delta_n) + \rho_n\frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{(\Delta z_n)^\alpha}\\
&\quad - \rho_n\frac{a_{n+1}(\Delta^2 z_{n+1})^\alpha}{(\Delta z_{n+1})^\alpha(\Delta z_n)^\alpha}\Delta((\Delta z_n)^\alpha)\\
&\le \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} + \rho_n\Delta(a_n\delta_n) + \rho_n\frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{(\Delta z_n)^\alpha}\\
&\quad - \alpha\rho_n\frac{a_{n+1}(\Delta^2 z_{n+1})^\alpha}{(\Delta z_{n+1})^\alpha}\frac{\Delta^2 z_n}{\Delta z_n}.
\end{aligned} \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\begin{aligned}
\Delta w_n &\le \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\alpha\rho_n}{a_n^{1/\alpha}}\Big(\frac{w_{n+1}}{\rho_{n+1}} - a_{n+1}\delta_{n+1}\Big)^{1+1/\alpha} + \rho_n\Delta(a_n\delta_n)\\
&\quad + \rho_n\frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{(\Delta z_n)^\alpha}, \quad n \ge N.
\end{aligned} \tag{2.10}$$

where we used $a_n^{1/\alpha}\Delta^2 z_n$ is nonincreasing, and Δz_n is nondecreasing for all $n \ge N$. From (2.10) and (2.8), we have

$$\Delta w_n \le \Delta\rho_n u_n - \frac{\alpha\rho_n}{a_n^{1/\alpha}} u_n^{1+1/\alpha} + \Delta(\rho_n a_n\delta_n) + \rho_n\frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{(\Delta z_n)^\alpha} \tag{2.11}$$

where $u_n = \frac{w_{n+1}}{\rho_{n+1}} - a_{n+1}\delta_{n+1} > 0$. Now using the inequality

$$Cu - Du^{1+1/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{C^{\alpha+1}}{D^\alpha}, \quad D > 0 \tag{2.12}$$

in (2.11), with $C = \Delta\rho_n$ and $D = \frac{\alpha\rho_n}{a_n^{1/\alpha}}$, we obtain

$$\Delta w_n \le \frac{a_n(\Delta\rho_n)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho_n)^\alpha} + \Delta(\rho_n a_n\delta_n) + \rho_n\frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{(\Delta z_n)^\alpha}. \tag{2.13}$$

Define another function v_n by

$$v_n = \rho_n\Big(\frac{a_{n-k}(\Delta^2 z_{n-k})^\alpha}{(\Delta z_{n-k})^\alpha} + a_{n-k}\delta_{n-k}\Big). \tag{2.14}$$

Then $v_n > 0$ for all $n \ge N$, and from (2.14) and Lemma 2.6, we obtain

$$\begin{aligned}
\Delta v_n &= \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} + \rho_n\Delta(a_{n-k}\delta_{n-k}) + \rho_n\Delta\Big(\frac{a_{n-k}(\Delta^2 z_{n-k})^\alpha}{(\Delta z_{n-k})^\alpha}\Big)\\
&\le \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} + \rho_n\Delta(a_{n-k}\delta_{n-k}) + \rho_n\Delta\Big(\frac{a_{n-k}(\Delta^2 z_{n-k})^\alpha}{(\Delta z_{n-k})^\alpha}\Big)\\
&\quad - \alpha\frac{\rho_n}{a_{n-k}^{1/\alpha}}\Big(\frac{w_{n+1}}{\rho_{n+1}} - a_{n+1-k}\delta_{n+1-k}\Big)^{1+1/\alpha}, \quad n \ge N,
\end{aligned} \tag{2.15}$$

where we have again used $a_n^{1/\alpha}\Delta^2 z_n$ is nonincreasing, and Δz_n is nondecreasing for all $n \geq N$. From (2.15) and (2.14), we have

$$\Delta v_n \leq \Delta\rho_n u_n - \frac{\alpha\rho_n}{a_{n-k}^{1/\alpha}} u_n^{1+1/\alpha} + \Delta(\rho_n a_{n-k}\delta_{n-k}) + \rho_n \frac{\Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha)}{(\Delta z_{n-k})^\alpha} \tag{2.16}$$

where $u_n = \frac{w_{n+1}}{\rho_{n+1}} - a_{n+1-k}\delta_{n+1-k} > 0$. Now using the inequality (2.12) to (2.16) we obtain

$$\Delta v_n \leq \frac{a_{n-k}(\Delta\rho_n)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho_n)^{\alpha+1}} + \Delta(\rho_n a_{n-k}\delta_{n-k}) + \rho_n \frac{\Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha)}{(\Delta z_{n-k})^\alpha}. \tag{2.17}$$

