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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLINEAR DIRAC-POISSON SYSTEMS

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Abstract. This article concerns the nonlinear Dirac-Poisson system

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u+(V(x)+a)\beta u+\omega u-\phi u=F_u(x,u),$$
$$-\Delta\phi=4\pi|u|^2,$$

in $\mathbb{R}^3$, where $V(x)$ is a potential function and $F(x,u)$ is an asymptotically quadratic nonlinearity modeling various types of interaction. Since the effects of the nonlocal term, we use some special techniques to deal with the nonlocal term. Moreover, the existence of infinitely many stationary solutions is obtained for system with periodicity assumption via variational methods.

1. Introduction and main results

In this article we study the nonlinear Maxwell-Dirac system

$$\begin{gathered} i\hbar\partial_t\psi=\sum_{k=1}^{3}\alpha_k(-ic\hbar\partial_k+A_k)\psi+mc^2\beta\psi-A_0\psi,\\ \partial_t A_0+\sum_{k=1}^{3}\partial_k A_k=0,\quad \partial_t^2A_0-\Delta A_0=4\pi|\psi|^2,\\ \partial_t^2A_k-\Delta A_k=4\pi(\alpha_k\psi)\bar{\psi},\quad k=1,2,3, \end{gathered}\tag{1.1}$$

in $\mathbb{R}\times\mathbb{R}^3$, where $\partial_k=\frac{\partial}{\partial x_k}$, $\psi(t,x)\in\mathbb{C}^4$, c is the speed of light, m is the mass of the electron, $\hbar$ is the Planck's constant, $\mathbf{A}:=(A_1,A_2,A_3):\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^3$ is the magnetic field, $A_0:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ is the electric field, and $u\bar{v}$ denotes the usual scalar product of $u,v\in\mathbb{C}^4$. Furthermore, $\alpha_1,\alpha_2,\alpha_3$ and β are the 4×4 Pauli-Dirac matrices:

$$\beta=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \alpha_k=\begin{pmatrix} 0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad k=1,2,3,$$

with

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

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The Maxwell-Dirac system plays an important role in quantum electrodynamics. It is used to describe the interaction of a particle with its self-generated electromagnetic field, and it has been widely employed in many areas such as quantum cosmology, atomic physics, nuclear physics and gravitational physics (see [19, 36]).

In this article, we consider the electrostatic case, namely

$$A_0 = \phi(x), \quad A_k = 0, \quad k = 1, 2, 3, \quad x \in \mathbb{R}^3,$$

and for standing wave function

$$\psi(t, x) = u(x)e^{i\theta t/\hbar}, \quad \theta \in \mathbb{R}, \quad u : \mathbb{R}^3 \to \mathbb{C}^4.$$

In the case of zero magnetic field (i.e. $A_k = 0, k = 1, 2, 3$) and non-trivial electric potential $\phi(x)$, the Maxwell-Dirac system (1.1) has the form

$$\begin{gathered} -i\sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + \omega u - \phi u = 0, \\ -\Delta\phi = 4\pi|u|^2, \end{gathered} \tag{1.2}$$

in $\mathbb{R}^3$, where $a = mc/\hbar$, $\omega = \theta/c\hbar$. In [21, 37], this system is called the Dirac-Poisson system.

In the past decade, system (1.2) has been studied for a long time and many results are available concerning the Cauchy problem, see for instance, [10, 11, 22, 25, 26, 28, 35] and the references therein. As we know, the existence of stationary solutions of the Maxwell-Dirac system has been an open problem for a long time, see [27]. As far as variational methods are concerned, there is a pioneering work by Esteban et al. [20] in which a multiplicity result was studied when $\omega \in (0, a)$. After that, Abenda [1] studied the case $\omega \in (-a, a)$ and obtained the existence result of soliton-like solutions. And a strong localization result was obtained in [33]. In [29], Garrett Lisi gave numerical evidence of the existence of bounded states by using an axially symmetric ansatz. In the survey paper [21], there are more detailed descriptions for equations and systems related to Dirac operator.

We emphasize that the works mentioned above mainly concerned with the autonomous system with null self-coupling. Besides, the idea to consider a nonlinear self-coupling, in quantum electrodynamics, gives the description of models of self-interacting spinor fields, see [23, 24]. Due to the special physical importance, in the present paper, we consider the Dirac-Poisson system with the general self-coupling nonlinearity

$$\begin{gathered} -i\sum_{k=1}^{3} \alpha_k \partial_k u + (V(x) + a)\beta u + \omega u - \phi u = F_u(x, u), \\ -\Delta\phi = 4\pi|u|^2, \end{gathered} \tag{1.3}$$

in $\mathbb{R}^3$, where $V(x)$ is a potential function and $F(x, u)$ is a nonlinear function modeling various types of interaction.

Recently, for system (1.3) with magnetic field, Chen and Zheng [15] studied the system with non-periodic potential and superquadratic nonlinearity, and the existence of least energy stationary solutions was obtained. An asymptotically quadratic nonperiodic problem was considered in [43]. Zhang et al. [42, 44] considered the more general periodic problem, and obtained the existence of ground state solutions by using linking and concentration compactness arguments. Besides, for other related topics including the superquadratic singular perturbation problem and concentration phenomenon of semi-classical states, see, for instance [17, 18, 19, 45] and the references therein.

Motivated by the above facts, in this paper we are concerned with system (1.3) with non-autonomous asymptotically quadratic nonlinearity and periodicity condition. To the best of our knowledge, there has been no work concerning on multiplicity result in the general case up to now. The main purpose of this paper is to study the existence and multiplicity of stationary solutions via variational methods. Before stating our main result, we first make the following assumptions:

- (A1) $\omega \in (-a, a)$;
- (A2) $V \in C^1(\mathbb{R}^3, \mathbb{R}^+)$, and $V(x)$ is 1-periodic in x_k, $k = 1, 2, 3$;
- (A3) $F(x, u) \in C^1(\mathbb{R}^3 \times \mathbb{C}^4, \mathbb{R}^+)$ and $F(x, u)$ is 1-periodic in x_k, $k = 1, 2, 3$;
- (A4) $F_u(x, u) = o(|u|)$ as $|u| \to 0$ uniformly in x;
- (A5) $F_u(x, u) - G(x)u = o(|u|)$ as $|u| \to \infty$ uniformly in x, and $\inf_{x\in\mathbb{R}^3} G(x) > a + \omega + \sup_{\mathbb{R}^3} V$, where $G \in C(\mathbb{R}^3, \mathbb{R})$ is 1-periodic in x_k, $k = 1, 2, 3$;
- (A6) $\widetilde{F}(x, u) \geq 0$, and there exists $\delta_1 \in (0, a - |\omega|)$ such that $\widetilde{F}(x, u) \geq \delta_1$ whenever $|F_u(x, u)| \geq (a - |\omega| - \delta_1)|u|$, where $\widetilde{F}(x, u) := \frac{1}{2}F_u(x, u)u - F(x, u)$.

