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RESONANT (p, q)-EQUATIONS WITH ROBIN BOUNDARY CONDITION

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Abstract. We consider a nonlinear nonhomogeneous Robin problem that has the sum of a p-Laplacian and a q-Laplacian (a (p, q)-equation). The reaction term is a Caratheodory function which is resonant at $\pm\infty$ with respect to any nonprincipal variational eigenvalue of the Robin p-Laplacian. Using variational methods and Morse theory (critical groups), we show the existence of at least three nontrivial smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2 boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous elliptic equation

$$\begin{gathered} -\Delta_p u(z) - \Delta_q u(z) = f(z, u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta_1(z)|u|^{p-2}u + \beta_2(z)|u|^{q-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here $1 < q < p < +\infty$, and for $1 < r < \infty$ by Δ_r we denote the r-Laplace differential operator

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du) \quad \text{for all } u \in W^{1,r}(\Omega).$$

The reaction term $f(z, x)$ is a Caratheodory function (that is, for all $x \in \mathbb{R}$ $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$ $x \to f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits $(p-1)$-linear growth near $\pm\infty$ and interacts with the non-principal part of the variational spectrum of the Robin p-Laplacian (resonant problem). In the boundary condition $\frac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative of u defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to \Big(|Du|^{p-2} + |Du|^{q-2}\Big)\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This generalized normal derivative is dictated by the nonlinear Green's identity (see Gasinski-Papageorgiou [9, p.210]). It is also used by Lieberman [14].

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In this article combining variational methods based on the critical point theory and Morse theory (critical groups), we prove a multiplicity theorem, showing the existence of at least three nontrivial smooth solutions. Note that the differential operator $u \to \Delta_p u + \Delta_q u$ is nonhomogeneous.

Equations driven by the sum of a p-Laplacian and of a q-Laplacian are known as (p, q)-equations and arise in problems of mathematical physics. We refer to Benci-D'Avenia-Fortunato-Pisani [4] (quantum physics) and Cherfils-Ilyason [7] (plasma physics). There have been existence and multiplicity results for such equations. We mention the works of Aizicovici-Papageorgiou-Staicu [2], Benouhiba-Belyacine [5], Bobkov-Tanaka [6], Cingolani-Degiovanni [8], Gasinski-Papageorgiou [10], Marano-Mosconi-Papageorgiou [15], Marano-Papageorgiou [16], Mugnai-Papageorgiou [19], Papageorgiou-Radulescu [20], [21], Papageorgiou-Winkert [25], Sun [27], Sun-Zhang-Su [28], Tanaka [29], Yang-Yin [30] (Dirichlet problems) and Papageorgiou-Radulescu [22] (Neumann and Robin problems). From the aforementioned papers resonant problems are examined in Gasinski-Papageorgiou [10], Papageorgiou-Radulescu [20] and Sun [27] and the resonance is with respect to the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$.

2. Preliminary results and hypotheses

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the pair (X, X^*).

Let $\varphi \in C^1(X, \mathbb{R})$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds: "every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence".

This is a compactness type condition on the functional φ. It leads to a deformation lemma, from which one can derive the minimax theory for the critical values of φ. Prominent in that theory, is the so-called "mountain pass theorem" of Ambrosetti-Rabinowitz [3]. Here we state the result in a slightly more general form (see Gasinski-Papageorgiou [9, p.648]).

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$, $\rho > 0$, $\|u_1 - u_0\| > \rho$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = \rho] = m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq m_\rho$ and c is a critical value of φ (i.e., there exists $u^* \in X$ such that $\varphi'(u^*) = 0$ and $\varphi(u^*) = c$ in X^*).

For the analysis of problem (1.1) we will use the Sobolev spaces $W^{1,r}(\Omega)$ $1 < r < \infty$. We know that this is a Banach space with norm

$$\|u\| = \left[\|u\|_r^r + \|Du\|_r^r\right]^{1/r} \quad \text{for all } u \in W^{1,r}(\Omega).$$

The Banach space $W^{1,r}(\Omega)$ is uniformly convex, thus reflexive.

We will also use the subspace $C^1(\overline{\Omega})$. We will exploit the fact that $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \quad \text{for all } z \in \overline{\Omega}\}$$

This cone has a nonempty interior containing the set

$$D_+ = \{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \overline{\Omega}\}$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the boundary Lebesgue spaces $L^\tau(\partial\Omega)$ $1 \le \tau \le \infty$.

From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,\tau}(\Omega) \to L^\tau(\partial\Omega)$, known as the "trace map" such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,\tau}(\Omega) \cap C(\overline{\Omega}).$$

The trace map assigns boundary values to any Sobolev function. We know that $\gamma_0(\cdot)$ is compact into $L^\theta(\partial\Omega)$ for all $\theta \in [1, \frac{(N-1)p}{N-p})$ if $\tau < N$ and into $L^\theta(\partial\Omega)$ for all $\theta \in [1, +\infty)$ if $\tau \ge N$. Moreover, we have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{\tau'},\tau}(\partial\Omega) \quad (\frac{1}{\tau} + \frac{1}{\tau'} = 1) \text{ and } \ker\gamma_0 = W_0^{1,\tau}(\Omega).$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$, are understood in the sense of traces. For $1 < r < \infty$, let $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ be defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,r}(\Omega).$$

For this operator we have the following result (see Motreanu-Motreanu-Papageorgiou [17, p.40]).