It follows from (2.13), (2.17) and (2.7) that

$$\begin{aligned}
&\Delta w_n + p^\alpha \Delta v_n \\
&\leq \rho_n \Big\{ \frac{\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha \Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha)}{(\Delta z_n)^\alpha} \Big\} \\
&\quad + \Delta(\rho_n a_n \delta_n + p^\alpha \rho_n a_{n-k}\delta_{n-k}) + \frac{(\Delta\rho_n)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho_n^\alpha}(a_n + p^\alpha a_{n-k}) \\
&\leq \frac{-M}{2^{\alpha-1}} \rho_n Q_n \frac{z_{n-l}^\alpha}{(\Delta z_n)^\alpha} + G_n, \quad n \geq N_1 \geq N.
\end{aligned}$$

Now using Lemma 2.4 in the above inequality, and then summing the resulting inequality from $N_2 \geq N_1$ to $n-1$, we obtain

$$\sum_{s=N_2}^{n-1} \Big[M 2^{1-\alpha} \rho_s Q_s \Big(\frac{B(s-l, N_1)}{A(s,N)}\Big)^\alpha - G_s \Big] \leq w_{N_2} + p^\alpha v_{N_2} < \infty$$

which contradicts (2.5).

Case (ii): Let $n \geq N \in \mathbb{N}(n_0)$ be fixed, and summing the inequality (2.7) from n to j, we have

$$\begin{aligned}
&a_{j+1}(\Delta^2 z_{j+1})^\alpha - a_n(\Delta^2 z_n)^\alpha + p^\alpha a_{j+1-k}(\Delta^2 z_{j+1-k})^\alpha \\
&- p^\alpha a_{n-k}(\Delta^2 z_{n-k})^\alpha + \frac{M}{2^{\alpha-1}} \sum_{t=n}^{j} Q_t z_{t-l}^\alpha \leq 0.
\end{aligned}$$

Since $\{a_j(\Delta^2 z_j)^\alpha\}$ is positive and decreasing, the above inequality implies that, as $j \to \infty$,

$$-\Delta^2 z_{n-k} + \Big(\frac{M}{2^{\alpha-1}(1+p^\alpha)}\Big)^{1/\alpha} \Big(\frac{1}{a_{n-k}} \sum_{t=n}^{\infty} Q_t z_{t-l}^\alpha\Big)^{1/\alpha} \leq 0.$$

Summing again from n to j and rearranging, we obtain

$$-\Delta z_{j+1-k} + \Delta z_{n-k} + \Big(\frac{M}{2^{\alpha-1}(1+p^\alpha)}\Big)^{1/\alpha} \sum_{t=n}^{j} \Big(\frac{1}{a_{t-k}} \sum_{s=n}^{t} Q_s\Big)^{1/\alpha} z_{t-l} \leq 0.$$

Since $\{\Delta z_j\}$ is negative and increasing, as $j \to \infty$, we have

$$\Delta z_{n-k} + \Big(\frac{M}{2^{\alpha-1}(1+p^\alpha)}\Big)^{1/\alpha} \sum_{t=n}^{\infty} \Big(\frac{1}{a_{t-k}} \sum_{s=n}^{t} Q_s\Big)^{1/\alpha} z_{t-l} \leq 0.$$

Summing the above inequality from $n+k$ to j and rearranging, we obtain

$$z_{j+1-k} - z_n + \Big(\frac{M}{2^{\alpha-1}(1+p^\alpha)}\Big)^{1/\alpha} \sum_{t=n}^{j} \Big[\sum_{s=n}^{t} \Big(\frac{1}{a_{s-k}} \sum_{i=s}^{t} Q_i\Big)^{1/\alpha} z_{t-l}\Big] \le 0.$$

Since $\{z_n\}$ is positive and decreasing, we have from the last inequality as $j \to \infty$,

$$\sum_{t=n+k}^{\infty} \Big[\sum_{s=n}^{t} \Big(\frac{1}{a_{s-k}} \sum_{i=s}^{t} Q_i\Big)^{1/\alpha}\Big] z_{t-l} \le \Big(\frac{2^{\alpha-1}(1+p^\alpha)}{M}\Big)^{1/\alpha} z_n,$$

or

$$\sum_{t=n+k}^{n+l} \Big[\sum_{s=n}^{t} \Big(\frac{1}{a_{s-k}} \sum_{i=s}^{t} Q_i\Big)^{1/\alpha}\Big] \le \Big(\frac{2^{\alpha-1}(1+p^\alpha)}{M}\Big)^{1/\alpha}$$

which contradicts (2.6) as $n \to \infty$. This completes the proof. □

By using the inequality in Lemma 2.2 instead of Lemma 2.1, we obtain the following result.