Observe that, because of the periodicity of V, F, if u is a solution of system (1.3), then so is $k * u$ for all $k \in \mathbb{Z}^3$, where $(k * u)(x) = u(x + k)$. Two solutions u_1 and u_2 are said to be geometrically distinct if $k * u_1 \neq u_2$ for all $k \in \mathbb{Z}^3$. The main result of this paper is the following theorem.

Theorem 1.1. *Assume that* (A1)–(A6) *are satisfied. Then system* (1.3) *has at least one nontrivial stationary solutions. Moreover, if $F(x, u)$ is even in u. Then system* (1.3) *has infinitely many geometrically distinct solutions.*

There have been a large number of works on the existence of stationary solutions of nonlinear Schrödinger-Poisson system arising in the non-relativistic quantum mechanics, see, for example, [2, 5, 14, 32, 34, 46]. It is quite natural to ask if certain similar results can be obtain for nonlinear Dirac-Poisson system arising in the relativistic quantum mechanics, we will give an answer for Dirac-Poisson system in the present paper. Mathematically, the two problems possess different variational structures, the mountain pass and the linking structures respectively. Contrary to the Schrödinger operator, the Dirac operator not only has unbounded positive continuous spectrum but also has unbounded negative continuous spectrum, and the corresponding energy functional is strongly indefinite. On the other hand, the main difficulty when dealing with this problem is the lack of compactness of Sobolev embedding, hence our problem poses more challenges in the calculus of variation. In order to overcome these difficulties, we will turn to the linking and concentration compactness arguments (see [7, 30, 31]).

Recently, there have been some works focused on existence of stationary solutions for nonlinear Dirac equation but not for Dirac-Poisson system, see, for example [8, 16, 39, 40, 41]. Particularly, under the conditions (A3)–(A5) and the following condition

A6') $\widetilde{F}(x, u) > 0$ if $u \neq 0$, and $\widetilde{F}(x, u) \to \infty$ as $|u| \to \infty$ uniformly in x.

Zhao and Ding [39] obtained the existence of infinitely many geometrically distinct solutions. We point out that condition (A6') plays an important role in the arguments that showing any $(C)_c$-sequence is bounded in [39]. However, F does not satisfy the above property under the conditions we assumed. Hence, we shall use new tricks to show any $(C)_c$-sequence is bounded in the present paper, which is different from the arguments in [39]. Moreover, the condition (A6) is weaker than the one (A6') and there are some functions satisfying (A6), but not (A6'), see Remark 1.2.

Compared with the Dirac equation, the Dirac-Poisson system becomes more complicated because of the effects of nonlocal term. This will need more delicate analysis and some new tricks to get the result. It is worth pointing out that although some ideas were used before for Dirac equation, the adaptation to the procedure to our problem is not trivial at all. Hence our result can be viewed as extension to the result in [8, 39] from Dirac equation to Dirac-Poisson system.

Remark 1.2. Let $F(x,u)=\frac{1}{2}b(x)|u|^2\big(1-\frac{1}{1+|u|^\sigma}\big)$, where > 0, $b\in C(\mathbb{R}^3,\mathbb{R})$ and is 1-periodic in x_k, $k=1,2,3$, and $\inf_{\mathbb{R}^3} b > a+\omega+\sup_{\mathbb{R}^3} V$. Then

$$F_u(x,u)=b(x)\Big(\frac{1}{1+|u|^\sigma}+\frac{\sigma|u|^\sigma}{(1+|u|^\sigma)^2}\Big)u,\quad \widetilde{F}(x,u)=\frac{b(x)\sigma|u|^{\sigma+2}}{2(1+|u|^\sigma)^2}\geq 0.$$

It is easy to see that F satisfies (A3)–(A6), but it does not satisfy (A6') when $\sigma\geq 2$.

The remainder of this article is organized as follows. In section 2, we formulate the variational setting, and present two critical point theorems required. In section 3, we will use the linking and concentration compactness principle to prove our main result.

2. Variational setting and abstract theorem

Below by $|\cdot|_q$ we denote the usual L^q-norm, $(\cdot,\cdot)_2$ denote the usual L^2 inner product, c, C_i stand for different positive constants. For convenience, let

$$A:=-i\sum_{k=1}^{3}\alpha_k\partial_k+(V+a)\beta.$$

be the Dirac operator. It is well known that A is a selfadjoint operator acting on $L^2:=L^2(\mathbb{R}^3,\mathbb{C}^4)$ with $\mathcal{D}(A)=H^1:=H^1(\mathbb{R}^3,\mathbb{C}^4)$ (see [16, Lemma 7.2 a)]). Let $|A|$ and $|A|^{1/2}$ denote respectively the absolute value of A and the square root of $|A|$, and let $\{\mathcal{F}_\lambda:-\infty\leq\lambda\leq+\infty\}$ be the spectral family of A. Set $U=id-\mathcal{F}_0-\mathcal{F}_{0-}$. Then U commutes with A, $|A|$ and $|A|^{1/2}$, and $A=U|A|$ is the polar decomposition of A. Let $\sigma(A)$, $\sigma_c(A)$ be the spectrum, the continuous spectrum of A, respectively. In order to establish a variational setting for the system (1.3), we have the following lemma.

Lemma 2.1 ([16, Lemma 7.3])**.** *Suppose* (A2) *holds. Then*

$$\sigma(A)=\sigma_c(A)\subset(-\infty,-a]\cup[a,\infty)$$

and $\inf\sigma(|A|)\leq a+\sup_{\mathbb{R}^3}V$.

From Lemma 2.1 it follows that the space L^2 possesses the orthogonal decomposition:

$$L^2=L^-\oplus L^+,\quad u=u^-+u^+$$

such that A is negative definite on L^- and positive definite on L^+. Let $E := \mathcal{D}(|A|^{1/2})$ be the domain of $|A|^{1/2}$. We introduce on E the inner product

$$(u,v) = (|A|^{1/2}u, |A|^{1/2}v)_2 + \omega(u, v^+ - v^-)_2$$

and the induced norm

$$\|u\| = (u,u)^{1/2} = \Big(\big||A|^{1/2}u\big|_2^2 + \omega(|u^+|_2^2 - |u^-|_2^2)\Big)^{1/2}.$$

It is clear that E possesses the following decomposition

$$E = E^- \oplus E^+ \quad \text{and} \quad E^\pm = E \cap L^\pm.$$

Then

$$\begin{gathered} Au = -|A|u, \quad \forall u \in E^-, \quad Au = |A|u, \quad \forall u \in E^+, \\ u = u^- + u^+, \quad \forall u \in E. \end{gathered}$$

Hence E^+ and E^- are orthogonal with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products.