Proposition 2.2. *The map $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is*

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,r}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle \le 0 \Rightarrow u_n \to u \text{ in } W^{1,r}(\Omega).$$

Let $f_0 : \Omega \times X \to \mathbb{R}$ be a Caratheodory function such that

$$|f_0(z,x)| \le \alpha_0(z)[1 + |x|^{\tau-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $\alpha_0 \in L^\infty(\Omega)_+ = \{\alpha \in L^\infty(\Omega) : \alpha(z) \ge 0 \text{ for a.a. } z \in \Omega\}$ and $\tau \in (1, p^*]$ where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \le p \end{cases}$$

(the critical Sobolev exponent for p). Also, let $k_0 \in C^{0,\eta}(\partial\Omega \times \mathbb{R})$ with $\eta \in (0,1)$ and

$$0 \le k_0(z,x) \le c_1(|x|^q + |x|^p) \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R},$$

with $c_1 > 0$, $1 < q < p$. We set

$$F_0(z,x) = \int_0^x f_0(z,s)ds \quad \text{and} \quad K_0(z,x) = \int_0^x k_0(z,s)ds$$

and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q + \int_{\partial\Omega} K_0(z,u)d\sigma - \int_\Omega F_0(z,u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

The next proposition is a special case of a more general result of Papageorgiou-Radulescu [24].

Proposition 2.3. *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C^1(\overline{\Omega})$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \le \rho_1.$$

As we already mentioned in the introduction, one of our tools, are critical groups. So, let us recall their definition. Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}, \quad K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\},$$
$$\varphi^c = \{u \in X : \varphi(u) \le c\}.$$

Suppose that (Y_1, Y_2) is a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Let $u \in K_\varphi^c$ be isolated. The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{u\}) \text{ for all } k \in \mathbb{N}_0,$$

where U is an isolating neighborhood of u, that is, $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition of critical groups is independent of the particular choice of the isolating neighborhood U.

Suppose that φ satisfies the C-condition and assume that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed, let $c' < c < \inf \varphi(K_\varphi)$. From Motreanu-Motreanu-Papageorgiou [17, Corollary 5.3.5, p.115], we have that: if $\varphi^{c'}$ is a strong deformation retract of φ^c, then

$$H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [17, Corollary 6.15, p.145 of]).

The next result is a useful tool for computing the critical groups at infinity. It extends an earlier analogous result for Hilbert spaces of Liang-Su [13].

Proposition 2.4. *If X is a Banach space, $(t, u) \to h_t(u)$ belongs in $C^1([0,1] \times X, \mathbb{R})$, maps bounded sets to bounded sets, the maps $u \to (h_t)'(u)$ and $t \to \partial_t h_t(u)$ are both locally Lipschitz, h_0, h_1 satisfy the C-condition*

$$|\partial_t h_t(u)| \le c_2(\|u\|^q + \|u\|^p) \quad \text{for all } u \in X,$$

with $c_2 > 0$, $1 < q < p < \infty$ and there exist $\theta_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that

$$h_t(u) \le \theta_0 \Rightarrow (1 + \|u\|)\|(h_t)'(u)\|_* \ge \delta_0[\|u\|^q + \|u\|^p] \quad \text{for all } t \in [0,1],$$

then $C_k(h_0, \infty) = C_k(h_1, \infty)$ for all $k \in \mathbb{N}_0$.

Proof. Since $h \in C^1([0,1] \times X, \mathbb{R})$, it admits a pseudogradient vector field $\widehat{v}_t(u)$ (see Gasinski-Papageorgiou [9, Theorem 5.1.19, p.616]). In fact from the construction of the pseudogradient vector field, we have

$$\widehat{v}_t(u) = (\partial_t h_t(u), v_t(u)),$$

with $v_t(\cdot)$ being a pseudogradient vector field corresponding to the function $h_t(\cdot)$. Therefore, for all $t \in [0,1]$ and all $u \in X\backslash K_{h_t}$, we have

$$\|(h_t)'(u)\|_*^2 \le \langle (h_t)'(u), v_t(u)\rangle \quad \text{and} \quad \|v_t(u)\| \le 2\|(h_t)'(u)\|_* \tag{2.1}$$

For $t \in [0,1]$ we consider the vector field $g_t : X\backslash K_{h_t} \to X$ defined by

$$g_t(u) = -\frac{|\partial_t h_t(u)|}{\|(h_t)'(u)\|_*^2} v_t(u) \quad \text{for all } u \in X\backslash K_{h_t}. \tag{2.2}$$

This is a locally Lipschitz vector field. Choose $\theta \le \theta_0$ such that

$$h_0^\theta \ne \emptyset \quad \text{or} \quad h_1^\theta \ne \emptyset$$

If no such $\theta \le \theta_0$ can be found, it means that both h_0, h_1 are bounded below. Since by hypothesis they satisfy the C-condition, we have

$$C_k(h_0,\infty) = C_k(h_1,\infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [17, Proposition 6.64, p.161]). Hence the conclusion of the proposition holds.

