Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 98, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NAVIER-STOKES EQUATIONS IN THE HALF-SPACE IN VARIABLE EXPONENT SPACES OF CLIFFORD-VALUED FUNCTIONS

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Communicated by Vicentiu Radulescu

Abstract. In this article, we study the steady generalized Navier-Stokes equations in a half-space in the setting of variable exponent spaces. We first establish variable exponent spaces of Clifford-valued functions in a half-space. Then, using this operator theory together with the contraction mapping principle, we obtain the existence and uniqueness of solutions to the stationary Navier-Stokes equations and Navier-Stokes equations with heat conduction in a half-space under suitable hypotheses.

1. Introduction

Since Kováčik and Rákosník [24] first studied the spaces $L^{p(x)}$ and $W^{k,p(x)}$, more and more attention are paid to Lebesgue and Sobolev variable exponent spaces and their applications to differential equations. See [7, 8] for basic properties of variable exponent spaces and [21, 33] for recent overviews of differential equations with variable growth. It is well-known that one of the reasons that forced the rapid expansion of the theory of variable exponent function spaces has been the models of electrorheological fluids introduced by Rajagopal and Růžička [29, 30], which can be described by the boundary-value problem for the generalized Navier-Stokes equations. In the setting of variable exponent spaces, Diening et al. [5] proved the existence and uniqueness of strong and weak solutions of the Stokes system and Poisson equations for bounded domains, the whole-space and the half-space, respectively.

In the previous decades, the study of these spaces has been stimulated by problems in elastic mechanics, calculus of variations and differential equations with variable growth conditions, see [9, 12, 31, 32, 34, 35, 36] and references therein.

As a powerful tool for solving elliptic boundary value problems in the plane, the methods of complex function theory play an important role. One way to extend these ideas to higher dimension is to begin with a generalization of algebraic and geometrical properties of the complex numbers. In this way, Hamilton studied the algebra of quaternion in 1843. Further generalizations were introduced by Clifford

2010 *Mathematics Subject Classification.* 30G35, 35J60, 35Q30, 46E30, 76D03.
Key words and phrases. Clifford analysis; variable exponent; Navier-Stokes equations; half-space.

Submitted January 28, 2017. Published April 5, 2017.

in 1878. He initiated the so-called geometric algebras or Clifford algebras, which are generalizations of the complex numbers, the quaternions, and the exterior algebras, see [19]. After that, Clifford algebras have important applications in a variety of fields including geometry, theoretical physics and digital image processing. Clifford analysis as an active branch of mathematics concerned with the study of Dirac equation or of a generalized Cauchy-Riemann system, in which solutions are defined on domains in the Euclidean space and take values in Clifford algebras, see the monograph of Brackx et al. [1]. It is worthy mentioning that Gürlebeck and Sprößig [14, 15] developed an operator calculus, which is analogous to the known complex analytic approach in the plane and based on three operators: a Cauchy-Riemann-type operator, a Teodorescu transform, and a generalized Cauchy-type integral operator, to investigate elliptic boundary value problems of fluid dynamics over bounded and unbounded domains, especially the Navier-Stokes equations and related equations. Of course, there are a number of unsolved basic problems involving the Navier-Stokes equations. This is mainly due to the problem concerning the solvability of the corresponding linear Stokes equations over domains, see [2, 16]. As Galdi [20] pointed out, the study of the Stokes problem in the half-space possesses an independent interest and it will be fundamental for the treatment of other linear and nonlinear problems when the region of flow is either an exterior domain or a domain with a suitable unbounded boundary.

On the one hand, Diening et al. [4] studied the following model introduced in [29, 30] to describe motions of electrorheological fluids:

$$\begin{aligned} -\operatorname{div}\mathcal{M}(\mathrm{D}u) + (u\cdot\nabla)u + \nabla\pi = f \quad x\in\Omega \\ \operatorname{div} u = 0 \quad x\in\Omega \\ u = 0 \quad x\in\partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain with Lipschitz boundary $\partial\Omega$, $f\in(W_0^{1,p(x)}(\Omega))^* = W^{-1,p'(x)}(\Omega)$, $2n/(n+2) < p_- \le p_+ < \infty$ and the operator $\mathcal{M}$ satisfies certain natural variable growth conditions. The authors obtained the existence of weak solutions in $(W_0^{1,p(x)}(\Omega))^n \times L_0^s(\Omega)$, here $s := \min\{(p_+)', np_-/2(n-p_-)\}$ if $p_- < n$ and $s := (p_+)'$ otherwise, $L_0^s(\Omega) := \{\pi\in L^s(\Omega) : \int_\Omega \pi dx = 0\}$. Diening et al. [5] studied the Stokes and Poisson problem in the context of variable exponent spaces in bounded domains and in the whole space. In the half-space case, the authors employed a localization technique to reduce the interior and the boundary regularity to regularity results on the half-space. While it should be pointed out that our attempt is to give a unified approach to deal with physical problems modelled by the generalized Navier–Stokes equations, which is quite different with approaches of some authors, for example, we refer to the monograph [3].

On the other hand, it is natural to focus on the A-Dirac equations if one interests in extending the classical Dirac equations. In [26, 27], Nolder first introduced the general nonlinear A-Dirac equations $DA(x, Du) = 0$ which arise in the study of many phenomena in physical sciences. Moreover, he developed some tools for the study of weak solutions to nonlinear A-Dirac equations in the space $W_0^{1,p}(\Omega, \mathrm{C}\ell_n)$. Inspired by his works, Fu and Zhang in [10, 11] were devoted to the the existence of weak solutions for the general nonlinear A-Dirac equations with variable growth. For this purpose, the authors established a theory of variable exponent spaces of Clifford-valued functions with applications to homogeneous and non-homogeneous

A-Dirac equations, see also [37]. Recently, Fu et al. [9, 28, 38] established a Hodge-type decomposition of variable exponent Lebesgue spaces of Clifford-valued functions with applications to the Stokes equations, the Navier-Stokes equations and the A-Dirac equations $DA(Du) = 0$. By using the Hodge-type decomposition and variational methods, Molica Bisci et al. [25] studied the properties of weak solutions to the homogeneous and nonhomogeneous A-Dirac equations with variable growth.

Motivated by the above works, we study of Navier-Stokes equations in a half-space in variable exponent spaces of Clifford-valued functions. To the best of our knowledge, this is the first time to investigate Navier-Stokes equations over unbounded domains in such spaces. To this end, we first establish variable exponent spaces of Clifford-valued functions in the half-space. Then, using an iteration method which requires the solution of a Stokes-problem in every step of iteration, we study the existence and uniqueness of Navier-Stokes equations in a half-space. There is no doubt that we encounter serious difficulties, for instance, the Sobolev embedding is not compact in a half-space, and operator theory in variable exponent spaces of Clifford-valued functions in a half-space is still unknown. Anyway, our attempt would be a meaningful exploration in the study of fluid dynamics, and the whole treatment applies to a much larger class of elliptic problems.

This article is organized as follows. In Section 2, we start with a brief summary of basic knowledge of Clifford algebras and then investigate basic properties of variable exponent spaces of Clifford-valued functions in a half-space. In Section 3, with the help of the results of Diening et al. [5], we prove the existence and uniqueness of the Stokes equations in the context of variable exponent spaces in a half-space. In Section 4, we present an iterative method for the solution of the stationary Navier-Stokes equations. Using the contraction mapping principle, we prove the existence and uniqueness of solutions to the Navier-Stokes equations in $W_0^{1,p(x)}(\mathbb{R}^N_+, \mathrm{C}\ell_n) \times L^{p(x)}(\mathbb{R}^N_+, \mathbb{R})$ under certain hypotheses. In Section 5, using the contracting mapping principle, we obtain the existence and uniqueness of solutions for the Navier-Stokes problem with heat conduction under some appropriate assumptions.

2. Preliminaries

2.1. **Clifford algebras.** We first recall some related notions and results concerning Clifford algebras. For a detailed account we refer to [14, 15, 26, 27].