Lemma 2.2 ([16, Lemma 7.4]). *Suppose* (A1)–(A2) *hold. Then* $E = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ *with equivalent norms, and* E *embeds continuously into* L^p *for all* $p \in [2,3]$ *and compactly into* $L^p_{\rm loc}$ *for all* $p \in [1,3)$*. Moreover*

$$(a - |\omega|)|u|_2^2 \le \|u\|^2, \quad \forall u \in E.$$

Let $\mathcal{D}^{1,2} := \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R})$ be the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ with respect to the norm

$$\|u\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x.$$

It is well known system (1.3) can be reduced to a single equation with nonlocal term. Actually, for each $u \in E$, the linear functional T_u in $\mathcal{D}^{1,2}$ defined by

$$T_u(v) = 4\pi \int_{\mathbb{R}^3} |u|^2 v \mathrm{d}x, \quad v \in \mathcal{D}^{1,2},$$

is continuous. In fact, since $u \in L^q$ for all $q \in [2,3]$, one has $|u|^2 \in L^{6/5}$ for all $u \in E$, and Hölder inequality and Sobolev inequality imply that

$$\begin{aligned} |T_u(v)| = 4\pi \Big| \int_{\mathbb{R}^3} |u|^2 v \mathrm{d}x \Big| &\le 4\pi \Big(\int_{\mathbb{R}^3} \big||u|^2\big|^{6/5} \mathrm{d}x \Big)^{5/6} \Big(\int_{\mathbb{R}^3} |v|^6 \mathrm{d}x \Big)^{1/6} \\ &\le 4\pi S^{-1/2} \big||u|^2\big|_{6/5} \|v\|_{\mathcal{D}}, \end{aligned} \tag{2.1}$$

where S is the Sobolev embedding constant. It follows from the Lax-Milgram theorem that there exists a unique $\phi_u \in \mathcal{D}^{1,2}$ such that

$$\int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v \mathrm{d}x = 4\pi \int_{\mathbb{R}^3} |u|^2 v \mathrm{d}x, \quad \forall v \in \mathcal{D}^{1,2}, \tag{2.2}$$

that is ϕ_u satisfies the Poisson equation

$$-\Delta \phi_u = 4\pi |u|^2$$

and it holds

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} \mathrm{d}y = \frac{1}{|x|} * |u|^2.$$

By (2.1) and (2.2), it is easy to see that

$$\|\phi_u\|_{\mathcal{D}}^2 = 4\pi \int_{\mathbb{R}^3} \phi_u |u|^2 \mathrm{d}x \le 4\pi S^{-1/2} \big||u|^2\big|_{6/5} \|\phi_u\|_{\mathcal{D}} \tag{2.3}$$

and

$$\int_{\mathbb{R}^3} \phi_u |u|^2 \mathrm{d}x \le S^{-1/2} \big||u|^2\big|_{6/5} \|\phi_u\|_{\mathcal{D}} \le 4\pi S^{-1} |u|_{12/5}^4. \tag{2.4}$$

Substituting ϕ_u in (1.3), we are led to the equation

$$-i \sum_{k=1}^{3} \alpha_k \partial_k u + (V(x)+a)\beta u + \omega u - \phi_u u = F_u(x,u). \tag{2.5}$$

Next, on E we define the functional

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Gamma(u) - \Psi(u) \tag{2.6}$$

for $u = u^+ + u^- \in E$, where

$$\Gamma(u) = \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 \mathrm{d}x = \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(y)|^2|u(x)|^2}{|x-y|} \mathrm{d}y\mathrm{d}x,$$
$$\Psi(u) = \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x.$$

Moreover, our hypotheses imply that $\Phi \in C^1(E,\mathbb{R})$, and a standard argument shows that critical points of Φ are solutions of system (1.3) (see [16, 38]).

To find critical points of Φ, we shall use the following abstract theorems which are taken from [7] and [16].

Let E be a Banach space with direct sum $E = X \oplus Y$ and corresponding projections P_X, P_Y onto X, Y. Let $\mathcal{S} \subset (X)^*$ be a dense subset, for each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R}, \quad p_s(u) : |s(x)| + \|y\| \quad \text{for } u = x + y \in E.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the topology induced by semi-norm family $\{p_s\}$, w^* denote the weak*-topology on E^*. Now, some notations are needed. For a functional $\Phi \in C^1(E,\mathbb{R})$ we write $\Phi_a = \{u \in E | \Phi(u) \ge a\}$, $\Phi^b = \{u \in E | \Phi(u) \le b\}$ and $\Phi_a^b = \Phi_a \cap \Phi^b$. Recall that a sequence $\{u_n\} \subset E$ is said to be a $(C)_c$-sequence if $\Phi(u_n) \to c$ and $(1+\|u_n\|)\Phi'(u_n) \to 0$; Φ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence. A set $\mathcal{O} \subset E$ is said to be a $(C)_c$-attractor if for any $\varepsilon, \delta > 0$ and any $(C)_c$-sequence $\{u_n\}$ there is n_0 such that $u_n \in U_\varepsilon(\mathcal{O} \cap \Phi_{c-\delta}^{c+\delta})$ for $n \ge n_0$. Given an interval $I \subset \mathbb{R}$, $\mathcal{O}$ is said to be a $(C)_I$-attractor if it is a $(C)_c$-attractor for all $c \in I$. Φ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in E one has $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$, and Φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty} \Phi'(u_n)w = \Phi'(u)w$ for each $w \in E$.

Suppose

- (A7) for any $c \in \mathbb{R}$, superlevel Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous;
- (A8) for any $c > 0$, there exists $\xi > 0$ such that $\|u\| < \xi \|P_Y u\|$ for all $u \in \Phi_c$;
- (A9) there exists $r > 0$ such that $\varrho := \inf \Phi(S_r \cap Y) > 0$, where $S_r := \{u \in E : \|u\| = r\}$;

(A10) there is an increasing sequence $Y_n \subset Y$ of finite-dimensional subsequences and a sequence $\{R_n\}$ of positive numbers such that, letting $E_n = X \oplus Y_n$ and $B_n = B_{R_n} \cap E_n$, $\sup \Phi(E_n) < \infty$ and $\sup \Phi(E_n \setminus B_n) < \inf \Phi(B_r \cap Y)$, where $B_r := \{u \in E : \|u\| \leq r\}$;

(A11) for any interval $I \subset (0, \infty)$ there is a $(C)_I$-attractor $\mathcal{O}$ with $P_X \mathcal{O}$ bounded and $\inf\{\|P_Y(z-w)\| : z, w \in \mathcal{O}, \|P_Y(z-w)\| \neq 0\} > 0$.

Now we state the following critical point theorems which will be used later (see [7, 16]).

Theorem 2.3. *Let* (A7)–(A9) *be satisfied and suppose there are* $R > r > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup \Phi(\partial Q) \leq \varrho$ *where* $Q := \{u = x + te : x \in X, t \geq 0, \|u\| < R\}$. *Then* Φ *has a* $(C)_c$*-sequence with* $\varrho \leq c \leq \sup \Phi(Q)$.