To fix things, we assume that $h_0^\theta \ne \emptyset$ (the reasoning is similar if $h_1^\theta \ne \emptyset$). Let $u \in h_0^\theta$ and consider the abstract Cauchy problem

$$\frac{d\xi}{dt} = g_t(\xi) \quad \text{on } [0,1],\ \tau(0) = u. \tag{2.3}$$

Since the vector field is locally Lipschitz, problem (2.3) admits a local flow (see Gasinski-Papageorgiou [9, Theorem 5.1.21, p.618]). We denote this local flow by $\xi(t,u)$. For the sake of notational simplicity, in the next calculation we drop the u-dependence in the expression of the local flow, since it does not play any role. We have

$$\begin{aligned}
&\frac{d}{dt} h_t(\xi(t)) \\
&= \langle (h_t)'(\xi(t)), \frac{d\xi}{dt}\rangle + \partial_t h_t(\xi(t)) \\
&= \langle (h_t)'(\xi(t)), -\frac{|\partial_t h_t(\xi(t))|}{\|(h_t)'(\xi(t))\|_*} v_t(\xi(t))\rangle + \partial_t h_t(\xi(t)) \quad \text{(see (2.2) and(2.3))} \\
&\le -|\partial_t h_t(\xi(t))| + \partial_t h_t(\xi(t)) \ \text{ (see (2.1))} \\
&\le 0.
\end{aligned}$$

Hence for $t > 0$ small, we have

$$h_t(\xi(t)) \le h_0(\xi(0)) = h_0(u) \le \theta \le \theta_0$$

(see (2.3) and recall that $u \in h_0^\theta$), which implies

$$(1 + \|\xi(t)\|)\|(h_t)'(\xi(t))\|_* \ge \delta_0\big[\|\xi(t))\|^q + \|\xi(t)\|^p\big]. \tag{2.4}$$

Therefore,

$$\begin{aligned}
\|g_t(\xi(t))\| &\le \frac{|\partial_t h_t(\xi(t))|}{\|(h_t)'(\xi(t))\|_*^2}\|v_t(\xi(t))\| \quad \text{(see (2.2))} \\
&\le \frac{c_2[\|\xi(t)\|^q + \|\xi(t)\|^p]}{\|(h_t)'(\xi(t))\|_*^2} 2\|(h_t)'(\xi(t))\|_* \quad \text{(by hypothesis and (2.1))} \\
&\le \frac{2c_2}{\delta_0}(1 + \|\xi(t)\|) \quad \text{(see (2.4))}
\end{aligned}$$

It follows that the local flow $\xi(\cdot, u)$ is in fact global on $[0,1]$ (see [9, theorem 5.1.22, p.618]).

We go back in denoting the flow by $\xi(t,u)$. For every $t \in [0,1]$, $\xi(t,u)$ is an homeomorphism. Hence $\xi(1,\cdot)$ is a homeomorphism of h_0^θ onto a subset D_0 of h_1^θ. Reversing the time (that is replacing t by $1-t$) and using the corresponding global flow $\xi_*(t,v)$, we have that h_1^θ is a homeomorphic to a subset D_1 of h_0^θ. We set

$$\eta(t,u) = \xi_*(t, \xi(t,u)) \quad \text{for all } (t,u) \in [0,1] \times h_0^\theta.$$

We have that

$$\eta(0,\cdot) \text{ is homotopy equivalent to } id|_{D_0}(\cdot) \text{ and } \eta(1,\cdot) = (\xi_*)_1 \circ \xi_1. \tag{2.5}$$

In a similar fashion, if we set

$$\eta_*(t,v) = \xi(t, \xi_*(t,v)) \quad \text{for all } (t,v) \in [0,1] \times h_1^\theta,$$

then we have that

$$\eta_*(0,\cdot) \text{ is homotopy equivalent to } id|_{D_1}(\cdot) \text{ and } \eta_*(1,\cdot) = \xi_1 \circ (\xi_*)_1. \tag{2.6}$$

Recall that

$$\{D_0, h_0^\theta\} \text{ and } \{D_1, h_1^\theta\} \text{ are homeomorphic pairs.} \tag{2.7}$$

Then from (2.5), (2.6), (2.7) it follows that: h_0^θ and h_1^θ are homotopy equivalent, which implies that

$$H_k(X, h_0^\theta) = H_k(X, h_1^\theta) \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [17, Proposition 6.11, p.143]), which implies

$$C_k(h_0, \infty) = C_k(h_1, \infty) \quad \text{for all } k \in \mathbb{N}_0$$

(choosing $\theta \in \mathbb{R}$ even smaller if necessary). □

Next let us recall some basic facts concerning the spectrum of the Robin r-Laplacian. So let $\beta \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha < 1$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_r u(z) = \widehat{\lambda}|u(z)|^{r-2}u(z) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_r} + \beta(z)|u|^{r-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.8}$$