Theorem 2.8. *Consider the sequences A and B defined in Lemma 2.4. Let $0 < \alpha \le 1$ and $l \ge k$. Assume condition (2.6) holds. Further assume that there exist a positive nondecreasing real sequence $\{\rho_n\}$ and a nonnegative real sequence $\{\delta_n\}$ such tat*

$$\limsup_{n\to\infty} \sum_{s=N_2}^{n-1} \Big[M\rho_s Q_s \Big(\frac{B(s-l, N_1)}{A(s,N)}\Big)^{\alpha} - G_s\Big] = \infty$$

for sufficiently large $N \in \mathbb{N}(n_0)$, and for some $N_2 > N_1 > N$, then every solution of equation (1.1) is oscillatory.

The proof of the above theorem is similar to that of Theorem 2.7, and hence it is omitted. Next, we present an easily verifiable oscillation condition for equation (1.1).

Theorem 2.9. *Let $\alpha \ge 1$, and assume that condition (2.6) with $l \ge k$. If there exists a positive nondecreasing real sequence $\{\rho_n\}$ such that*

$$\sum_{n=N_1}^{\infty} \Big[\rho_{n+1}\Big(\frac{M}{2^{\alpha-1}} Q_n - d(1+p^\alpha)\Big) + d(1+p^\alpha)\rho_n\Big] = \infty \tag{2.18}$$

for every $d > 0$, and for some $N_1 \ge N$, then every solution of (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \ge n_1$. From the definition of z_n, we have $z_n > 0$ for all $n \ge N \in \mathbb{N}(n_1)$, where N is chosen so that Lemma 2.3 holds for all $n \ge N$.

Case(i): Define

$$w_n = \rho_n \frac{a_n(\Delta^2 z_n)^\alpha}{z_{n-l}^\alpha}, \quad n \ge N. \tag{2.19}$$

Then $w_n > 0$ for all $n \ge N$, from (2.19), we have

$$\Delta w_n \le \rho_{n+1} \frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{z_{n-l}^\alpha} + \Delta\rho_n \frac{a_n(\Delta^2 z_n)^\alpha}{z_{n-l}^\alpha}, \quad n \ge N. \tag{2.20}$$

Define another function

$$v_n = \rho_n \frac{a_{n-k}(\Delta^2 z_{n-k})^\alpha}{z_{n-l}^\alpha}, \quad n \geq N. \tag{2.21}$$

Then $v_n > 0$ for all $n \geq N$, and from (2.21), we have

$$\Delta v_n \leq \rho_{n+1} \frac{\Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha)}{z_{n-l}^\alpha} + \Delta\rho_n \frac{a_{n-k}(\Delta^2 z_{n-k})}{z_{n-l}^\alpha}, \quad n \geq N. \tag{2.22}$$

Combining (2.20) and (2.22), and then using (2.7), we obtain

$$\begin{aligned}\Delta w_n + p^\alpha \Delta v_n &\leq \frac{-M}{2^{\alpha-1}} \rho_{n+1} Q_n + \Delta\rho_n \frac{a_N(\Delta^2 z_N)^\alpha}{z_{N-l}^\alpha} + p^\alpha \Delta\rho_n \frac{a_{N-k}(\Delta^2 z_{N-k})^\alpha}{z_{N-l}^\alpha} \\ &\leq \frac{-M}{2^{\alpha-1}} \rho_{n+1} Q_n + d(1+p^\alpha)\Delta\rho_n, \quad n \geq N_1 \geq N,\end{aligned}$$

where $d = \frac{a_{N-k}(\Delta^2 z_{N-k})^\alpha}{z_{N-l}^\alpha} > 0$ is a constant. Summing the last inequality from N_1 to m, we obtain

$$\sum_{n=N_1}^{m} \Big[\rho_{n+1}\Big(\frac{MQ_n}{2^{\alpha-1}} - d(1+p^\alpha)\Big) + d(1+p^\alpha)\rho_n\Big] \leq w_{N_1} + p^\alpha v_{N_1}.$$

which contradicts condition (2.18) as $m \to \infty$.