Theorem 2.4. *Assume* Φ *is even with* $\Phi(0) = 0$ *and let* (A7)–(A11) *be satisfied. Then* Φ *possesses an unbounded sequence of positive critical values.*

3. Proof of main results

First, let $r > 0$, set $B_r := \{u \in E : \|u\| \leq r\}$, $S_r := \{u \in E : \|u\| = r\}$. From assumptions (A3)–(A5), for any $\epsilon > 0$, there exist positive constants r_ϵ, C_ϵ such that

$$
\begin{gathered}
|F_u(x,u)| \leq \epsilon|u| \quad \text{for all } 0 \leq |u| \leq r_\epsilon, \\
|F_u(x,u)| \leq \epsilon|u| + C_\epsilon|u|^{p-1} \quad \text{for all } (x,u), \\
|F(x,u)| \leq \epsilon|u|^2 + C_\epsilon|u|^p \quad \text{for all } (x,u),
\end{gathered} \tag{3.1}
$$

where $p \in (2,3)$. Before proving our result, we need some preliminary results.

Lemma 3.1. Γ *and* Ψ *are non-negative, weakly sequentially lower semi-continuous,* Γ' *and* Ψ' *are weakly sequentially continuous. Moreover, there is* $\xi > 0$ *such that for any* $c > 0$,

$$
\|u\| \leq \xi\|u^+\|, \quad \textit{for all } u \in \Phi_c.
$$

Proof. By a standard argument of [38], Ψ and Ψ' are obvious. So it is sufficient to show that Γ and Γ' have the above property. Clearly, Γ is non-negative. Let $u_n \rightharpoonup u$ in E, we can assume, up to a subsequence, that $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$. It follows from Fatou's lemma that

$$
\Gamma(u) \leq \liminf_{n\to\infty} \Gamma(u_n).
$$

Hence Γ is weakly sequentially lower semi-continuous.

Next, we show that Γ' is weakly sequentially continuous. Let $u_n \rightharpoonup u$ in E, we can assume, up to a subsequence, that $u_n \to u$ in $L^s_{\rm loc}$ for all $s \in [1,3)$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$. It is not difficult to prove that

$$
\begin{gathered}
\Gamma'(u_n)\varphi = \int_{\mathbb{R}^3} \phi_{u_n} u_n \bar{\varphi} \mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u u \bar{\varphi} \mathrm{d}x = \Gamma'(u)\varphi, \\
|\Gamma'(u)\varphi| \leq C_0 \|u\|^3 \|\varphi\|.
\end{gathered}
$$

for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Since C_0^∞ is dense in E, for any $v \in E$ we take $\varphi_n \in C_0^\infty$ such that $\|\varphi_n - v\| \to 0$. Note that

$$
|(\Gamma'(u_n) - \Gamma'(u))\varphi_n| \to 0.
$$

Thus, by the above facts we obtain

$$
\big|\Gamma'(u_n)v - \Gamma'(u)v\big|
$$

$$
\begin{aligned}
&= |\Gamma'(u_n)v - \Gamma'(u_n)\varphi_n + \Gamma'(u_n)\varphi_n - \Gamma'(u)\varphi_n + \Gamma'(u)\varphi_n - \Gamma'(u)v| \\
&\le |(\Gamma'(u_n) - \Gamma'(u))\varphi_n| + |(\Gamma'(u_n) - \Gamma'(u))(v - \varphi_n)| \\
&\le |(\Gamma'(u_n) - \Gamma'(u))\varphi_n| + C_0(\|u_n\|^3 + \|u\|^3)\|\varphi_n - v\| \to 0.
\end{aligned}
$$

Therefore, we have shown that Γ' is weakly sequentially continuous.

On the other hand, for any $c > 0$ and $u \in \Phi_c$, using the fact that $\Gamma, \Psi \ge 0$ one has

$$
0 < c \le \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2).
$$

This yields $\|u^-\| \le \|u^+\|$, and hence $\|u\| \le \sqrt{2}\|u^+\|$. Thus we obtain the second conclusion. □

Lemma 3.2. *Let* (A3)–(A5) *be satisfied. Then there exists* $r > 0$ *such that* $\varrho := \inf \Phi(S_r \cap E^+) > 0$.

Proof. Observe that $|u|_p^p \le c_p\|u\|^p$ for all $u \in E$ by Lemma 2.2. For any $u \in E^+$, by (2.4) and (3.1) we have

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \Gamma(u) - \Psi(u) \\
&\ge \frac{1}{2}\|u\|^2 - C_1\|u\|^4 - c_2\epsilon\|u\|^2 - C_\epsilon c_p\|u\|^p \\
&= (\frac{1}{2} - c_2\epsilon)\|u\|^2 - C_1\|u\|^4 - C_\epsilon c_p\|u\|^p.
\end{aligned}
$$

Since $p \in (2, 3)$, choosing suitable $r > 0$ we see that the desired conclusion holds. □

As a consequence of Lemma 2.1 we have

$$
a \le \inf \sigma(A) \cap [0, \infty) \le a + \sup_{\mathbb{R}^3} V.
$$

Let $\Lambda := \inf_{x\in\mathbb{R}^3} G(x)$. By (A5), we take a positive number μ such that

$$
a + \sup_{\mathbb{R}^3} V < \mu < \Lambda - \omega. \tag{3.2}
$$

Since A is invariant under the action of $\mathbb{Z}^3$ by (A2), the subspace $Y_0 := (\mathcal{F}_\mu - \mathcal{F}_0)L^2$ is infinite-dimensional, and

$$
(a + \omega)|u|_2^2 < \|u\|^2 < (\mu + \omega)|u|_2^2 \quad \text{for all } u \in Y_0. \tag{3.3}
$$

Let $\{\mu_n\} \subset \sigma(A)$ satisfy $\mu_0 := a < \mu_1 < \mu_2 < \dots \le \mu$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we take an element $e_n \in (\mathcal{F}_{\mu_n} - \mathcal{F}_{\mu_{n-1}})L^2$ with $\|e_n\| = 1$ and define $Y_n := \operatorname{span}\{e_1, \dots, e_n\}$, $E_n := E^- \oplus Y_n$.

Lemma 3.3. *Let* (A3)–(A5) *be satisfied and* $r > 0$ *be given by Lemma 3.2. Then* $\sup \Phi(E_n) < \infty$, *and there is a sequence* $R_n > 0$ *such that* $\sup \Phi(E_n \setminus B_n) < \inf \Phi(B_n)$, *where* $B_n := \{u \in E_n : \|u\| \le R_n\}$.