This eigenvalue problem was studied by Papagerogiou-Radulescu [23]. A number $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of the negative Robin r-Laplacian, if problem (2.8) admits a nontrivial solution $\widehat{u}$. The nontrivial solution $\widehat{u}$ is an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. There is a smallest eigenvalue $\widehat{\lambda}_1(r)$ which has the following properties:

- $\widehat{\lambda}_1(r) \geq 0$ and it is isolated (in fact if $\beta = 0$, then $\widehat{\lambda}_1(r) = 0$, while if $\beta \neq 0$, then $\widehat{\lambda}_1(r) > 0$)
- $\widehat{\lambda}_1(r)$ is simple (this means that, if $\widehat{u}, \widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1(r)$, then $\widehat{u} = c\widehat{v}$ for some $c \in \mathbb{R}\backslash\{0\}$).
- $$\widehat{\lambda}_1(r) = \inf\Big[\frac{\|Du\|_r^r + \int_{\partial\Omega} \beta(z)|u|^r\,d\sigma}{\|u\|_r^r} : u \in W^{1,r}(\Omega),\ u \neq 0\Big]. \tag{2.9}$$

In (2.9) the infimum is attained on the corresponding one dimensional eigenspace. From (2.9) it is easy to see that the elements of this eigenspace have fixed sign. By $\widehat{u}_1(r)$ we denote the L^r-normalized (that is, $\|\widehat{u}_1(r)\|_r = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(r)$. The nonlinear regularity theory and the nonlinear maximum principle (see for example, Gasinski-Papageorgiou [9, pp.737-738]) imply that $\widehat{u}_1(r) \in D_+$. Since the spectrum $\widehat{\sigma}(r)$ of (2.8) is closed and $\widehat{\lambda}_1(r)$ is isolated, then the second eigenvalue $\widehat{\lambda}_2(r)$ is well defined by

$$\widehat{\lambda}_2(r) = \inf[\widehat{\lambda} \in \widehat{\sigma}(r) : \widehat{\lambda} > \widehat{\lambda}_1(r)].$$

By $\operatorname{ind}(\cdot)$ we denote the $\mathbb{Z}_2$-cohomological index of Fadell-Rabinowitz (see [8]). Using $\operatorname{ind}(\cdot)$ and the Ljusternik-Schnirelmann minimax scheme, we can define a whole sequence $\{\widehat{\lambda}_k(r)\}_{k\in\mathbb{N}}$ of dinstict eigenvalues of (2.8), by setting

$$\widehat{\lambda}_k(r) = \inf[\sup_{u\in A}\{\|Du\|_r^r + \int_{\partial\Omega} \beta(z)|u|^r\,d\sigma\} : A \subseteq M \text{ symmetric}, \operatorname{ind}(A) \geq k, k \in \mathbb{N}]$$

with $M = \{u \in W^{1,r}(\Omega) : \|u\|_r = 1\}$. Evidently M is a C^1-Banach manifold. We have $\widehat{\lambda}_k(r) \to +\infty$ as $k \to +\infty$ and these eigenvalues are known as "variational eigenvalues" of (2.8). If $k = 1, 2$, then $\widehat{\lambda}_k(r)$ are as defined earlier. We do not know if this sequence exhausts $\widehat{\sigma}(r)$. This is the case if $r = 2$ (linear eigenvalue problem) and if $N = 1$ (ordinary differential equations). We mention that, if $\widehat{u}$ is an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}_k(r)$, $k \geq 2$, then $\widehat{u} \in C^1(\overline{\Omega})$ (by the nonlinear regularity theory) and it is nodal (that is, sign-changing),

Also, for the nonprincipal eigenvalues, the corresponding eigenspaces are only cones and not linear subspaces of $W^{1,r}(\Omega)$ and the latter cannot be expressed as a direct sum of these eigenspaces. For these reasons, when $r \neq 2$ (nonlinear eigenvalue problem), it is difficult to deal with problems resonant with respect to any nonprincipal eigenvalue.

As an easy consequence of the properties of the principal eigenvalue $\widehat{\lambda}_1(r)$, we have the following result (see Mugnai-Papageorgiou [18, Lemma 4.11]).

Lemma 2.5. *If $\theta \in L^\infty(\Omega)$, $\theta(z) \leq \widehat{\lambda}_1(r)$ for a.a. $z \in \Omega$ and $\theta \neq \widehat{\lambda}_1(r)$, then there exists $\widehat{c} > 0$ such that*

$$\|Du\|_r^r + \int_{\partial\Omega} \beta(z)|u|^r\,d\sigma - \int_\Omega \theta(z)|u|^r\,dz \geq \widehat{c}\|u\|^r$$

for all $u \in W^{1,p}(\Omega)$.

For $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,r}(\Omega)$ we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,r}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-$$

Let us introduce our hypotheses on the data of (1.1).