Let $\mathrm{C}\ell_n$ be the real universal Clifford algebras over $\mathbb{R}^n$. Denote $\mathrm{C}\ell_n$ by

$$\mathrm{C}\ell_n = \mathrm{span}\{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2, \dots, \mathrm{e}_n, e_1e_2, \dots, \mathrm{e}_{n-1}\mathrm{e}_n, \dots, \mathrm{e}_1\mathrm{e}_2\cdots\mathrm{e}_n\}$$

where $\mathrm{e}_0 = 1$(the identity element in $\mathbb{R}^n$), $\{\mathrm{e}_1, \mathrm{e}_2, \dots, \mathrm{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $\mathrm{e}_i\mathrm{e}_j + \mathrm{e}_j\mathrm{e}_i = -2\delta_{ij}\mathrm{e}_0$. Thus the dimension of $\mathrm{C}\ell_n$ is 2^n. For $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ with $1 \le i_1 < i_2 < \dots < i_n \le n$, put $\mathrm{e}_I = \mathrm{e}_{i_1}\mathrm{e}_{i_2}\cdots\mathrm{e}_{i_r}$, while for $I = \emptyset$, $\mathrm{e}_\emptyset = \mathrm{e}_0$. For $0 \le r \le n$ fixed, the space $\mathrm{C}\ell_n^r$ is defined by

$$\mathrm{C}\ell_n^r = \mathrm{span}\{\mathrm{e}_I : |I| := \mathrm{card}(I) = r\}.$$

The Clifford algebras $\mathrm{C}\ell_n$ is a graded algebra as

$$\mathrm{C}\ell_n = \oplus_r \mathrm{C}\ell_n^r.$$

Any element $a \in \mathrm{C}\ell_n$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \dots + [a]_n$$

where $[\ ]_r : \mathrm{C}\ell_n \to \mathrm{C}\ell_n^r$ denotes the projection of $\mathrm{C}\ell_n$ onto $\mathrm{C}\ell_n^r$. In particular, by $\mathrm{C}\ell_n^2 = \mathbb{H}$ we denote the algebra of real quaternion. It is customary to identify $\mathbb{R}$ with $\mathrm{C}\ell_n^0$ and identify $\mathbb{R}^n$ with $\mathrm{C}\ell_n^1$ respectively. This means that each element x of $\mathbb{R}^n$ may be represented by

$$x = \sum_{i=1}^{n} x_i \mathrm{e}_i.$$

For $u \in \mathrm{C}\ell_n$, we denotes by $[u]_0$ the scalar part of u, that is the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{\mathrm{e}_{i_1}\mathrm{e}_{i_2}\cdots\mathrm{e}_{i_r}} = (-1)^{\frac{r(r+1)}{2}} \mathrm{e}_{i_1}\mathrm{e}_{i_2}\cdots\mathrm{e}_{i_r}.$$

We denote

$$(A, B) = \big[\overline{A}B\big]_0.$$

Then an inner product is thus obtained, giving to the norm $|\cdot|$ on $\mathrm{C}\ell_n$ given by

$$|A|^2 = \big[\overline{A}A\big]_0.$$

From [15], we know that this norm is submultiplicative: $|AB| \le C(n)|A||B|$, where $C(n)$ is a positive constant depending only on n and no more than $2^{n/2}$.

In what follows, we let $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ and $\Sigma = \partial\mathbb{R}_+^n$. A Clifford-valued function $u : \mathbb{R}_+^n \to \mathrm{C}\ell_n$ can be written as $u = \Sigma_I u_I \mathrm{e}_I$, where the coefficients $u_I : \mathbb{R}_+^n \to \mathbb{R}$ are real-valued functions.

The Dirac operator on the Euclidean space used here is introduced by

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} := \sum_{j=1}^{n} e_j \partial_j.$$

If u is a real-valued function defined on $\mathbb{R}_+^n$, then $Du = \nabla u = (\partial_1 u, \partial_2 u, \dots, \partial_n u)$. Moreover, $D^2 = -\Delta$, where Δ is the Laplace operator which operates only on coefficients. A function is left monogenic if it satisfies the equation $Du(x) = 0$ for each $x \in \mathbb{R}_+^n$. A similar definition can be given for right monogenic function. An important example of a left monogenic function is the generalized Cauchy kernel

$$G(x) = \frac{1}{\omega_n} \frac{\overline{x}}{|x|^n},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator. Basic properties of left monogenic functions one can refer to [11, 17] and references therein.

2.2. **Variable exponent spaces of Clifford-valued functions.** Next we recall some basic properties of variable exponent spaces of Clifford-valued functions. In what follows, we use the short notation $L^{p(x)}(\mathbb{R}_+^N)$, $W^{1,p(x)}(\mathbb{R}_+^N)$, etc., instead of $L^{p(x)}(\mathbb{R}_+^N, \mathbb{R})$, $W^{1,p(x)}(\mathbb{R}_+^N, \mathbb{R})$, etc. Throughout this paper we always assume (unless declared specially)

$$p \in \mathcal{P}^{\log}(\mathbb{R}_+^n) \text{and} 1 < p_- := \inf_{x \in \overline{\mathbb{R}_+^n}} p(x) \le p(x) \le \sup_{x \in \overline{\mathbb{R}_+^n}} p(x) =: p_+ < \infty. \tag{2.1}$$

where $\mathcal{P}^{\log}(\mathbb{R}_+^n)$ is the set of exponent p satisfying the so-called log-Hölder continuity, i.e.,

$$|p(x) - p(y)| \le \frac{C}{\log(e + |x-y|^{-1})}, \quad |p(x) - p(\infty)| \le \frac{C}{\log(e + |x|^{-1})}$$

hold for all $x, y \in \mathbb{R}^N_+$, where $p(\infty) = \lim_{|x|\to\infty} p(x)$, see [3, 5]. Let $\mathcal{P}(\mathbb{R}^n_+)$ be the set of all Lebesgue measurable functions $p : \mathbb{R}^n_+ \to (1, \infty)$. Given $p \in \mathcal{P}(\mathbb{R}^n_+)$ we define the conjugate function $p'(x) \in \mathcal{P}(\mathbb{R}^n_+)$ by

$$p'(x) = \frac{p(x)}{p(x) - 1} \quad \text{for each } x \in \mathbb{R}^N_+.$$

The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^n_+)$ is defined by

$$L^{p(x)}(\mathbb{R}^n_+) = \big\{u \in \mathcal{P}(\mathbb{R}^n_+) : \int_{\mathbb{R}^n_+} |u|^{p(x)} dx < \infty\big\},$$

with the norm

$$\|u\|_{L^{p(x)}(\mathbb{R}^n_+)} = \inf\big\{t > 0 : \int_{\mathbb{R}^n_+} |\frac{u}{t}|^{p(x)} dx \le 1\big\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^n_+)$ in a half-space is defined by

$$W^{1,p(x)}(\mathbb{R}^n_+) = \big\{u \in L^{p(x)}(\mathbb{R}^n_+) : |\nabla u| \in L^{p(x)}(\mathbb{R}^n_+)\big\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^n_+)} = \|\nabla u\|_{L^{p(x)}(\mathbb{R}^n_+)} + \|u\|_{L^{p(x)}(\mathbb{R}^n_+)}. \tag{2.2}$$

Denote $W^{1,p(x)}_0(\mathbb{R}^n_+)$ by the completion of $C^\infty_0(\mathbb{R}^n_+)$ in $W^{1,p(x)}(\mathbb{R}^n_+)$ with respect to the norm (2.2). The space $W^{-1,p(x)}(\mathbb{R}^n_+)$ is defined as the dual of the space $W^{1,p'(x)}_0(\mathbb{R}^n_+)$. For more details we refer to [3, 7, 8] and reference therein.

In the following, we say that $u \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ can be understood coordinate wise. For example, $u \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ means that $\{u_I\} \subset L^{p(x)}(\mathbb{R}^n_+)$ for $u = \Sigma_I u_I e_I \in \mathrm{C}\ell_n$ with the norm $\|u\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} = \sum_I \|u_I\|_{L^{p(x)}(\mathbb{R}^n_+)}$. In the same way, spaces $W^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, $W^{1,p(x)}_0(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, $C^\infty_0(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, etc., can be understood similarly. In particular, the space $L^2(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ can be converted into a right Hilbert $\mathrm{C}\ell_n$-module by defining the following Clifford-valued inner product (see [14, Definition 3.74])

$$\big(f, g\big)_{\mathrm{C}\ell_n} = \int_{\mathbb{R}^n_+} \overline{f(x)} g(x) dx. \tag{2.3}$$

Remark 2.1. Following the same arguments as in [10, 37], we can calculate easily that $\|u\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}$ is equivalent to the norm $\||u|\|_{L^{p(x)}(\mathbb{R}^n_+)}$. Furthermore, we also prove that for every $u \in W^{1,p(x)}_0(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, $\|Du\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}$ is an equivalent norm of $\|u\|_{W^{1,p(x)}_0(\mathbb{R}^n_+, \mathrm{C}\ell_n)}$.

Lemma 2.2 ([10])**.** *Assume that $p(x) \in \mathcal{P}(\mathbb{R}^n_+)$. Then*

$$\int_{\mathbb{R}^n_+} |uv| dx \le C(n, p) \|u\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \|v\|_{L^{p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}$$

for every $u \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and $v \in L^{p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$.