Proof. It is sufficient to prove that $\Phi(u) \to -\infty$ in E_n as $\|u\| \to \infty$. If not, then there are $M > 0$ and $\{u_n\} \subset E_n$ with $\|u_n\| \to \infty$ such that $\Phi(u_n) \ge -M$ for all n. Denote $v_n := \frac{u_n}{\|u_n\|}$, passing to a subsequence if necessary, $v_n \rightharpoonup v$, $v_n^- \rightharpoonup v^-$ and $v_n^+ \rightharpoonup v^+$. Since $\Gamma(u) \ge 0$ and $\Psi(u) \ge 0$,

$$
\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) \ge \frac{\Phi(u_n)}{\|u_n\|^2} \ge \frac{-M}{\|u_n\|^2}, \tag{3.4}
$$

which implies

$$\frac{1}{2}\|v_n^-\|^2 \le \frac{1}{2}\|v_n^+\|^2 + \frac{M}{\|u_n\|^2}. \tag{3.5}$$

We claim that $v^+ \neq 0$. Indeed, if not, (3.5) yields $\|v_n^-\| \to 0$. Thus $\|v_n\| \to 0$, which contradicts $\|v_n\| = 1$. It follows from (3.2) and (3.3) that

$$\begin{aligned}\|v^+\|^2 - \|v^-\|^2 - \int_{\mathbb{R}^3} G(x)v^2 \mathrm{d}x &\le \|v^+\|^2 - \|v^-\|^2 - \Lambda|v|_2^2 \\ &\le -(\Lambda - \mu - \omega)|v^+|_2^2 - \|v^-\|^2 - \Lambda|v^-|_2^2 < 0,\end{aligned}$$

then there exists a bounded set $\Omega \subset \mathbb{R}^3$ such that

$$\|v^+\|^2 - \|v^-\|^2 - \int_\Omega G(x)v^2 \mathrm{d}x < 0. \tag{3.6}$$

Letting $R(x,u) := F(x,u) - \frac{1}{2}G(x)u^2$, then $|R(x,u)| \le C_2|u|^2$ for some $C_2 > 0$ and $\frac{R(x,u)}{|u|^2} \to 0$ as $|u| \to \infty$ uniformly in x. Hence, by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega \frac{R(x,u_n)}{\|u_n\|^2} \mathrm{d}x = \lim_{n\to\infty} \int_\Omega \frac{R(x,u_n)}{|u_n|^2} |v_n|^2 \mathrm{d}x = 0. \tag{3.7}$$

Thus (3.4), (3.6) and (3.7) imply

$$\begin{aligned}0 &\le \lim_{n\to\infty} \Big(\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \frac{1}{4}\int_{\mathbb{R}^3} \frac{\phi_{u_n}|u_n|^2}{\|u_n\|^2}\mathrm{d}x - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|^2}\mathrm{d}x\Big) \\ &\le \lim_{n\to\infty} \Big(\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}\mathrm{d}x\Big) \\ &\le \frac{1}{2}\Big(\|v^+\|^2 - \|v^-\|^2 - \int_\Omega G(x)v^2\mathrm{d}x\Big) < 0.\end{aligned}$$

Now the desired conclusion is obtained from this contradiction. □

As a consequence, we have the following result.

Lemma 3.4. *Let* (A3)–(A5) *be satisfied. Then there is* $R_0 > r > 0$*, such that* $\Phi|_{\partial Q} \le \varrho$*, where* $\varrho > 0$ *be given by Lemma 3.2,* $Q := \{u = u^- + se : u^- \in E^-, s \ge 0, \|u\| \le R_0\}$*.*

Next we discuss the properties of the $(C)_c$-sequences. Since the presence of nonlocal term $\Gamma(u)$, it is not easy to prove the boundedness of the $(C)_c$-sequence for the functional Φ. Motivated by Ackermann [3], we give a delicate estimate for the norm of $\Gamma'(u)$ by using some special techniques, it is very important in our arguments.

Lemma 3.5. *For any* $u \in E\backslash\{0\}$*, there exists* $C > 0$ *such that*

$$\Gamma'(u)u > 0 \quad \text{and} \quad \|\Gamma'(u)\|_{E^*} \le C\big(\sqrt{\Gamma'(u)u} + \Gamma'(u)u\big),$$

where E^* *denotes the dual space of* E*.*

Proof. Clearly, $\Gamma'(u)u = 4\Gamma(u) > 0$ for any $u \in E\backslash\{0\}$. Now we show the second conclusion. Since Γ is the unique nonlocal term in Φ, from the argument in Ackermann [3](see also [4]), we have

$$\int_{\mathbb{R}^3} \big(\frac{1}{|x|} * |u|^2\big)|v|^2\mathrm{d}x \le C_3\Big(\int_{\mathbb{R}^3} \big(\frac{1}{|x|} * |u|^2\big)|u|^2\mathrm{d}x \int_{\mathbb{R}^3} \big(\frac{1}{|x|} * |v|^2\big)|v|^2\mathrm{d}x\Big)^{1/2}$$

for all $u, v \in E$ and some $C_3 > 0$. Hence using this, (2.4) and Hölder inequality, we can obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\frac{1}{|x|} * |u|^2)|uv|\mathrm{d}x \\
&\le \Big(\int_{\mathbb{R}^3} (\frac{1}{|x|} * |u|^2)|u|^2\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}^3} (\frac{1}{|x|} * |u|^2)|v|^2\mathrm{d}x\Big)^{1/2} \\
&\le C_4\Big(\int_{\mathbb{R}^3} (\frac{1}{|x|} * |u|^2)|u|^2\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}^3} (\frac{1}{|x|} * |u|^2)|u|^2\mathrm{d}x\Big)^{1/4} \\
&\quad \times \Big(\int_{\mathbb{R}^3} (\frac{1}{|x|} * |v|^2)|v|^2\mathrm{d}x\Big)^{1/4} \\
&\le C_5\Big(\int_{\mathbb{R}^3} (\frac{1}{|x|} * |u|^2)|u|^2\mathrm{d}x\Big)^{3/4}\|v\|,
\end{aligned}
$$

which implies

$$
|\Gamma'(u)v| \le C_5\left(\Gamma'(u)u\right)^{3/4}\|v\| \le C\left(\sqrt{\Gamma'(u)u} + \Gamma'(u)u\right)\|v\|.
$$

This shows the second conclusion. □

Lemma 3.6. *Suppose that* (A3)–(A6) *are satisfied. Then any* $(C)_c$*-sequence of* Φ *is bounded.*

Proof. Let $\{u_n\} \subset E$ be such that

$$
\Phi(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\Phi'(u_n) \to 0. \tag{3.8}
$$

Then, there is constant $C_0 > 0$ such that

$$
C_0 \ge \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n = \Gamma(u_n) + \int_{\mathbb{R}^3} \widetilde{F}(x, u_n)\mathrm{d}x. \tag{3.9}
$$

Suppose to the contrary that $\{u_n\}$ is unbounded. Setting $v_n := u_n/\|u_n\|$, then $\|v_n\| = 1$ and $|v_n|_s \le c_s\|v_n\| = c_s$ for all $s \in [2, 3]$. Observe that

$$
\Phi'(u_n)(u_n^+ - u_n^-) = \|u_n\|^2\Big(1 - \frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2} - \int_{\mathbb{R}^3} \frac{F_u(x, u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\mathrm{d}x\Big).
$$