(H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that $f(z,0) = 0$, for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $\alpha_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leq \alpha_\rho(z) \quad \text{for a.a. } z \in \Omega \text{ all } |x| \leq \rho;$$

(ii) there exists an integer $m \geq 2$ such that

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = \widehat{\lambda}_m(p) \text{ uniformly for a.a. } \quad z \in \Omega;$$

(iii) if $F(z,x)=\int_0^x f(z,s)ds$, then

$$0<c_0\le \liminf_{x\to\pm\infty}\frac{f(z,x)x-pF(z,x)}{|x|^{p-1}}\quad \text{uniformly for a.a. } z\in\Omega;$$

(iv) there exists a function $\theta\in L^\infty(\Omega)_+$ such that $\theta(z)\le\widehat{\lambda}_1(q)$ for a.a. $z\in\Omega$, with strict inequality on a set of positive measure,

$$\limsup_{x\to 0}\frac{qF(z,x)}{|x|^q}\le\theta(z)\quad \text{uniformly for a.a. } z\in\Omega$$

note that if $\beta_2\equiv 0$, then $\widehat{\lambda}_1(q)=0$).

Remark 2.6. Hypothesis (H1)(ii) implies that at $\pm\infty$ we have resonance with respect to a nonprincipal variational eigenvalue. We can write that

$$f(z,x)=\widehat{\lambda}_m(p)|x|^{p-2}x+f_0(z,x) \tag{2.10}$$

with a Caratheodory function $f_0(z,x)$ such that

$$\lim_{x\to\pm\infty}\frac{f_0(z,x)}{|x|^{p-2}x}=0 \text{ uniformly for a.a. } z\in\Omega. \tag{2.11}$$

If we set $F_0(z,x)=\int_0^x f_0(z,s)ds$. Then

$$F(z,x)=\frac{\widehat{\lambda}_m(p)}{p}|x|^p+F_0(z,x)$$

and we have

$$0<c_0\le\liminf_{x\to\pm\infty}\frac{f_0(z,x)x-pF_0(z,x)}{|x|^{p-1}}\quad \text{uniformly for a.a. } z\in\Omega; \tag{2.12}$$

see hypothesis (H1)(iii).

Example 2.7. The following function satisfies hypotheses (H1). For the sake of simplicity we drop the z-dependence

$$f(x)=\begin{cases}\theta|x|^{q-2}x & \text{if } |x|\le 1\\ \widehat{\lambda}_m(p)|x|^{p-2}x+|x|^{\tau-2}x+\widehat{c} & \text{if } 1<|x|\end{cases}$$

with $0<\theta<\widehat{\lambda}_1(q)$, $\widehat{c}=\theta-(\widehat{\lambda}_m(p)+1)$, $1<q<p$, $p-1\le\tau<p$.

The hypotheses on the boundary coefficient $\beta(\cdot)$ are the following

(H2) $\beta_1\in C^{0,\alpha}(\partial\Omega)$, $\beta_2\in C^{0,\eta}(\partial\Omega)$ with $0<\alpha,\eta<1$ and $\beta_1(z),\beta_2(z)\ge 0$ for all $z\in\partial\Omega$.

(H3) For every $\rho>0$ there exists $\widehat{\xi}_\rho>0$ such that

$$f(z,x)x+\widehat{\xi}_\rho(|x|^\rho+|x|^q)\ge 0\quad \text{for a.a. } x\in\Omega,\ \ |x|\le\rho.$$

Remark 2.8. Note that when $\beta_1=\beta_2=0$ in (H2), we recover the Neumann problem. Also note that the example given earlier satisfies (H3).

In what follows for $\tau\in(1,+\infty)$, we introduce the C^1-functional $\gamma_\tau:W^{1,\tau}(\Omega)\to\mathbb{R}$ defined by

$$\gamma_\tau(u)=\|Du\|_\tau^\tau+\int_{\partial\Omega}\beta(z)|u|^\tau\,d\sigma\quad \text{for all } u\in W^{1,\tau}(\Omega).$$

Let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\gamma_q(u) - \int_\Omega F(z,u)dz \text{ for all } u \in W^{1,p}(\Omega),$$

γ_p with $\beta = \beta_1$ and γ_q with $\beta = \beta_2$. Evidently $\varphi \in C^1(W^{1,p}(\Omega))$.