The proof of Case(ii) is similar to that of in Theorem 2.7, and we omit it. The proof is complete. □

From Lemma 2.2, similar to the proof of Theorem 2.9, we obtain the following result.

Theorem 2.10. *Let $0 < \alpha \leq 1$, assume condition (2.6) with $l \geq k$. If there exists a positive nondecreasing real sequence $\{\rho_n\}$ such that*

$$\sum_{n=N_1}^{\infty} [\rho_{n+1}(MQ_n - d(1+p^\alpha)) + d(1+p^\alpha)\rho_n] = \infty$$

for every constant $d > 0$, then every solution of equation (1.1) is oscillatory.

Next, we present some oscillation criteria using Lemma 2.5.

Theorem 2.11. *Consider the sequences E defined in Lemma 2.5. Let $\alpha \geq 1$, assume that condition (2.6) with $l > k$. If*

$$\limsup_{n\to\infty} \sum_{s=n-l+k}^{n-1} Q_s E^\alpha(s-l, N) > \Big(\frac{l-k}{l-k+1}\Big)^{l-k+1} \frac{2^{\alpha-1}(1+p^\alpha)}{M} \tag{2.23}$$

then every solution of (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1$. From the definition of z_n, we have $z_n > 0$ for all $n \geq N \in \mathbb{N}(n_1)$, where N is chosen so that Lemma 2.3 holds for all $n \geq N$.

Case(i): From Lemma 2.5, we have

$$z_{n-l}^\alpha \geq a_{n-l}(\Delta^2 z_{n-l})^\alpha E^\alpha(n-l, N), \quad n \geq N. \tag{2.24}$$

Using (2.24) in (2.7), we obtain

$$\begin{aligned} &\Delta(a_n(\Delta^2 z_n)^\alpha) + p^\alpha \Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha) \\ &+ \frac{M}{2^{\alpha-1}} Q_n E^\alpha(n-l,N) a_{n-l}(\Delta^2 z_{n-l})^\alpha \le 0 \end{aligned} \tag{2.25}$$

for all $n \ge N$. Set

$$w_n = a_n(\Delta^2 z_n)^\alpha + p^\alpha a_{n-k}(\Delta^2 z_{n-k})^\alpha, \quad n \ge N.$$

Then $w_n > 0$, and

$$w_n \le (1+p^\alpha) a_{n-k}(\Delta^2 z_{n-k})^\alpha,\ n \ge N. \tag{2.26}$$

Combining (2.26) with (2.25), we have

$$\Delta w_n + \frac{M}{2^{\alpha-1}(1+p^\alpha)} Q_n E^\alpha_{n-l,N} w_{n+k-l} \le 0. \tag{2.27}$$

In view of [1, Theorem 6.20.5], condition (2.22) implies that the inequality (2.27) has no positive solution, which is a contradiction.

The proof of Case(ii) is similar to that of Case(ii) of Theorem 2.9. This completes the proof. □

From Lemma 2.2, similar to the proof of Theorem 2.11, we obtain the following result.

Theorem 2.12. *Let $0 < \alpha \le 1$, and assume that condition (2.6) with $l > k$. If*

$$\limsup_{n\to\infty} \sum_{s=n-l+k}^{n-1} Q_s E^\alpha(s-l,N) > \Big(\frac{l-k}{l-k+1}\Big)^{l-k+1} \frac{(1+p^\alpha)}{M}$$

then every solution of equation (1.1) is oscillatory.

Our final result is concern with the case when

$$\sum_{n=N}^{\infty} Q_n < \infty. \tag{2.28}$$

Theorem 2.13. *Let $\alpha \ge 1$, and assume that condition (2.6) with $l \ge k$. If (2.28), and*

$$\limsup_{n\to\infty} E^\alpha(n-l,N) \sum_{s=n}^{\infty} Q_s > \frac{2^{\alpha-1}(1+p^\alpha)}{M} \tag{2.29}$$

hold, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \ge n_1$. From the definition of z_n, we have $z_n > 0$ for all $n \ge N \in \mathbb{N}(n_1)$, where N is chosen so that Lemma 2.3 holds for all $n \ge N$.