Lemma 2.3 ([10, 11])**.** *If $p(x) \in \mathcal{P}(\mathbb{R}^n_+)$, then $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and $W^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ are reflexive Banach spaces.*

Based on the Cauchy kernel $G(x)$ we can introduce the Teodorescu operator. There exist a number of applications and methods based on the properties of this Teodorescu operator. But in our case of considering the domain Ω as an unbounded domain, the main problem in applying this operator is that the Cauchy kernel does not have good enough behaviour near infinity. For example, the Teodorescu operator is an unbounded operator over the usual function spaces on Ω. In this paper, we will follow the idea from [2, 18] by using add-on terms to the Cauchy kernel. More precisely, we choose a fixed point z lying in the complement of $\mathbb{R}^N_+$. Then we consider the following operators.

Definition 2.4 ([2, 14, 15])**.** (i) Let $u \in C(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. The Teodorescu operator is defined by

$$Tu(x) = \int_{\mathbb{R}^n_+} K_z(x,y)u(y)dy,$$

where $K_z(x,y) = G(x-y) - G(y-z)$, $G(x)$ is the above-mentioned generalized Cauchy kernel.

(ii) Let $u \in C^1(\mathbb{R}^n_+, \mathrm{C}\ell_n) \cap C(\overline{\mathbb{R}^n_+}, \mathrm{C}\ell_n)$. The *boundary operator* is defined by

$$Fu(x) = \int_{\Sigma} K_z(x,y)\alpha(y)u(y)dS_y,$$

where $\alpha(y)$ denotes the outward normal unit vector at y.

(iii) Let $u \in L^1_{\rm loc}(\mathbb{R}^n)$. Then the Hardy-Littlewood maximal operator is defined by

$$Mu(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |u(y)|dy.$$

for all $x \in \mathbb{R}^n$, where the supremum is taken over all cubes (or ball) $Q \subset \mathbb{R}^n$ which contain x.

The Teodorescu operator was first introduced in [18] and the operator properties in the scale of $W^{k,2}$-spaces were given in [23], see also [2] for the corresponding operator properties in the $W^{k,q}$-spaces over unbounded domains. Its main advantage is a faster decay to infinity of the kernel.

Lemma 2.5. (see [3]) *Let $x \in \mathbb{R}^n$, $\delta > 0$ and $u \in L^1_{\rm loc}(\mathbb{R}^n)$. Then*

$$\int_{B(x,\delta)} \frac{1}{|x-y|^{n-1}}|u(y)|dy \le C(\delta)Mu(x).$$

where $C(\delta) > 0$ is a positive constant. Moreover, if $u \in L^{p(x)}(\mathbb{R}^n)$ with $\|u\|_{p(x)} \le 1$, then

$$\int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{1}{|x-y|^{n-1}}|u(y)|dy \le C(n,p,\delta,|B|).$$

where $C(n,p,\delta,|B|)$ is a positive constant.

Lemma 2.6 ([3])**.** *Let $p(x)$ satisfy* (2.1). *Then M is bounded in $L^{p(x)}(\mathbb{R}^n)$.*

Lemma 2.7 ([18])**.** *Let $u \in C^1(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. Then*

$$\partial_k Tu(x) = \frac{1}{\omega_n}\int_{\mathbb{R}^n_+} \frac{\partial}{\partial x_k}G(x-y)u(y)dy + \frac{u(x)}{n}\bar{e}_k.$$

Lemma 2.8 ([3])**.** *Let Φ be a Calderón-Zygmund operator with Calderón-Zygmund kernel K on $\mathbb{R}^n \times \mathbb{R}^n$. Then Φ is bounded on $L^{p(x)}(\mathbb{R}^n)$.*

Lemma 2.9. *The following operators are continuous linear operators:*

(i) $T : L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \to W^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$.

(ii) $\widetilde{T} : W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \to L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$.

Proof. (i) We divide the proof into two parts:

Step 1: The operator $\partial_k \mathrm{T} : L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \to L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ is continuous. By Lemma 2.7 we have for $u \in C_0^\infty(\mathbb{R}^n_+, \mathrm{C}\ell_n)$

$$\partial_k \mathrm{T} u(x) = \frac{1}{\omega_n} \int_{\mathbb{R}^n_+} \frac{\partial}{\partial x_k} K_z(x,y) u(y) dy + \frac{u(x)}{n} \overline{e}_k.$$

Let $\mathcal{K}(x,y) = \frac{1}{\omega_n} \frac{\partial}{\partial x_k} K_z(x,y)$. Since $\frac{1}{\omega_n} \frac{\partial}{\partial x_k} K_z(x,y) = \frac{1}{\omega_n} \frac{\partial}{\partial x_k} G(x-y)$ and

$$\frac{\partial}{\partial x_k} G(x-y) = \frac{1}{|x-y|^n} \Big(\overline{e}_k - n \sum_{i=1}^n \frac{(x_k - y_k)(x_i - y_i)}{|x-y|^2} \overline{e}_i \Big),$$

we obtain

$$\big| \frac{\partial}{\partial x_k} G(x-y) \big| \le \frac{n^2+1}{|x-y|^n}, \quad (k = 1, \dots, n).$$

Notice that

$$\int_{S_1} \Big(\overline{e}_k - n \sum_{i=1}^n \frac{(x_k - y_k)(x_i - y_i)}{|x-y|^2} \overline{e}_i \Big) dS = 0,$$

where $S_1 = \{ y \in \mathbb{R}^n_+ : |x-y| = 1 \}$. Then it is easy to verify that $\mathcal{K}(x,y)$ satisfies the following properties:

(a) $|\mathcal{K}(x,y)| \le C|x-y|^{-n}$;

(b) $\mathcal{K}\big(t(x,y)\big) = t^{-n} K(x,y), t > 0$;

(c) $\int_{S_1} \mathcal{K}(x,y) dS = 0$.

Now we define $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Then $\mathcal{K}(x,y)$ satisfies the conditions of Calderón-Zygmund kernal on $\mathbb{R}^n \times \mathbb{R}^n$. By Theorem 2.13, we know the inequality can be extended to $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. Therefore, we obtain by Lemma 2.5 and Lemma 2.6

$$\| \frac{1}{\omega_n} \int_\Omega \partial_{k,x} G(x-y) u(y) dy \|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \le C(n,p) \|u\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \tag{2.4}$$

On the other hand,

$$\| \frac{u(x)}{n} \overline{e}_k \|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \le \frac{1}{n} \|u\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \tag{2.5}$$

Combining (2.3) with (2.5), we obtain

$$\|\partial_k \mathrm{T} u\|_{L^{p(x)}(\mathbb{R}^n, Cl_n)} \le C(n,p) \|u\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}.$$

Step 2: The operator $\mathrm{T} : L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \to L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ is continuous. We define $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Since

$$|G(x-y) + G(y-z)| \le \frac{1}{\omega_n} \Big(\frac{1}{|x-y|^{n-1}} + \frac{1}{|y-z|^{n-1}} \Big),$$

we have

$$|Tu(x)| \le C(n) \int_{\mathbb{R}^n_+} (|G(x-y)| + |G(y-z)|) \, |u(y)| dy$$

$$\leq C\Big(\int_{\mathbb{R}^n_+}\frac{1}{|x-y|^{n-1}}|u(y)|dy+\int_{\mathbb{R}^n_+}\frac{1}{|y-z|^{n-1}}|u(y)|dy\Big).$$

Then by Lemmas 2.5 and 2.6, we obtain

$$\|Tu\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}\leq C(n,p)\|u\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}.$$

Finally, combining Step 1 with Step 2, we have

$$\begin{aligned}\|Tu\|_{W^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}&=\|Tu\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}+\sum_{k=1}^n\|\partial_k \mathrm{T}u\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}\\&\leq C(n,p)\|u\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}.\end{aligned}$$

Then we obtain the desired conclusion (i).