3. Solutions of constant sign

We introduce the following truncations-perturbations of the reaction term $f(z,\cdot)$:

$$\begin{gathered} \widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x \le 0 \\ f(z,x) + x^{p-1} & \text{if } 0 < x; \end{cases} \\ \widehat{f}_-(z,x) = \begin{cases} f(z,x) + |x|^{p-2}x & \text{if } x < 0 \\ 0 & \text{if } 0 \le x \end{cases} \end{gathered} \tag{3.1}$$

Both are Caratheodory functions. We set

$$\widehat{F}_+(z,x) = \int_0^x \widehat{f}_+(z,s)ds \quad \text{and} \quad \widehat{F}_-(z,x) = \int_0^x \widehat{f}_-(z,s)ds$$

and consider the C^1-functionals $\widehat{\varphi}_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\pm(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\gamma_q(u) + \frac{1}{p}\|u\|_p^p - \int_\Omega \widehat{F}_\pm(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega.)$$

Proposition 3.1. *If hypotheses* (H1)(i)–(iii), (H2) *hold then the functionals* $\widehat{\varphi}_\pm$ *satisfy the C-condition.*

Proof. We do the proof for the functional $\widehat{\varphi}_+$ the proof for $\widehat{\varphi}_-$ being similar. So, let $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\widehat{\varphi}_+(u_n)| \le M_1 \quad \text{for some } M_1 > 0, \text{ and all } n \in \mathbb{N}, \tag{3.2}$$

$$(1 + \|u_n\|)\widehat{\varphi}'_+(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \quad \text{as } n \to \infty. \tag{3.3}$$

From (3.3) we have

$$\begin{aligned} &\Big|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_{\partial\Omega} (\beta_1(z)|u_n|^{p-2}u_n + \beta_2(z)|u_n|^{q-2}u_n)h\,d\sigma \\ &+ \int_\Omega |u_n|^{p-2}u_n h\,dz - \int_\Omega \widehat{f}_+(z,u_n)h\,dz\Big| \\ &\le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+. \end{aligned} \tag{3.4}$$

In (3.4) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p + \|Du_n^-\|_q^q + \|u_n^-\|_p^p \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}$$

(see (3.1) and hypothesis (H2)) which implies

$$u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.5}$$

Claim 1: $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded.

We argue indirectly. So, suppose that Claim 1 is not true. Then by passing to a subsequence if necessary, we may assume that $\|u_n^+\| \to \infty$. We set $y_n = \frac{u_n^+}{\|u_n^+\|}$ $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.6}$$

From (3.4) and (3.5), we have

$$\begin{aligned}
&\Big|\langle A_p(u_n^+), h\rangle + \langle A_q(u_n^+), h\rangle + \int_{\partial\Omega} \Big(\beta_1(z)(u_n^+)^{p-1} + \beta_2(z)(u_n^+)^{q-1}\Big) h \, d\sigma \\
&+ \int_\Omega (u_n^+)^{p-1} h \, dz - \int_\Omega \widehat{f}_+(z, u_n^+) h \, dz\Big| \\
&\leq \varepsilon_n' \|h\| \quad \text{for all } n \in \mathbb{N}, \text{ with } \varepsilon_n' \to 0^+,
\end{aligned}$$

which implies

$$\begin{aligned}
&\Big|\langle A_p(y_n) + \frac{1}{\|u_n^+\|^{p-q}} A_q(y_n), h\rangle + \int_{\partial\Omega} \Big(\beta_1(z) y_n^{p-1} + \frac{1}{\|u_n^+\|^{p-q}} \beta_2(z) y_n^{q-1}\Big) h \, d\sigma \\
&+ \int_\Omega y_n^{p-1} h \, dz - \int_\Omega \frac{\widehat{f}_+(z, u_n^+)}{\|u_n^+\|^{p-1}} h \, dz\Big| \\
&\leq \varepsilon_n' \frac{\|h\|}{\|u_n^+\|^{p-1}} \quad \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{3.7}$$

In (3.7) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.6) and the fact that $q < p$. Then

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0$$

which implies

$$y_n \to y \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to \infty, \tag{3.8}$$

thus $\|y\| = 1$ and $y \geq 0$, see Proposition 2.2.

Hypotheses (H1)(i),(ii) imply that

$$|f(z, x)| \leq c_3(1 + |x|^{p-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_3 > 0,$$

which in turn implies

$$\Big\{\frac{\widehat{f}_+(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\Big\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded (see (3.1))}.$$

Therefore using hypothesis (H1)(ii) we have (at least for a subsequence) such that

$$\frac{\widehat{f}_+(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} (\widehat{\lambda}_m(p) + 1) y^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ as } n \to \infty \tag{3.9}$$

(see Aizicovici-Papageorgiou [1, proof of Proposition 30]).

If in (3.7) we pass to the limit as $n \to \infty$ and use (3.8) and (3.9) and the facts that $q < p$, $y \geq 0$, then

$$\langle A_p(y), h\rangle + \int_{\partial\Omega} \beta_1(z) y^{p-1} h \, d\sigma = \widehat{\lambda}_m(p) \int_\Omega y^{p-1} h \, dz \quad \text{for all } h \in W^{1,p}(\Omega) \text{ (see (3.1))}$$

which implies

$$
\begin{gathered}
-\Delta_p y(z) = \widehat{\lambda}_m(p)|y(z)|^{p-2}y(z) \quad \text{for a.a. } z \in \Omega, \\
\frac{\partial y}{\partial \eta_p} + \beta(z)y^{p-1} = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{3.10}
$$

(see Papageorgiou-Radulescu [23]). Since $m \geq 2$, from (3.10) we infer that y must be nodal, which contradicts (3.8). This proves Claim 1.