Case(i): For this case, we define w_n as in Theorem 2.9 with $\rho_n \equiv 1$, to obtain

$$\Delta\Big(\frac{a_n(\Delta^2 z_n)^\alpha}{z^\alpha_{n-l}}\Big) \le \frac{\Delta(a_n(\Delta^2 z_n)^\alpha)}{z^\alpha_{n-l}}, \quad n \ge N. \tag{2.30}$$

Next we define v_n as in Theorem 2.9 with $\rho_n \equiv 1$, to obtain

$$\Delta\Big(\frac{a_{n-k}(\Delta^2 z_{n-l})^\alpha}{z^\alpha_{n-l}}\Big) \le \frac{\Delta(a_{n-k}(\Delta^2 z_{n-k})^\alpha)}{z^\alpha_{n-l}}, \quad n \ge N. \tag{2.31}$$

Multiplying inequality (2.31) by p^α and adding with (2.30), and then using (2.7), we obtain

$$\Delta\Big(\frac{a_n(\Delta^2 z_n)^\alpha + p^\alpha a_{n-k}(\Delta^2 z_{n-k})^\alpha}{z_{n-l}^\alpha}\Big) \le \frac{-M}{2^{\alpha-1}} Q_n, \quad n \ge N.$$

Summing the last inequality from N to m, and then using the nonincreasing property of $\{a_n(\Delta z_n)^\alpha\}$, we obtain

$$\frac{M}{2^{\alpha-1}} \sum_{s=N}^{m} Q_s \le (1+p^\alpha)\frac{a_{N-l}(\Delta^2 z_{N-l})^\alpha}{z_{N-l}^\alpha}.$$

Since the right-hand side of the last inequality is independent of m, we have

$$\frac{M}{2^{\alpha-1}(1+p^\alpha)} \sum_{s=N}^{\infty} Q_s \le \frac{a_{N-l}(\Delta^2 z_{N-l})^\alpha}{z_{N-l}^\alpha}. \tag{2.32}$$

In view of condition (2.28), it follows from (2.32) that

$$\frac{M}{2^{\alpha-1}(1+p^\alpha)} \sum_{s=n}^{\infty} Q_s \le \frac{a_{n-l}(\Delta^2 z_{n-l})^\alpha}{z_{n-l}^\alpha}, \ n \ge N. \tag{2.33}$$

Now using Lemma 2.5 in (2.33), we have

$$E^\alpha(n-l,N) \sum_{s=n}^{\infty} Q_s \le \frac{2^{\alpha-1}(1+p^\alpha)}{M}, \quad n \ge N,$$

which contradicts (2.29). The proof of Case (ii) is similar to that of Theorem 2.7. This completes the proof. ☐

From Lemma 2.2, similar to the proof of Theorem 2.13, we obtain the following result.

Theorem 2.14. *Let $0 < \alpha \le 1$, and assume that condition* (2.6) *with $l \ge k$. If* (2.28)*, and*

$$\limsup_{n\to\infty} E^\alpha(n-l,N) \sum_{s=n}^{\infty} Q_s > \frac{(1+p^\alpha)}{M}$$

hold, then every solution of (1.1) *is oscillatory.*

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the third-order neutral type difference equation

$$\Delta(n(\Delta^2(x_n + p x_{n-1}))^3) + \lambda n x_{n-2}^3 = 0, \ n \ge 1. \tag{3.1}$$

Here $a_n = n$, $p_n = p > 0$, $q_n = \lambda n$, $\lambda > 0$, $k = 1$, $l = 2$, $m = 1$ and $Q_n = \lambda(n-1)$. Since $A(n,1) \le (n-1)$ and $B(n,2) \ge (n-2)$, and by taking $\rho_n \equiv 1$ and $\delta_n \equiv 0$, we see that condition (2.5) is clearly satisfied. Further, we have

$$\begin{aligned}
&\sum_{t=n+1}^{n+2} \sum_{s=n}^{t} \Big(\frac{1}{s-1} \sum_{i=s}^{t} \lambda(i-1)\Big)^{1/3} \\
&= \sum_{t=n+1}^{n+2} \sum_{s=n}^{t} \Big(\frac{\lambda}{s-1}\Big(\frac{t(t-1)}{2} - \frac{(s-1)(s-2)}{2}\Big)^{1/3}\Big),
\end{aligned}$$

$$\limsup_{n\to\infty} \sup \sum_{t=n+1}^{n+2} \sum_{s=n}^{t} \Big(\frac{\lambda}{2}\Big(\frac{t(t-1)}{s-1} - s + 2\Big)\Big)^{1/3} = \lambda(2 + 3^{1/3} + 2^{4/3}).$$

Therefore if

$$\lambda > \frac{4(1+p^3)}{(2+2^{4/3}+3^{1/3})^3}\,,$$

then condition (2.6) is satisfied. Hence by Theorem 2.7, every solution of equation (3.1) is oscillatory provided $\lambda > 0.0189(1+p^3)$.