(ii) In view of [3, Proposition 12.3.2], we know that for each $f\in W^{-1,p(x)}(\mathbb{R}^n_+)$, there exists $f_k\in L^{p(x)}(\mathbb{R}^n_+), k=0,1,\dots,n$, such that

$$\langle f,\varphi\rangle=\sum_{k=0}^n\int_{\mathbb{R}^n_+}f_k\frac{\partial\varphi}{\partial x_k}dx, \tag{2.6}$$

for all $\varphi\in W_0^{1,p'(x)}(\mathbb{R}^n_+)$. Moreover, $\|f\|_{W^{-1,p(x)}(\mathbb{R}^n_+)}$ is equivalent to $\sum_{k=0}^n\|f_k\|_{L^{p(x)}(\mathbb{R}^n_+)}$. Obviously, for every $f\in W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)$ the equality (2.5) still holds for $f_k\in L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)$, $k=0,1,\dots,n$. Moreover, $\|f\|_{W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}$ is equivalent to $\sum_{k=0}^n\|f_k\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}$. On the other hand, by [3, Proposition 12.3.4], it is easy to show that the space $C_0^\infty(\mathbb{R}^n_+,\mathrm{C}\ell_n)$ is dense in $W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)$. Thus we may choose

$$u^j=u_0^j+\sum_{k=1}^n\frac{\partial u_k^j}{\partial x_k},$$

where $u_0^j,u_k^j\in C_0^\infty(\mathbb{R}^n_+,\mathrm{C}\ell_n)$, such that $\|u^j-f\|_{W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}\to 0$ and $\|u_k^j-f_k\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}\to 0$ as $j\to\infty$, where $k=0,1,\dots,n$. Here, we are using the fact that $C_0^\infty(\mathbb{R}^n_+,\mathrm{C}\ell_n)$ is dense in $L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)$(see [3]). Then we consider

$$Tu^j=\int_{\mathbb{R}^n_+}K_z(x,y)u^j(y)dy.$$

Then we have

$$\begin{aligned}Tu^j&=\int_{\mathbb{R}^n_+}K_z(x,y)\Big(u_0^j(y)+\sum_{k=1}^n\frac{\partial}{\partial y_k}u_k^j(y)\Big)dy\\&=\int_{\mathbb{R}^n_+}K_z(x,y)u_0^j(y)dy-\sum_{k=1}^n\int_{\mathbb{R}^n_+}\frac{\partial}{\partial y_k}K_z(x,y)u_k^j(y)dy.\end{aligned}$$

Since

$$\Big|\int_{\mathbb{R}^n_+}K_z(x,y)u_0^j(y)dy\Big|\leq\int_{\mathbb{R}^n_+}\frac{1}{|x-y|^{n-1}}\big|u_0^j(y)\big|dy+\int_{\mathbb{R}^n_+}\frac{1}{|y-z|^{n-1}}\big|u_0^j(y)\big|dy.$$

By Remark 2.1, Lemma 2.5 and Lemma 2.6, there exists a constant $C_0>0$ such that

$$\Big\|\int_{\mathbb{R}^n_+}K_z(x,y)u_0^j(y)dy\Big\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}\leq C_0\|u_0^j\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}. \tag{2.7}$$

Now let us extend $u_k^j(x)$ by zero to $\mathbb{R}^n \setminus \mathbb{R}_+^n$. Note that the position of z which is outside of a half space $\overline{\mathbb{R}_+^n}$ leads to the fact that $G(y-z)$ has no singularities for any $y \in \mathbb{R}_+^n$. Thus it is easy to show that $\frac{\partial}{\partial y_k} K_z(x,y)$ satisfies the conditions of Calderón-Zygmund kernel on $\mathbb{R}^n \times \mathbb{R}^n$. In view of Lemma 2.8, there exist positive constant $C_k (k = 1, \dots, n)$ such that

$$\| \int_{\mathbb{R}_+^n} \frac{\partial}{\partial y_k} K_z(x,y) u_k^j(y) \|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)} \le C_k \|u_k^j\|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)}. \tag{2.8}$$

Combining (2.7) with (2.8), we have

$$\|Tu^j\|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)} \le C_0 \|u_0^j\|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)} + \sum_{k=1}^n C_k \|u_k^j\|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)}.$$

Letting $j \to \infty$, by means of the Continuous Linear Extension Theorem, the operator T can be uniquely extended to a bounded linear operator $\widetilde{T}$ such that for all $f \in W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$, there exists a constant $\widetilde{C} > 0$ such that

$$\begin{aligned} \|\widetilde{T} f\|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)} &\le C\Big(\|f_0\|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)} + \sum_{k=1}^n \|f_k\|_{L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)} \Big) \\ &\le \widetilde{C} \|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)}. \end{aligned}$$

Hence claim (ii) follows. □

Lemma 2.10. *The following operators are continuous linear operators:*

(i) $D : W^{1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n) \to L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$.
(ii) $\widetilde{D} : L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$.

Proof. (i) The proof is similar to that of [11, Lemma 2.6], so we omit it.

(ii) We consider the following Dirichlet problem of the Poisson equation with homogeneous boundary data

$$\begin{gathered} -\Delta u = f, \quad \text{in } \mathbb{R}_+^n \\ u = 0, \quad \text{on } \Sigma \end{gathered} \tag{2.9}$$

It is easy to see that for all $f \in W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$ problem (2.9) still has a unique weak solution $u \in W^{1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$, see Diening, Lengeler and Ružička [5]. We denote by Δ_0^{-1} the solution operator. On the other hand, it is clear that the operator

$$\Delta : W^{1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$$

is continuous, so we obtain from Lemma 2.9 that the operator $\widetilde{D} = -\Delta T$: $L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$ is continuous, where the operator $\widetilde{D}$ can be considered as a unique continuous linear extension of the operator D. □

Lemma 2.11. *Let $p(x) \in \mathcal{P}(\mathbb{R}_+^n)$.*

(i) *If $u \in W^{1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$, then the Borel-Pompeiu formula $Fu(x) + TDu(x) = u(x)$ holds for all $x \in \mathbb{R}_+^n$.*
(ii) *If $u \in L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$, then the equation $DTu(x) = u(x)$ holds for all $x \in \mathbb{R}_+^n$.*

Proof. Let us denote by $C_0^\infty(\mathbb{R}_+^n)$ the space of all restrictions of functions from $C_0^\infty(\mathbb{R}^n)$ to $\mathbb{R}_+^n$. Furthermore, suppose $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^n})$. Now, let us consider a point $y \in \Omega$ and the open ball $B(0,r)$ with origin 0, radius r, and boundary $S(0,r)$. If r is sufficiently large such that y lies in the domain $\Omega(r) = B(0,r) \cap \mathbb{R}_+^n$. For this domain, we have

$$F_{S(0,r)}\varphi(y) = \varphi(y) - T_{\Omega(r)}D\varphi(y).$$

see [25] for more details. This can be written in the form

$$\begin{aligned}
&\lim_{r\to\infty}\Big(\int_{\Sigma\cap B(0,r)} + \int_{S(0,r)\cap\mathbb{R}_+^n}\Big)K_z(x,y)\alpha(y)u(y)dS_y\\
&= \varphi(y) - \lim_{r\to\infty} T_{\Omega(r)}D\varphi(y)
\end{aligned}$$

Since

$$\lim_{r\to\infty}\int_{\Sigma\cap B(0,r)} K_z(x,y)\alpha(y)u(y)dS_y = \int_\Sigma K_z(x,y)\alpha(y)u(y)dS_y$$

and

$$\lim_{r\to\infty} T_{\Omega(r)}D\varphi(y) = T_{\mathbb{R}_+^n}D\varphi(y), \quad \lim_{r\to\infty}\int_{S(0,r)\cap\mathbb{R}_+^n} K_z(x,y)\alpha(y)u(y)dS_y = 0,$$

we obtain the Borel-Pompeiu formula in case of $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^n})$. Finally, the desired result (i) follows immediately from the density document.

(ii) Using the same idea with (i), we can get directly the desired result from [25, Lemma 2.6]. □

Lemma 2.12. *Let $p(x)$ satisfy* (2.1).

(i) *If $u \in L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$, then $\widetilde{T}\widetilde{D}u(x) = u(x)$ for all $x \in \mathbb{R}_+^n$.*

(ii) *If $u \in W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$, then $\widetilde{D}\widetilde{T}u(x) = u(x)$ for all $x \in \mathbb{R}_+^n$.*

Proof. (i) follows from Lemma 2.11 (i) and the denseness of $W_0^{1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$ in $L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$.

(ii) follows from Lemma 2.11 (ii) and the denseness of $C_0^\infty(\overline{\mathbb{R}_+^n}, \mathrm{C}\ell_n)$ in the space $W^{-1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$, see [3, Proposition 12.3.4] for the details. □

Gürlebeck and Sprößig [14, 15] showed that the orthogonal decomposition of the space $L^2(\Omega)$ holds in the hyper-complex function theory:

$$L^2(\Omega, \mathrm{C}\ell_n) = (\ker D \cap L^2(\Omega, \mathrm{C}\ell_n)) \oplus DW_0^{1,2}(\Omega, \mathrm{C}\ell_n) \tag{2.10}$$

with respect to the Clifford-valued product (2.3). Note that $\ker D$ denotes the set of all monogenic functions on Ω. This decomposition has a number of applications, especially to the theory of partial differential equations, see [6] for the complex case and [14] for the hyper-complex case. Kähler [22] extended the orthogonal decomposition (2.10) to the spaces $L^p(\Omega)$ in form of a direct decomposition in the case of Clifford analysis. In [7], Fu et al. extended the direct decomposition to the case of variable exponent Lebesgue spaces in bounded smooth domains.