From (3.5) and Claim 1, we have that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$
u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.11}
$$

In (3.4) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.11). Then we have

$$
\begin{gathered}
\lim_{n\to\infty}[\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0, \\
\Rightarrow \limsup_{n\to\infty}[\langle A_p(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle] \leq 0,
\end{gathered}
$$

(recall that $A_q(\cdot)$ is monotone), which implies

$$
\begin{gathered}
\limsup_{n\to\infty}[\langle A_p(u_n), u_n - u\rangle \leq 0 \quad \text{(see (3.11))}, \\
\Rightarrow u_n \to u \ \text{ in } \ W^{1,p}(\Omega) \quad \text{(see Proposition 2.2)}.
\end{gathered}
$$

This proves that the functional $\widehat{\varphi}_+$ satisfies the C-condition. Similarly for the functional $\widehat{\varphi}_-$. ☐

Proposition 3.2. *If hypotheses (H1)(i)–(iii), (H2) hold, then the energy functional φ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$
|\varphi(u_n)| \leq M_2 \quad \text{for some } M_2 > 0, \text{and all } n \in \mathbb{N}, \tag{3.12}
$$

$$
(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{3.13}
$$

From (3.13) we have

$$
\begin{aligned}
&\Big|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_{\partial\Omega} (\beta_1(z)|u_n|^{p-2}u_n + \beta_2(z)|u_n|^{q-2}u_n)h\, d\sigma \\
&- \int_\Omega f(z, u_n)h dz\Big| \\
&\leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+.
\end{aligned}
\tag{3.14}
$$

In (3.14) we choose $h = u_n \in W^{1,p}(\Omega)$. Then

$$
-\gamma_p(u_n) - \gamma_q(u_n) + \int_\Omega f(z, u_n)u_n\, dz \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.15}
$$

Also from (3.12) we have

$$
\gamma_p(u_n) + \frac{p}{q}\gamma_q(u_n) - \int_\Omega pF(z, u_n)\, dz \leq pM_2 \quad \text{for all } n \in \mathbb{N}. \tag{3.16}
$$

Adding (3.15) and (3.16), we obtain

$$
(\frac{p}{q} - 1)\gamma_q(u_n) + \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\, dz \leq M_3 \tag{3.17}
$$

for some $M_3 > 0$ and all $n \in \mathbb{N}$,

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)] \le M_3 \quad \text{for all } n \in \mathbb{N},$$

(recall that $q < p$ and $\gamma_q \ge 0$).

Claim 2: $\{u_n\}_n \ge 1 \subseteq W^{1,p}(\Omega)$ is bounded,

We argue by contradiction. So, suppose that Claim 2 is not true. We may assume that $\|u_n\| \to \infty$. We set $y_n = \frac{u_n}{\|u_n\|}$ $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.18}$$

From (3.14) we have

$$\begin{aligned} &\Big|\langle A_p(y_n) + \frac{1}{\|u_n\|^{p-2}} A_q(y_n), h\rangle \\ &+ \int_{\partial\Omega} \Big(\beta_1(z)|y_n|^{p-2}y_n + \frac{\beta_2(z)}{\|u_n\|^{p-q}}|y_n|^{q-2}y_n\Big) h\,dz - \int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}} h\,dz\Big| \\ &\le \frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.19}$$

In (3.19) we choose $h = y_n - y \in W^{1,p}(\Omega)$, we pass to the limit as $n \to \infty$ and use the fact that $q < p$. Then $\lim_{n\to\infty}\langle A_p(y_n), y_n - y\rangle = 0$ which implies

$$y_n \to y \quad \text{in } W^{1,p}(\Omega) \text{ and so } \|y\| = 1 \quad \text{(see Proposition 2.2)} \tag{3.20}$$

From (3.20) it follows that we can find $D \subseteq \Omega$ measurable with $|D|_N > 0$ (here $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$) such that

$$|u_n(z)| \to +\infty \quad \text{for all } z \in D. \tag{3.21}$$

Hypotheses (H1)(i), (iii) imply that we can find $c_4 > 0$ such that

$$-c_4 \le f(z,x)x - pF(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } \mathbb{R}. \tag{3.22}$$

Then

$$\begin{aligned} &\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz \\ &= \int_{\Omega\setminus D} [f(z,u_n)u_n - pF(z,u_n)]dz + \int_D [f(z,u_n)u_n - pF(z,u_n)]\,dz \\ &\ge -c_4|\Omega\setminus D|_N + \int_D [f(z,u_n)u_n - pF(z,u_n)]\,dz \end{aligned} \tag{3.23}$$

for all $n \in \mathbb{N}$ (see (3.22)). From (3.21), hypothesis (H1)(iii) and Fatou's lemma, we have

$$\int_D [f(z,u_n)u_n - pF(z,u_n)]\,dz \to +\infty,$$

which implies

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]\,dz \to +\infty \quad \text{(see (3.23))}. \tag{3.24}$$

Comparing (3.17) and (3.24) we have a contradiction. This proves the claim 2.