Hence

$$
\frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2} + \int_{\mathbb{R}^3} \frac{F_u(x, u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\mathrm{d}x \to 1. \tag{3.10}
$$

Let

$$
\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |v_n|^2\mathrm{d}x.
$$

If $\delta = 0$, by Lions' concentration compactness principle in [31] or [38, Lemma 1.21], then $v_n \to 0$ in L^s for any $s \in (2, 3)$. Set

$$
\Omega_n := \big\{x \in \mathbb{R}^3 : \frac{|F_u(x, u_n)|}{|u_n|} \le a - |\omega| - \delta_1\big\}. \tag{3.11}
$$

Then by Lemma 2.2 and (3.11), we have

$$
\int_{\Omega_n} \frac{|F_u(x, u_n)|}{|u_n|}|v_n||v_n^+ - v_n^-|\mathrm{d}x \le (a - |\omega| - \delta_1)|v_n|_2^2 \le 1 - \frac{\delta_1}{a - |\omega|}. \tag{3.12}
$$

Choose $q > 3$, then $q' := q/(q-1) \in (1, 3/2)$. Hence, by (A4)–(A6) and (3.9), we have

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus\Omega_n} \frac{|F_u(x,u_n)|}{|u_n|}|v_n||v_n^+ - v_n^-|\mathrm{d}x \\ &\le C_6 \int_{\mathbb{R}^3\setminus\Omega_n} \Big(\widetilde{F}(x,u_n)\Big)^{1/q} |v_n||v_n^+ - v_n^-|\mathrm{d}x \\ &\le C_6\Big(\int_{\mathbb{R}^3\setminus\Omega_n} \widetilde{F}(x,u_n)\mathrm{d}x\Big)^{1/q}|v_n|_{2q'}|v_n^+ - v_n^-|_{2q'} \\ &\le C_7|v_n|_{2q'}|v_n^+ - v_n^-|_{2q'} = o(1). \end{aligned} \tag{3.13}$$

From (3.9), for the nonlocal term, we easily show that

$$\frac{\Gamma(u_n)}{\|u_n\|} \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

Moreover, by Lemma 3.5, we have

$$\begin{aligned} \Big|\frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\Big| &\le \frac{\|\Gamma'(u_n)\|_{E^*}\|u_n^+ - u_n^-\|}{\|u_n\|^2} \\ &\le C_8\Big|\frac{\big(\sqrt{\Gamma'(u_n)u_n} + \Gamma'(u_n)u_n\big)\|u_n^+ - u_n^-\|}{\|u_n\|^2}\Big| \\ &\le C_9\Big|\frac{\sqrt{\Gamma'(u_n)u_n} + \Gamma'(u_n)u_n}{\|u_n\|}\Big| \\ &= C_9\Big(\frac{1}{\sqrt{\|u_n\|}}\sqrt{\frac{4\Gamma(u_n)}{\|u_n\|}} + \frac{4\Gamma(u_n)}{\|u_n\|}\Big) = o(1). \end{aligned} \tag{3.15}$$

By (3.10), (3.12), (3.13) and (3.15) we have

$$\begin{aligned} 1 + o(1) &= \frac{\Gamma'(u_n)(u_n^+ - u_n^-)}{\|u_n\|^2} + \int_{\mathbb{R}^3} \frac{F_u(x,u_n)(u_n^+ - u_n^-)}{\|u_n\|^2}\mathrm{d}x \\ &\le 1 - \frac{\delta_1}{a - |\omega|} + o(1). \end{aligned}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^3$ such that $\int_{B_{1+\sqrt{3}}(k_n)} |v_n|^2\mathrm{d}x > \delta/2$. Let $\tilde{v}_n(x) = v_n(x+k_n)$. Since $V(x)$ is 1-periodic in each of x_1, x_2, x_3. Then $\|\tilde{v}_n\| = \|v_n\| = 1$, and

$$\int_{B_{1+\sqrt{3}}(0)} |\tilde{v}_n|^2\mathrm{d}x > \frac{\delta}{2}. \tag{3.16}$$

Passing to a subsequence, we have $\tilde{v}_n \rightharpoonup \tilde{v}$ in E, $\tilde{v}_n \to \tilde{v}$ in $L^s_{\rm loc}$, for all $s \in [1,3)$, $\tilde{v}_n \to \tilde{v}$ a.e. on $\mathbb{R}^3$. Obviously, (3.16) implies that $\tilde{v} \neq 0$.

Now we define $\tilde{u}_n(x) = u_n(x + k_n)$, then $\tilde{u}_n/\|u_n\| = \tilde{v}_n \to \tilde{v}$ a.e. on $\mathbb{R}^3$, $\tilde{v} \neq 0$. For $x \in \Omega_0 := \{y \in \mathbb{R}^3 : \tilde{v}(y) \neq 0\}$, we have $\lim_{n\to\infty} |\tilde{u}_n(x)| = \infty$. For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, setting $\varphi_n(x) = \varphi(x - k_n)$, then

$$\begin{aligned} \Phi'(u_n)\varphi_n &= (u_n^+ - u_n^-, \varphi_n) - \Gamma'(u_n)\varphi_n - \int_{\mathbb{R}^3} F_u(x,u_n)\varphi_n\mathrm{d}x \\ &= \|u_n\|\Big((v_n^+ - v_n^-, \varphi_n) - \frac{\Gamma'(u_n)\varphi_n}{\|u_n\|} - \int_{\mathbb{R}^3} \frac{F_u(x,u_n)\varphi_n}{\|u_n\|}\mathrm{d}x\Big) \end{aligned}$$

$$= \|u_n\|\Big((\tilde{v}_n^+ - \tilde{v}_n^-, \varphi) - \frac{\Gamma'(\tilde{u}_n)\varphi}{\|\tilde{u}_n\|} - \int_{\mathbb{R}^3} \frac{F_u(x,\tilde{u}_n)\varphi}{\|\tilde{u}_n\|}\mathrm{d}x\Big).$$

This yields

$$(\tilde{v}_n^+ - \tilde{v}_n^-, \varphi) - \frac{\Gamma'(\tilde{u}_n)\varphi}{\|\tilde{u}_n\|} - \int_{\mathbb{R}^3} \frac{F_u(x,\tilde{u}_n)\varphi}{\|\tilde{u}_n\|}\mathrm{d}x \to 0. \tag{3.17}$$