Theorem 2.13. *The space $L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)$ allows the Hodge-type decomposition*

$$L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n) = (\ker \widetilde{D} \cap L^{p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n)) \oplus DW_0^{1,p(x)}(\mathbb{R}_+^n, \mathrm{C}\ell_n) \tag{2.11}$$

with respect to the Clifford-valued product (2.3).

Proof. Similar to the proof of [22, Theorem 6], we first show that the intersection of $(\ker \widetilde{D} \cap L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n))$ and $DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ is empty. Suppose f belongs to both $\ker \widetilde{D} \cap L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and $DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, then $\widetilde{D}f = 0$. Because f belongs to $DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, there exists a function $v \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ such that $Dv = f$. Hence, we obtain that $-\Delta v = 0$ and $v = 0$ on Σ. From the uniqueness of Δ_0^{-1} we obtain $v = 0$. Consequently, $f = 0$. Therefore, the sum of the two subspaces is a direct one.

Now let $u \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. Then $u_2 = D\Delta_0^{-1}\widetilde{D}u \in DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. Let $u_1 = u - u_2$. Then $u_1 \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. Furthermore, we take $u_k \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ such that $u_k \to u$ in $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, then by the denseness of $W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ in $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and Lemma 2.2, we have for any $\varphi \in W_0^{1,p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$

$$\begin{aligned}
\big(u_1, D\varphi\big)_{\mathrm{C}\ell_n} &= \big(u - u_2, D\varphi\big)_{\mathrm{C}\ell_n} \\
&= \lim_{k\to\infty} \big(Du_k - DD\Delta_0^{-1}Du_k, \varphi\big)_{\mathrm{C}\ell_n} \\
&= \lim_{k\to\infty} \big(Du_k - Du_k, \varphi\big)_{\mathrm{C}\ell_n} = 0,
\end{aligned}$$

which implies that $u_1 \in \ker \widetilde{D}$. Since $u \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ is arbitrary, the desired result follows. $\square$

From this decomposition we can get the following two projections

$$\begin{gathered}
P : L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \to \ker \widetilde{D} \cap L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n), \\
Q : L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \to DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n).
\end{gathered}$$

Moreover, we have

$$Q = D\Delta_0^{-1}\widetilde{D}, \quad P = I - Q.$$

Corollary 2.14. *The space $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \cap \operatorname{im} Q$ is a closed subspace of $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$.*

The proof can be easily done by combining Theorem 2.13, Lemma 2.3 with Lemma 2.10. We refer the reader to [38, Lemma 2.6] for a similar argument.

Corollary 2.15. $\big(L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \cap \operatorname{im} Q\big)^* = L^{p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \cap \operatorname{im} Q$. *Namely, the linear operator*

$$\Phi : DW_0^{1,p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \to \big(DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)\big)^*$$

given by

$$\Phi(Du)(D\varphi) = (D\varphi, Du)_{Sc} := \int_{\mathbb{R}^n_+} \big[\overline{D\varphi} Du\big]_0 dx$$

is a Banach space isomorphism.

Proof. In terms of Lemma 2.14, $DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and $DW_0^{1,p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ are reflexive Banach spaces since they are closed subspaces in $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and $L^{p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ respectively. The linearity of Φ is clear. For injectivity, suppose

$$\Phi(Du)(D\varphi) = (D\varphi, Du)_{Sc} = 0 \tag{2.12}$$

for all $\varphi \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and some $u \in W_0^{1,p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. For any $\omega \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, according to (2.11), we may write $\omega = \alpha + \beta$ with $\alpha \in \ker \widetilde{D} \cap$

$L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and $\beta \in DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. Thus we obtain

$$(\omega, Du)_{Sc} = (\alpha + \beta, Du)_{Sc} = (\beta, Du)_{Sc}.$$

This and (2.12) gives $(\omega, Du)_{Sc} = 0$. This means that $Du = 0$. It follows that Φ is injective. To get subjectivity, let $f \in \big(DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)\big)^*$. By the Hahn-Banach Theorem, there is $F \in \big(L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)\big)^*$ with $\|F\| = \|f\|$ and $F|_{DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} = f$. Moreover, there exists $\varphi \in L^{p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ such that $F(u) = (u, \varphi)_{Sc}$ for any $u \in L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. According to (2.11), we can write $\varphi = \eta + D\alpha$, where $\eta \in \ker \widetilde{D} \cap L^{p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, $D\alpha \in DW_0^{1,p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. For any $Du \in DW_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, we have

$$f(Du) = (Du, \varphi)_{Sc} = (Du, D\alpha)_{Sc} = \Phi(D\alpha)(Du).$$

Consequently, $\Phi(D\alpha) = f$. It follows that Φ is surjective. By [10, Theorem 3.1] we have

$$|\Phi(Du)(D\varphi)| \le C\|D\varphi\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}\|Du\|_{L^{p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}.$$

This means that Φ is continuous. Furthermore, it is immediate that Φ^{-1} is continuous from the Inverse Function Theorem. This ends the proof of Lemma 2.3. $\square$

3. Stokes equations in the half-space

In the section, we consider the Stokes system which consists in finding a solution (u, π) for

$$-\Delta u + \frac{1}{\mu}\nabla\pi = \frac{\rho}{\mu}f \quad \text{in } \mathbb{R}^n_+, \tag{3.1}$$

$$\operatorname{div} u = f_0 \quad \text{in } \mathbb{R}^n_+, \tag{3.2}$$

$$u = v_0 \quad \text{on } \Sigma. \tag{3.3}$$

With $\int_\Omega f_0 dx = \int_{\partial\Omega} n \cdot v_0 dx$ the necessary condition for the solvability is given. Here, u is the velocity, π the hydrostatic pressure, ρ the density, μ the viscosity, f the vector of the external forces and the scalar function f_0 a measure of the compressibility of fluid. The boundary condition (3.3) describes the adhesion at the boundary of the domain Ω for $v_0 = 0$. This system describes the stationary flow of a homogeneous viscous fluid for small Reynold's numbers. For more details, we refer to [2, 14, 15, 16, 20].

In this paper, for $f = \sum_{i=1}^n f_i \mathrm{e}_i$ and $u = \sum_{i=1}^n u_i \mathrm{e}_i$, we consider the following Stokes system in the hyper-complex formulation (see [16, 17]):

$$\widetilde{D}Du + \frac{1}{\mu}D\pi = \frac{\rho}{\mu}f \quad \text{in } \mathbb{R}^n_+, \tag{3.4}$$

$$[Du]_0 = 0 \quad \text{in } \mathbb{R}^n_+, \tag{3.5}$$

$$u = 0 \quad \text{on } \Sigma. \tag{3.6}$$

Definition 3.1. We say that $(u, \pi) \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \times L^{p(x)}(\mathbb{R}^n_+)$ is a solution of (3.4)–(3.6) provided that it satisfies the system (3.4)–(3.6) for all $f \in W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$.

Definition 3.2. The operator $\widetilde{\nabla} : L^{p(x)}(\mathbb{R}^n_+) \to (W^{-1,p(x)}(\mathbb{R}^n_+))^n$ is defined by

$$\langle \widetilde{\nabla} f, \varphi\rangle = -\langle f, \operatorname{div}\varphi\rangle := -\int_{\mathbb{R}^n_+} f \operatorname{div}\varphi dx$$

for all $f \in L^{p(x)}(\mathbb{R}^n_+)$ and $\varphi \in (C_0^\infty(\mathbb{R}^n_+))^n$.

Theorem 3.3. *Suppose $f \in W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. Then the Stokes system (3.4)–(3.6) has a unique solution $(u,\pi) \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \times L^{p(x)}(\mathbb{R}^n_+)$ of the form*

$$u + \frac{1}{\mu} TQ\pi = \frac{\rho}{\mu} TQ\widetilde{T} f,$$

with respect to the estimate

$$\|Du\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} + \frac{1}{\mu}\|Q\pi\|_{L^{p(x)}(\mathbb{R}^n_+)} \le C\frac{\rho}{\mu}\|Q\widetilde{T} f\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}.$$

Here, $C \ge 1$ is a constant and the hydrostatic pressure π is unique up to a constant.