Example 3.2. Consider the third-order neutral delay difference equation

$$\Delta((n+1)(\Delta^2(x_n + 2x_{n-1}))^3) + 64(2n+3)x_{n-3}^3 = 0, \quad n \geq 1. \tag{3.2}$$

Here $a_n = (n+1)$, $p_n = 2$, $q_n = 64(2n+3)$, $\alpha = 3$, $k = 1$, $l = 3$, $M = 1$ and $Q_n = 64(2n+1)$. By taking $\rho_n \equiv 1$, we see that condition (2.18) is clearly satisfied. Further it is easy to verify that condition (2.6) is also satisfied. Therefore by Theorem 2.9, every solution of (3.2) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of (3.2).

Example 3.3. Consider the third-order neutral type difference equation

$$\Delta(\frac{1}{(n+1)^3}(\Delta^2(x_n + px_{n-1}))^3) + \frac{\lambda}{n(n+1)}x_{n-2}^3 = 0, \quad n \geq 1. \tag{3.3}$$

Here $a_n = \frac{1}{(n+1)^3}$, $p_n = p > 0$, $q_n = \frac{\lambda}{n(n+1)}$, $\lambda > 0$, $k = 1$, $l = 2$, $M = 1$ and $Q_n = \frac{\lambda}{n(n+1)}$. Since $E(n,1) = \frac{n^3-7n+6}{6}$, it is easy to see that condition (2.24) is satisfied. Further, we have

$$\sum_{t=n+1}^{n+2} \sum_{s=n}^{t} s\Big(\sum_{i=s}^{t} \frac{\lambda}{i(i+1)}\Big)^{1/3} = \lambda^{1/3} \sum_{t=n+1}^{n+2} \sum_{s=n}^{t} s\Big(\frac{1}{s} - \frac{1}{t}\Big)^{1/3},$$

or

$$\limsup_{n\to\infty} \sum_{t=n+1}^{n+2} \sum_{s=n}^{t} \lambda^{1/3} s\Big(\frac{1}{s} - \frac{1}{t}\Big)^{1/3} = \infty.$$

Hence condition (2.6) is also satisfied. Therefore by Theorem 2.11, every solution of (3.3) is oscillatory provided $\lambda > 0$.

Example 3.4. Consider a third-order neutral delay difference equation

$$\Delta(\frac{1}{(n-1)^3}(\Delta^2(x_n + 2x_{n-1}))^3) + \frac{128}{(n-3)^3}x_{n-3}^3 = 0, \quad n \geq 5. \tag{3.4}$$

Here $a_n = \frac{1}{(n-1)^3}$, $p_n = 2$, $q_n = \frac{128}{(n-3)^3}$, $\alpha = 3$, $k = 1$, $l = 3$, $M = 1$ and $Q_n = \frac{128}{(n-3)^3}$. Since $E(n,5) = \frac{n^3-6n^2-n+210}{6}$, it is easy to see that condition (2.30) is satisfied. Further, one can easily that condition (2.6) is also satisfied. Therefore by Theorem 2.13, every solution of (3.4) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of (3.4).

Remark 3.5. From the results given in [14, 16, 10, 12, 15], one cannot conclude that all solutions of (3.1)–(3.4) are oscillatory.

3.1. **Conclusion.** In this article, we have established some new oscillation theorems for (1.1) when $0 \leq p_n \leq p < \infty$, and $\alpha \in (0, \infty)$. These results ensure that all solutions are just oscillatory. Therefore our results improve those in [14, 16, 10, 12, 15] since the results in these papers will not ensure that all solutions are oscillatory. Also one can extend the results in [6, 13, 11] to neutral type difference equations, and the details are left to the reader. It is also interesting to extend the results of the equation (1.1) when $-1 < p_n \leq 0$ and $\{p_n\}$ is oscillatory.

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