Observe that, by (3.14) and Lemma 3.5, we have

$$\begin{aligned}
\Big|\frac{\Gamma'(\tilde{u}_n)\varphi}{\|\tilde{u}_n\|}\Big| &\le \frac{\|\Gamma'(\tilde{u}_n)\|_{E^*}\|\varphi\|}{\|\tilde{u}_n\|} \\
&\le C_{10}\Big|\frac{\big(\sqrt{\Gamma'(\tilde{u}_n)\tilde{u}_n} + \Gamma'(\tilde{u}_n)\tilde{u}_n\big)\|\varphi\|}{\|\tilde{u}_n\|}\Big| \\
&= C_{10}\Big(\frac{1}{\sqrt{\|\tilde{u}_n\|}}\sqrt{\frac{4\Gamma(\tilde{u}_n)}{\|\tilde{u}_n\|}} + \frac{4\Gamma(\tilde{u}_n)}{\|\tilde{u}_n\|}\Big)\|\varphi\| = o(1).
\end{aligned} \tag{3.18}$$

On the other hand, $|\tilde{u}_n(x)| \to \infty$ since $v(x) \neq 0$. By (A5) and Lebesgue's dominated convergence theorem, it is easy to see that

$$\int_{\mathbb{R}^3} \frac{F_u(x,\tilde{u}_n)\varphi}{\|\tilde{u}_n\|}\mathrm{d}x \to \int_{\mathbb{R}^3} G(x)\tilde{v}\varphi\mathrm{d}x. \tag{3.19}$$

Hence, it follows from (3.17)–(3.19) that

$$(\tilde{v}^+ - \tilde{v}^-, \varphi) - \int_{\mathbb{R}^3} G(x)\tilde{v}\varphi\mathrm{d}x = 0. \tag{3.20}$$

This implies that $A\tilde{v} = (G(x) - \omega)\tilde{v}$. By the weak unique continuation property for Dirac operator [6] or [16, p.128], we deduce that $|\Omega_0| = \infty$. Hence, we can choose $\varepsilon_0 > 0$ such that $|\Omega_0'| \ge 3C_0/\delta_1$, where C_0 is given in (3.9) and

$$\Omega_0' := \{x \in \mathbb{R}^3 : |\tilde{v}(x)| \ge 2\varepsilon_0\}. \tag{3.21}$$

By Egoroff's theorem, we can find a set $\Omega_0'' \subset \Omega_0'$ with $|\Omega_0''| \ge 2C_0/\delta_1$ such that $\tilde{v}_n \to \tilde{v}$ uniformly on Ω_0''. So there is an integer $n_0 \ge 1$ such that

$$|\tilde{v}_n(x)| \ge \varepsilon_0, \quad \forall x \in \Omega_0'',\ n \ge n_0. \tag{3.22}$$

By (A5), there exists a $r_0 > 0$ such that

$$\frac{|F_u(x,u)|}{|u|} \ge G(x) - \frac{|R_u(x,u)|}{|u|} \ge a + \omega + \sup_{\mathbb{R}^3} V - \delta_1, \quad \forall x \in \mathbb{R}^3,\ |u| \ge r_0. \tag{3.23}$$

Combining (3.22) with (3.23), one has

$$\frac{|F_u(x,\tilde{u}_n)|}{|\tilde{u}_n|} \ge a - |\omega| - \delta_1, \quad \forall x \in \Omega_0'',\ n \ge n_1, \tag{3.24}$$

where $n_1 \in \mathbb{Z}$ such that $|\tilde{u}_n(x)| \ge r_0$ for $x \in \Omega_0''$ and $n \ge n_1$. It follows from (A6) and (3.24) that $\widetilde{F}(x,u_n) \ge \delta_1$ for $x \in \Omega_0''$ and $n \ge n_1$. Hence,

$$C_0 \ge \int_{\mathbb{R}^3} \widetilde{F}(x,u_n)\mathrm{d}x \ge \delta_1|\Omega_0''| \ge 2C_0, \quad \text{for } n \ge n_1,$$

a contradiction. Therefore $\{u_n\}$ is bounded in E. □

Let $\{u_n\} \subset E$ be a $(C)_c$-sequence of Φ, by Lemma 3.6, it is bounded, up to a subsequence, we may assume $u_n \rightharpoonup u$ in E, $u_u \to u$ in $L^s_{\rm loc}$ for all $s \in (1,3)$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$. Obviously, u is a critical point of Φ. Set $v_n := u_n - u$, then $v_n \rightharpoonup 0$ in E. Using Brezis-Lieb lemma in [9], we can prove the following results.

Lemma 3.7. *Let $\{u_n\}$ be a $(C)_c$-sequence of Φ at level c, and set $v_n := u_n - u$. Then, passing to a subsequence,*

$$\lim_{n\to\infty}\Big(\int_{\mathbb{R}^3}\Big(F(x,u_n)-F(x,u)-F(x,v_n)\Big)\mathrm{d}x\Big)=0,$$
$$\lim_{n\to\infty}\big(\Gamma(u_n)-\Gamma(u)-\Gamma(v_n)\big)=0,$$
$$\lim_{n\to\infty}\big(\Gamma'(u_n)\varphi-\Gamma'(u)\varphi-\Gamma'(v_n)\varphi\big)=0,$$
$$\lim_{n\to\infty}\Big(\int_{\mathbb{R}^3}\big(F_u(x,u_n)-F_u(x,u)-F_u(x,v_n)\big)\varphi\mathrm{d}x\Big)=0$$

uniformly in $\varphi \in E$.

Lemma 3.8. *Let $\{u_n\}$ be a $(C)_c$-sequence of Φ at level c, and set $v_n := u_n - u$. Then, passing to a subsequence,*

$$\Phi(v_n) \to c - \Phi(u) \text{ and } \Phi'(v_n) \to 0.$$

Let $\mathcal{K} := \{u \in E : \Phi'(u) = 0, u \neq 0\}$ be the set of nontrivial critical points of Φ.

Lemma 3.9. *Under the assumptions of Theorem 1.1, the following two conclusions hold*

(1) $\nu := \inf\{\|u\| : u \in \mathcal{K}\} > 0$;
(2) $\theta := \inf\{\Phi(u) : u \in \mathcal{K}\} > 0$.

Proof. (1) For any $u \in \mathcal{K}$, it holds

$$0 = \Phi'(u)(u^+ - u^-) = \|u\|^2 - \Gamma'(u)(u^+ - u^-) - \int_{\mathbb{R}^3} F_u(x,u)(u^+ - u^-)\mathrm{d}x\,.$$

This (2.4) and (3.1) imply

$$\|u\|^2 \le C\|u\|^4 + \epsilon\|u\|^2 + C_\epsilon\|u\|^p,$$

where $p \in (2,3)$. Choose ϵ small enough, hence

$$0 < (1-\epsilon)\|u\|^2 \le C\|u\|^4 + C_\epsilon\|u\|^{p-2},$$

which implies that $\|u\| > 0$.