Proof. We first prove that if $f \in W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, then we have the representation

$$TQ\widetilde{T} f = u + TQ\omega.$$

Indeed, let $\varphi_n \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ with $\varphi_n \to \varphi$ in $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$. By Lemma 2.11, we have

$$TQT(D\varphi_n) = TQ\varphi_n.$$

Since $W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ is dense in $L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, it follows that $TQ\widetilde{T}\widetilde{D}\varphi = TQ\varphi$. Thus, for $u \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ and $\pi \in L^{p(x)}(\mathbb{R}^n_+)$ we obtain

$$TQ\widetilde{T}(\frac{\rho}{\mu} f) = TQ\widetilde{T}(\widetilde{D}Du + \frac{1}{\mu}\widetilde{D}\pi) = u + \frac{1}{\mu} TQ\pi.$$

This implies that our system (3.4)–(3.5) is equivalent to the system

$$u + \frac{1}{\mu} QTQ\pi = \frac{\rho}{\mu} TQ\widetilde{T} f, \tag{3.7}$$

$$[Q\pi]_0 = [Q\widetilde{T} f]_0. \tag{3.8}$$

Obviously, the equality (3.4) is equivalent to the equality

$$Du + \frac{1}{\mu} Q\pi = \frac{\rho}{\mu} Q\widetilde{T} f. \tag{3.9}$$

Now we need to show that for each $f \in W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n)$, the function QTf can be decomposed into two functions Du and $Q\pi$. Suppose $Du + Q\pi = 0$ for $u \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n) \cap \ker\operatorname{div}$ and $\pi \in L^{p(x)}(\mathbb{R}^n_+)$. Then (3.5) gives $[Q\pi]_0 = 0$. Thus, $Q\pi = 0$. Hence, $Du = Q\pi = 0$. This means that $Du + Q\pi$ is a direct sum, which is a subset of $\operatorname{im} Q$.

Next we have to ask about the existence of a functional $\mathcal{F} \in (L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n) \cap \operatorname{im} Q)^*$ with $\mathcal{F}(Du) = 0$ and $\mathcal{F}(Q\pi) = 0$ but $\mathcal{F}(Q\widetilde{T} f) \ne 0$. This is equivalent to ask if there exists $g \in W^{-1,p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n)$, such that for all $u \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n) \cap$ ker div and $\omega \in L^{p(x)}(\mathbb{R}^n_+)$,

$$(Du, Q\widetilde{T} g)_{Sc} = 0, \tag{3.10}$$

$$(Q\pi, Q\widetilde{T} g)_{Sc} = 0, \tag{3.11}$$

but $(Q\widetilde{T}f, Q\widetilde{T}g)_{Sc} \neq 0$. Here, Lemmas 2.9 and Corollary 2.15 are employed.

Thus, let us consider the system (3.10) and (3.11) with $g \in W^{-1,p'(x)}(\Omega, \mathrm{C}\ell_n^1)$ for all open cubes $\Omega \subset \mathbb{R}^n_+$. Notice that, with the help of Lemma 2.10, (3.10) yields

$$(Du, Q\widetilde{T}g)_{Sc} = (u, \widetilde{D}Q\widetilde{T}g)_{Sc} = (u, \widetilde{D}\widetilde{T}g - \widetilde{D}P\widetilde{T}g)_{Sc} = (u, g)_{Sc} = 0,$$

which implies $g = \widetilde{\nabla}h = \widetilde{D}h$ with $h \in L^{p'(x)}_{\rm loc}(\mathbb{R}^n_+)$ because of [38, Lemma 2.8]. Furthermore, by Definition 3.2, it is easy to see that if $g \in W^{-1,p'(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n^1)$, then $h \in L^{p'(x)}(\mathbb{R}^n_+)$. Thus it follows from (3.11) and Lemma 2.3,

$$(Q\pi, Q\widetilde{T}g)_{Sc} = (Q\pi, Q\widetilde{T}\widetilde{D}h)_{Sc} = (Q\pi, Qh)_{Sc} = 0$$

holds for each $\pi \in L^{p(x)}(\mathbb{R}^n_+)$. Hence, $Q\pi = |Qh|^{p'(x)-2}Qh$ gives $Qh = 0$. Then we obtain

$$g = \widetilde{D}h = \widetilde{D}Qh + \widetilde{D}Ph = 0.$$

Furthermore, we obtain

$$(Q\widetilde{T}f, Q\widetilde{T}g)_{Sc} = 0, \quad \text{for all } f \in W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n^1).$$

Finally, (3.9) yields

$$\|Du\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} + \frac{1}{\mu}\|Q\pi\|_{L^{p(x)}(\mathbb{R}^n_+)} \geq \frac{\rho}{\mu}\|Q\widetilde{T}f\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}.$$

By the Norm Equivalence Theorem, we obtain

$$\|Du\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} + \frac{1}{\mu}\|Q\pi\|_{L^{p(x)}(\mathbb{R}^n_+)} \leq C\frac{\rho}{\mu}\|Q\widetilde{T}f\|_{L^{p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}.$$

By Remark 2.1, Lemma 2.9 and the boundedness of the operator Q, we obtain

$$\|u\|_{W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} + \frac{1}{\mu}\|Q\pi\|_{L^{p(x)}(\mathbb{R}^n_+)} \leq C\frac{\rho}{\mu}\|f\|_{W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}, \tag{3.12}$$

which implies the uniqueness of solution. Note that $Q\pi = 0$ implies $\pi \in \ker \widetilde{D}$. Therefore, π is unique up to a constant. The proof is complete. □

4. N-S equations in the half-space

In this section, we consider the time-independent Navier-Stokes equations in variable exponent spaces of Clifford-valued functions in a half-space:

$$-\Delta u + \frac{\rho}{\mu}(u \cdot \nabla)u + \frac{1}{\mu}\nabla\pi = \frac{\rho}{\mu}f \quad \text{in } \mathbb{R}^n_+, \tag{4.1}$$

$$\operatorname{div} u = f_0 \quad \text{in } \mathbb{R}^n_+, \tag{4.2}$$

$$u = v_0 \quad \text{on } \Sigma. \tag{4.3}$$

In addition to the case of the Stokes system, the main difference from the above-mentioned Stokes equations is the appearance of the non-linear convection term $(u \cdot \nabla)u$. In 1928, Oseen showed that one can get relatively good results if the convection term $(u \cdot \nabla)u$ is replaced by $(v \cdot \nabla)u$, where v is a solution of the corresponding Stokes equations. In 1965, Finn [13] proved the existence of solutions for small external forces with a spatial decreasing to infinity of order $|x|^{-1}$ for the case of $n = 3$, and used the Banach fixed-pointed theorem. Gürlebeck and Sprößig [14, 15, 17] solved this system by a reduction to a sequence of Stokes problems provided the external force f belongs to $L^p(\Omega, \mathbb{H})$ for a bounded domain Ω and $6/5 < p < 3/2$. Cerejeiras and Kähler [2] obtained the similar results provided the external force f belongs to $W^{-1,p}(\Omega, \mathrm{C}\ell_n)$ for an unbounded domain Ω and $n/2 \le p < \infty$, see also [38] for similar results of bounded domains in the variable exponents context. Now we would like to extend these results to the setting of variable exponent spaces in a half-space.

For $f = \sum_{i=1}^n f_i e_i$, $u = \sum_{i=1}^n u_i e_i$, we consider the following steady Navier-Stokes equations in the hyper-complex notation:

$$\widetilde{D}Du + \frac{1}{\mu} D\pi = \frac{\rho}{\mu} F(u) \quad \text{in } \mathbb{R}^n_+, \tag{4.4}$$

$$\big[Du\big]_0 = 0 \quad \text{in } \mathbb{R}^n_+, \tag{4.5}$$

$$u = 0 \quad \text{on } \Sigma, \tag{4.6}$$

with the non-linear part $F(u) = f - \big[uD\big]_0 u$. We first give the following lemma, which is crucial to the convergence of the iteration method.

Lemma 4.1. *Let $p(x)$ satisfy (2.1) and $n/2 \le p_- \le p(x) \le p_+ < \infty$. Then the operator $F : W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n) \to W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n)$ is a continuous operator and*

$$\|\big[uD\big]_0 u\|_{W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \le C_1 \|u\|^2_{W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)},$$

where $C_1 = C_1(n,p)$ is a positive constant.

Proof. Let $u = \sum_{i=1}^n u_i e_i \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell^1_n)$. Then

$$\|\big[uD\big]_0 u\|_{W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \le \sum_{i,j=1}^n \|u_i \partial_i u_j\|_{W^{-1,p(x)}(\mathbb{R}^n_+)}.$$

In view of the continuous embedding $L^{s(x)}(\mathbb{R}^n_+) \hookrightarrow W^{-1,p(x)}(\mathbb{R}^n_+)$ for $s(x) = np(x)/(n+p(x))$ (see [3]), we have

$$\|u_i \partial_i u_j\|_{W^{-1,p(x)}(\mathbb{R}^n_+)} \le C(n,p) \|u_i \partial_i u_j\|_{L^{s(x)}(\mathbb{R}^n_+)}.$$

By Hölder's inequality, we obtain

$$\begin{aligned} \|u_i \partial_i u_j\|_{L^{s(x)}(\mathbb{R}^n_+)} &\le C \sup_{\|\varphi_j\|_{L^{s'(x)}(\mathbb{R}^n_+)} \le 1} \int_{\mathbb{R}^n_+} |u_i \partial_i u_j| \, |\varphi_j| dx \\ &\le C \|u_i\|_{L^n(\mathbb{R}^n_+)} \|u_j\|_{W_0^{1,p(x)}(\mathbb{R}^n_+)}. \end{aligned}$$

According to the continuous embedding $W_0^{1,p(x)}(\mathbb{R}^n_+) \hookrightarrow L^n(\mathbb{R}^n_+)$ for $n/2 \le p_- \le p(x) \le p_+ < \infty$ (see [3]), we obtain

$$\|u_i \partial_i u_j\|_{L^{s(x)}(\mathbb{R}^n_+)} \le C(n,p) \|u_i\|_{W_0^{1,p(x)}(\mathbb{R}^n_+)} \|u_j\|_{W_0^{1,p(x)}(\mathbb{R}^n_+)}.$$

Finally, it is easy to obtain the desired estimate from above-mentioned inequalities. Hence, the continuity of the operator F follows immediately. $\square$

Remark 4.2. Actually, $n/2 \le p_-$ means $p_- \in (1, +\infty)$ for $n = 2$ while $p_- \in [n/2, +\infty)$ for $n > 2$. Evidently, Lemma 4.1 is a direct generalization of [2, Lemma 4.1] to the variable exponent context in a half-space.

Now we are in a position to prove our main result.

Theorem 4.3. *Let $p(x)$ satisfy (2.1) and $n/2 \le p_- \le p(x) \le p_+ < \infty$. Then the system (4.4)–(4.6) has a unique solution $(u,\pi) \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \times L^{p(x)}(\mathbb{R}^n_+, \mathbb{R})$ (π is unique up to a real constant) if the right-hand side f satisfies the condition*

$$\|f\|_{W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} < \frac{\nu^2}{4C_1C_4^2}, \tag{4.7}$$

with $\nu = \mu/\rho$, $C_4 = C_2(1+C_3)$, where $C_3 \ge 1$ indicated in (4.11) and

$$C_2 = \|T\|_{[L^{p(x)}\cap \operatorname{im} Q, W_0^{1,p(x)}]} \|Q\|_{[L^{p(x)}, L^{p(x)}\cap \operatorname{im} Q]} \|\widetilde{T}\|_{[W^{-1,p(x)}, L^{p(x)}\cap \operatorname{im} Q]}.$$

For any function $u_0 \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ with

$$\|u_0\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_2} - \mathcal{M}, \tag{4.8}$$

here, $\mathcal{M} = \sqrt{\frac{\nu^2}{4C_1^2C_4^2} - \frac{1}{C_1}\|f\|_{W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}}$, the iteration process

$$u_k + \frac{1}{\mu} TQ\pi_k = \frac{\rho}{\mu} TQ\widetilde{T}F(u_{k-1}), \quad k = 1,2,\dots \tag{4.9}$$

$$\frac{1}{\mu}\big[Q\pi_k\big]_0 = \frac{\rho}{\mu}\big[Q\widetilde{T}F(u_{k-1})\big]_0, \tag{4.10}$$

converges in $W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \times L^{p(x)}(\mathbb{R}^n_+, \mathbb{R})$.

Proof. The proof is similar to one of [15, theorem 4.6.8]. For the reader's convenience, we would like to give some details. Replacing f by $F(u_{k-1})$ in the proof of Theorem 3.3 we obtain the unique solvability of the Stokes equations (4.9)–(4.10) which we have to solve in each step. Moreover, we have the following estimate:

$$\|Du_k\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} + \frac{1}{\mu}\|Q\pi_k\|_{L^{p(x)}(\mathbb{R}^n_+)} \le C_3\frac{\rho}{\mu}\|Q\widetilde{T}F(u_{k-1})\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \tag{4.11}$$

where $C_3 \ge 1$ is a constant. The only remaining problem is the convergence of our iteration procedure. From Theorem 3.3 we know

$$Du_k + \frac{1}{\mu}Q\pi_k = \frac{\rho}{\mu}Q\widetilde{T}F(u_{k-1}).$$

Then it follows from (4.11) that

$$\frac{1}{\mu}\|Q(\pi_k - \pi_{k-1})\|_{L^{p(x)}(\mathbb{R}^n_+)} \le \frac{C_3}{\nu}\|Q\widetilde{T}(F(u_{k-1}) - F(u_{k-2}))\|_{L^{p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}.$$

Hence

$$\begin{aligned}
\|u_k - u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} &\le \frac{1}{\mu}\|TQ(\pi_k - \pi_{k-1})\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \\
&\quad + \frac{\rho}{\mu}\|TQ\widetilde{T}(F(u_{k-1}) - F(u_{k-2}))\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \\
&\le \frac{C_2(1+C_3)}{\nu}\|F(u_{k-1}) - F(u_{k-2})\|_{W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}.
\end{aligned}$$

In terms of Lemma 4.1, one obtain

$$\begin{aligned}
&\|F(u_{k-1}) - F(u_{k-2})\|_{W^{-1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \\
&\le C_1\|u_{k-1} - u_{k-2}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}(\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} + \|u_{k-2}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}).
\end{aligned}$$

Let $L_k = \frac{C_1C_4}{\nu}(\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}+\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)})$ with $C_4 = C_2(1+C_3)$. Then we obtain

$$\|u_k - u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} \le L_k\|u_{k-1} - u_{k-2}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}. \tag{4.12}$$

On the other hand, by (3.1) and Lemma 4.1, we have

$$\begin{aligned}\|u_k\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} &\le \frac{1}{\mu}\|TQ\pi_k\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} + \frac{\rho}{\mu}\|TQ\widetilde{T}F(u_{k-1})\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}\\ &\le \frac{C_1C_4}{\nu}\|u_{k-1}\|^2_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} + \frac{C_4}{\nu}\|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}.\end{aligned}$$

Now we have to ensure that

$$\|u_k\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} \le \|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}.$$

For this we notice that

$$\frac{C_1C_4}{\nu}\|u_{k-1}\|^2_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} + \frac{C_4}{\nu}\|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} \le \|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)},$$

which is equivalent to

$$\|u_{k-1}\|^2_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} - \frac{\nu}{C_1C_4}\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,Cl_n)} + \frac{1}{C_1}\|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} \le 0,$$

which is equivalent to

$$\Big(\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} - \frac{\nu}{2C_1C_4}\Big)^2 \le \frac{\nu^2}{(2C_1C_4)^2} - \frac{1}{C_1}\|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}.$$

According to the assumption (4.7), we have

$$\Big|\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} - \frac{\nu}{2C_1C_4}\Big| \le \mathcal{W}$$

with

$$\mathcal{M} = \Big(\frac{\nu^2}{4C_1^2C_4^2} - \frac{1}{C_1}\|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}\Big)^{1/2}.$$

This leads to the following condition for $\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}$,

$$\frac{\nu}{2C_1C_4} - \mathcal{M} \le \|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4} + \mathcal{M}.$$

Now assume that $\|u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4} - \mathcal{M}$. Then it follows that

$$\begin{aligned}&\|u_k\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}\\ &\le \frac{1}{\mu}\|TQ\pi_k\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} + \frac{\rho}{\mu}\|TQ\widetilde{T}F(u_{k-1})\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}\\ &\le \frac{C_1C_4}{\nu}\|u_{k-1}\|^2_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} + \frac{C_4}{\nu}\|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}\\ &\le \frac{C_1C_4}{\nu}\Big(\frac{\nu}{2C_1C_4} - \mathcal{M}\Big)^2 + \frac{C_4}{\nu}\|f\|_{W^{-1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}\\ &\le \frac{\nu}{2C_1C_4} - \mathcal{M}.\end{aligned}$$

Consequently, using the inequality $\|u_{k-2}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4} - \mathcal{M}$ and (4.12) we have

$$\|u_k - u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}_+^n,\mathrm{C}\ell_n)}$$

$$
\begin{aligned}
&\le \frac{2C_1C_4}{\nu}\Big(\frac{\nu}{2C_1C_4}-\mathcal{M}\Big)\|u_{k-1}-u_{k-2}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}\\
&\le \Big(1-\frac{2C_1C_4}{\nu}\mathcal{M}\Big)\|u_k-u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}.
\end{aligned}
$$

And

$$
L_k \le 1-\frac{2C_1C_4}{\nu}\mathcal{M} := \mu < 1.
$$

In this case one has

$$
\|u_k-u_{k-1}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \le \mu\|u_{k-1}-u_{k-2}\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \tag{4.13}
$$

with $0<\mu<1$ and fixed. The convergence of the sequence $\{u_k\}$ is therefore obtained by Banach's contraction mapping principle, and hence the convergence of the sequence $\{\pi_k\}$ immediately follows from (4.9). $\square$

Remark 4.4. Here, ν is the kinematic viscosity of the fluid. Our result states that under certain smallness condition of the external force, there exists a unique solution to the stationary Navier-Stokes equations.