(2) Suppose to the contrary that there exist a sequence $\{u_n\} \subset \mathcal{K}$ such that $\Phi(u_n) \to 0$. By the first conclusion, $\|u_n\| \ge \nu$. Clearly, $\{u_n\}$ is a $(C)_0$-sequence of Φ, and hence is bounded by Lemma 3.6. Moreover, $\{u_n\}$ is nonvanishing. By the invariance under translation of Φ, we can assume, up to a translation, that $u_n \rightharpoonup u \in \mathcal{K}$. Moreover, by Fatou's lemma and Lemma 3.5, we have

$$\begin{aligned}
0 = \lim_{n\to\infty}\Phi(u_n) &= \lim_{n\to\infty}\Big(\Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n\Big)\\
&= \lim_{n\to\infty}\Big(\Gamma(u_n) + \int_{\mathbb{R}^3}\widetilde{F}(x,u_n)\mathrm{d}x\Big)\\
&\ge \Gamma(u) + \int_{\mathbb{R}^3}\widetilde{F}(x,u)\mathrm{d}x > 0,
\end{aligned}$$

a contraction. This completes the proof. □

In the following lemma we discuss further the $(C)_c$-sequence. Let $[l]$ denote the integer part of $l \in \mathbb{R}$. Combining Lemma 3.8, Lemma 3.9 and a standard argument, we have the following lemma (see Coti-Zelati and Rabinowitz [12, 13]).

Lemma 3.10. *Under the assumptions of Theorem 1.1, let $\{u_n\} \subset E$ ba a $(C)_c$-sequence of Φ. Then either*

(i) $u_n \to 0$ *(and hence $c = 0$), or*

(ii) *$c \geq \theta$ and there exist a positive integer $l \leq [\frac{c}{\theta}]$, $u_1, \dots, u_l \in \mathcal{K}$ and sequences $\{a_n^i \subset \mathbb{Z}^3\}$, $i = 1, 2, \dots, l$, such that, after extraction of a subsequence of $\{u_n\}$,*

$$\|u_n - \sum_{i=1}^{l} a_n^i * u_i\| \to 0 \quad \text{and} \quad \sum_{i=1}^{l} \Phi(u_i) = c$$

and for $i \neq k$, $|a_n^i - a_n^k| \to \infty$.

Proof of Theorem 1.1. (Existence) With $X = E^-$ and $Y = E^+$. By Lemma 3.1, we see that (A7) and (A8) are satisfied. Lemma 3.2 implies that (A9) holds. Lemma 3.4 shows that Φ possesses the linking structure of Theorem 2.3. Therefore, using Theorem 2.3, there exists a sequence $\{u_n\} \subset E$ such that $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$. By Lemma 3.6, $\{u_n\}$ is bounded in E. Let

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |u_n|^2 \mathrm{d}x.$$

If $\delta = 0$, by Lions' concentration compactness principle in [31] or [38, Lemma 1.21], then $u_n \to 0$ in L^s for any $s \in (2, 3)$. Therefore, it follows from (2.4) and (3.1) that

$$\int_{\mathbb{R}^3} F(x, u_n)\mathrm{d}x \to 0, \quad \int_{\mathbb{R}^3} F_u(x, u_n)u_n \mathrm{d}x \to 0 \quad \text{and} \quad \Gamma(u_n) \to 0$$

as $n \to \infty$. Consequently,

$$\begin{aligned} c &= \lim_{n\to\infty} \Big(\Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n\Big) \\ &= \lim_{n\to\infty} \Big(\Gamma(u_n) + \int_{\mathbb{R}^3} \widetilde{F}(x, u_n)\mathrm{d}x\Big) = 0. \end{aligned}$$

This is a contradiction. Hence $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^3$ such that

$$\int_{B(k_n, 1+\sqrt{3})} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}.$$

Let us define $v_n(x) = u_n(x + k_n)$ so that

$$\int_{B(0, 1+\sqrt{3})} |v_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.25}$$

Since Φ and Φ' are $\mathbb{Z}^3$-translation invariant, we obtain $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|)\Phi'(v_n) \to 0. \tag{3.26}$$

Passing to a subsequence, we have $v_n \rightharpoonup v$ in E, $v_n \to v$ in L^s_{loc}, for all $s \in [1, 3)$ and $v_n \to v$ a.e. on $\mathbb{R}^3$. Hence it follows from (3.25) and (3.26) that $\Phi'(v) = 0$ and $v \neq 0$. This shows that $v \in \mathcal{K}$ is a nontrivial of system (1.2).

(Multiplicity) Φ is even provided $F(x,u)$ is even in u. Lemma 3.3 shows that Φ satisfies (A10). Next we assume

$$\mathcal{K}/\mathbb{Z}^3 \text{ is a finite set.} \tag{3.27}$$

In fact, if (3.27) is false, then the last conclusion of Theorem 1.1 holds automatically. In the sequel, we assume (3.27) holds. Let $\mathcal{F}$ be a set consisting of arbitrarily chosen representatives of the $\mathbb{Z}^3$-orbits of $\mathcal{K}$. Then $\mathcal{F}$ is a finite set by (3.27), and since Φ' is odd we may assume that $\mathcal{F} = -\mathcal{F}$. If $u \in \mathcal{K}$, then $\Phi(u) \geq \theta$ by (2) of Lemma 3.9. Hence there exists $\theta \leq \vartheta$ such that

$$\theta \leq \min_{\mathcal{F}} \Phi = \min_{\mathcal{K}} \Phi \leq \max_{\mathcal{K}} \Phi = \max_{\mathcal{F}} \Phi \leq \vartheta.$$

For $l \in \mathbb{N}$ and a finite set $\mathcal{A} \subset E$ we define

$$[\mathcal{A}, l] := \Big\{ \sum_{i=1}^{j} a_i * u_i \mid 1 \leq j \leq l, a_i \in \mathbb{Z}^3, u_i \in \mathcal{A} \Big\}.$$

As in Coti-Zelati and Rabinowitz [12, 13],

$$\inf\{\|u - u'\| : u, u' \in [\mathcal{A}, l]\} > 0. \tag{3.28}$$

Now we check (A11). Given a compact interval $I \subset (0,\infty)$ with $d := \max I$ and $\mathcal{O} = [\mathcal{F}, l]$. We have $P^+[\mathcal{F}, l] = [P^+\mathcal{F}, l]$. Thus from (3.28)

$$\inf\{\|u_1^+ - u_2^+\| : u_1, u_2 \in \mathcal{O}, u_1^+ \neq u_2^+\} > 0.$$

In addition, $\mathcal{O}$ is a $(C)_I$-attractor by Lemma 3.10 and $\mathcal{O}$ is bounded because $\|u\| \leq l \max\{\|\bar{u}\| : \bar{u} \in \mathcal{F}\}$ for all $u \in \mathcal{O}$. Therefore, by Theorem 2.4, Φ has a unbounded sequence of critical values which contradicts with the assumption (3.27), and hence Φ has infinitely many geometrically distinct nontrivial critical points. Therefore, our multiplicity result follows. This completes the proof. $\square$

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