Remark 4.5. Here we would like to point out that the obtained solutions in Theorem 3.3 and Theorem 4.3 are weak solutions, see [28, Theorem 4.2 and Theorem 5.1] for the similar proofs in the case of bounded domains, so we omit all the details.

Now it is straightforward to obtain the following results based on Theorem 4.3.

Corollary 4.6. *Under the assumptions in Theorem 4.3, we have the a-priori estimate*

$$
\|u\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4}-\mathcal{M}. \tag{4.14}
$$

An a-priori estimate for the term $\|Q\pi\|_{L^{p(x)}(\mathbb{R}^n_+)}$ is easy to obtain.

Corollary 4.7. *There exists the error estimate*

$$
\|u_k-u\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)} \le \frac{L^k}{1-L}\|u_0-u\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,\mathrm{C}\ell_n)}.
$$

In the case of $u_0=0$ we have

$$
\|u_k-u\|_{W_0^{1,p(x)}(\mathbb{R}^n_+,Cl_n)} \le \frac{L^k}{1-L}\Big(\frac{\nu}{2C_1C_4}-\mathcal{M}\Big).
$$

5. N-S equations with heat conduction in the half-space

In this section we will study the flow of a viscous fluid under the influence of temperature. Similar to [15] the above method for treating the stationary Navier-Stokes equations can be applied to more complicated problems. More specifically, we consider the following problem:

$$
-\Delta u+\frac{\rho}{\mu}(u\cdot\nabla)u+\frac{1}{\mu}\nabla\pi+\frac{\gamma}{\mu}gw=-f \quad \text{in } \mathbb{R}^n_+, \tag{5.1}
$$

$$
-\Delta w+\frac{m}{\kappa}(u\cdot\nabla)w=\frac{1}{\kappa}h \quad \text{in } \mathbb{R}^n_+, \tag{5.2}
$$

$$
\operatorname{div} u=0 \quad \text{in } \mathbb{R}^n_+, \tag{5.3}
$$

$$
u=0,\quad w=0 \quad \text{on } \Sigma. \tag{5.4}
$$

In addition to the case of Navier-Stokes equations, w denotes the temperature, γ the Grasshof number, m the Prandtl number, κ the number of temperature conductivity and g the vector $(0, 0, \dots, -1)^T$, where only the nth component is different from zero. For the detailed account about the Grasshof number, the Prandtl number and the Reynolds number, we refer to [16].

Remark 5.1. In the case of Ω a bounded domain and space $W_0^{k,2}(\Omega, \mathbb{H})$, the problem (5.1)–(5.4) was already studied by Gürlebeck and Sprößig [16]. In the case of Ω a unbounded domain and space $W_0^{1,p}(\Omega, \mathrm{C}\ell_n)$, the problem (5.1)–(5.4) was already investigated by Cerejeiras and Kähler [2], see also [38] for the corresponding results in the setting of variable exponents in bounded smooth domains.

In analogy to the case of the Navier-Stokes equations, we consider the following equivalent hyper-complex problem:

$$u + \frac{1}{\mu} TQ\pi = -TQ\widetilde{T}(F(u) - \frac{\gamma}{\mu} e_n w) \quad \text{in } \mathbb{R}^n_+, \tag{5.5}$$

$$\frac{1}{\mu}\big[Q\pi\big]_0 = \big[Q\widetilde{T}(F(u) - \frac{\gamma}{\mu} e_n w)\big]_0 \quad \text{in } \mathbb{R}^n_+, \tag{5.6}$$

$$w = -\frac{m}{\kappa} TQ\widetilde{T}\big[uD\big]_0 w + \frac{1}{\kappa} TQ\widetilde{T} h \quad \text{in } \mathbb{R}^n_+, \tag{5.7}$$

with $F(u) := f + \frac{\rho}{\mu}\big[uD\big]_0 u$. Then the problem can be solved by the following iteration process:

$$u_k + \frac{1}{\mu} TQ\pi_k = -TQ\widetilde{T}(F(u_{k-1}) - \frac{\gamma}{\mu} e_n w_{k-1}) \quad \text{in } \mathbb{R}^n_+, \tag{5.8}$$

$$\frac{1}{\mu}\big[Q\pi_k\big]_0 = \big[Q\widetilde{T}(F(u_{k-1}) - \frac{\gamma}{\mu} e_n w_{k-1})\big]_0 \quad \text{in } \mathbb{R}^n_+, \tag{5.9}$$

$$w_k = -\frac{m}{\kappa} TQ\widetilde{T}\big[u_k D\big]_0 w_k + \frac{1}{\kappa} TQ\widetilde{T} h \quad \text{in } \mathbb{R}^n_+. \tag{5.10}$$

Equations (5.8) and (5.9) represent an iteration similar to the case of the Navier-Stokes equations. Hence we have to study the solvability of equation (5.10). To this end, in analogy to [15], we give the following "inner" iteration:

$$w_k^i = -\frac{m}{\kappa} TQ\widetilde{T}(u_k \cdot \nabla) w_k^{i-1} + \frac{1}{\kappa} TQ\widetilde{T} h. \tag{5.11}$$

Similar to the proof of [38, Theorem 4.1], one can obtain the following result.

Theorem 5.2. *Let $u_k \in W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, where $p(x)$ satisfies* (2.1) *and $n/2 \le p_- \le p(x) \le p_+ < \infty$. Furthermore, suppose*

(i) $\|u_k\|_{W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} < \kappa / m C_1 C_2$;

(ii) $m\nu < 2\kappa(1 + C_3)$.

Then the iteration procedure (5.11) *converges in $W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$ to a unique solution of* (5.10) *and we have a-priori estimate*

$$\|w_k\|_{W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} \le \frac{2(1 + C_3) C_2}{2\kappa(1 + C_3) - m\nu} \|h\|_{W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)}.$$

Combining theorem 5.2 with our considerations in the case of the Navier-Stokes equations leads to the following theorem. See a similar proof in [38, Theorem 4.2].

Theorem 5.3. *Let $f \in W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, $h \in W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)$, where $p(x)$ satisfies* (2.1) *and $n/2 \le p_- \le p(x) \le p_+ < \infty$. Furthermore, assume*

(a) $\nu \parallel f \parallel_{W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} + C_5 \parallel h \parallel_{W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} < C_6$;

(b) $\| h \|_{W^{-1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n)} < C_7$;
(c) $m\nu < 2\kappa(1 + C_3)$,

where

$$C_5 = \frac{2\gamma(1 + C_3)C_2}{\mu(2\kappa(1 + C_3) - m\nu)}, \quad C_6 = \frac{3\nu^2}{16C_1C_4^2}, \gamma C_7 = \frac{\mu(2\kappa(1 + C_3) - m\nu)^2}{8\gamma m C_1 C_2^3 (1 + C_3)^3}.$$

Then the problem (5.5)–(5.7) *has a unique solution* (u, w, π) *in* $W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \times W_0^{1,p(x)}(\mathbb{R}^n_+, \mathrm{C}\ell_n) \times L^{p(x)}(\mathbb{R}^n_+, \mathbb{R})$, *where* u *and* w *are uniquely defined, and* π *uniquely up to a constant. Our iteration procedure* (5.8)–(5.10) *converges to the solution of* (5.5)–(5.7).

Acknowledgements. B. Zhang was supported by the Natural Science Foundation of Heilongjiang Province of China (No. A201306), by the Research Foundation of Heilongjiang Educational Committee (No. 12541667), and by the Doctoral Research Foundation of Heilongjiang Institute of Technology (No. 2013BJ15).

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