Using the claim 2, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.25}$$

In (3.14) we choose $h = u_n - u \in W^{1,p}(\Omega)$, we pass to the limit as $n \to \infty$ and use (3.25). Then

$$\begin{gathered}\lim_{n\to\infty}[\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0,\\ \Rightarrow \limsup_{n\to\infty}\langle A_p(u_n), u_n - u\rangle \le 0, \quad \text{(using the monotonicity of } A_q(\cdot) \text{ and (3.25))},\\ \Rightarrow u_n \to u \quad \text{in} \quad W^{1,p}(\Omega) \quad \text{(see Proposition 2.2)},\end{gathered}$$

Therefore φ satisfies the C-condition. □

Hypotheses (H1)(i), (ii) imply that

$$|f(z,x)| \le c_5(1 + |x|^{p-1}) \quad \text{for a.a. } z \in \Omega \text{ all } x \in \mathbb{R}, \text{ some } c_5 > 0\,. \tag{3.26}$$

Proposition 3.3. *If hypotheses* (H1)(iv), (3.26) *and* (H2) *hold, then* $u = 0$ *is a local minimizer of the functionals* $\widehat{\varphi}_\pm$ *and of* φ.

Proof. We do the proof for the functional $\widehat{\varphi}_+$, the proofs for $\widehat{\varphi}_-$ and φ being similar. Hypothesis (H1)(iv) implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \le \frac{1}{q}[\theta(z) + \varepsilon]|x|^q \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta. \tag{3.27}$$

On the other hand, given $r > p > q$, using (3.26) we see that we can find $c_6 = c_6(\varepsilon) > 0$ such that

$$F(z,x) \le c_6|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \ge \delta. \tag{3.28}$$

Since $\theta \in L^\infty(\Omega)_+$, from (3.27) and (3.28) it follows that

$$F(z,x) \le \frac{1}{q}[\theta(z) + \varepsilon]|x|^q + c_6|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.29}$$

For $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}\widehat{\varphi}_+(u) &= \frac{1}{p}\gamma_p(u) + \frac{1}{q}\gamma_q(u) + \frac{1}{p}\|u^-\|_p^p - \int_\Omega F(z,u^+)\,dz \quad \text{(see (3.1))}\\ &\ge \frac{1}{p}\gamma_p(u^+) + \frac{1}{p}\big[\gamma_p(u^-) + \|u^-\|_p^p\big]\\ &\quad + \frac{1}{q}\Big[\gamma_q(u^+) - \int_\Omega \theta(z)(u^+)^q\,dz - \varepsilon\|u^+\|^q\Big] - c_7\|u\|^r\end{aligned}$$

for some $c_7 > 0$ (see (3.29)). Using Lemma 2.5 and choosing $\varepsilon > 0$ small, we have

$$\widehat{\varphi}_+(u) \ge \frac{1}{p}\|u^-\|^p + \frac{c_8}{q}\|u^+\|^q - c_7\|u\|^r$$

for some $c_8 > 0$. If $\|u\| \le 1$, then $\|u^+\|$, $\|u^-\| \le 1$ and so $\|u^+\|^q \ge \|u^+\|^p$. Hence

$$\begin{aligned}\widehat{\varphi}_+(u) &\ge \frac{1}{p}\|u^-\|^p + \frac{c_8}{q}\|u^+\|^p - c_7\|u\|^r\\ &\ge c_9\|u\|^p - c_7\|u\|^r \quad \text{for some } c_9 > 0.\end{aligned} \tag{3.30}$$

Since $r > p$, if we choose $\rho \in (0,1)$ small, then from (3.30) we see that

$$\widehat{\varphi}_+(u) \ge 0 \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| \le \rho,$$

which implies that $u = 0$ is a local minimizer of $\widehat{\varphi}$. Similar argument works for the functionals $\widehat{\varphi}_-$ and φ. □

Remark 3.4. We can avoid the use of (3.26) and instead assume that

$$|f(z,x)| \le c_{10}(1+|x|^{p^*-1}) \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ some } c_{10}>0.$$

For $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \le \delta$, we have

$$\widehat{\varphi}_+(u) \ge \frac{1}{p}\|u^-\|^p + \frac{c_8}{q}\|u^+\|^q \quad \text{(see (3.27) and Lemma 2.5)}$$

By taking $\delta > 0$ even smaller, we have

$$\widehat{\varphi}_+(u) \ge \frac{1}{p}\|u^-\|^p + c_{11}\|u^+\|^p \ge c_{12}\|u\|^p$$

which implies that $u = 0$ is a local $C^1(\overline{\Omega})$-minimizer of $\widehat{\varphi}_+$, and that $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of $\widehat{\varphi}_+$ (see Proposition 2.4).

Now we are ready to produce two constant sign solutions (one positive and the other negative).

Proposition 3.5. *If hypotheses* (H1)–(H3) *hold, then problem* (1.1) *has at least two nontrivial constant sign smooth solutions*

$$u_0 \in D_+ \quad \textit{and} \quad v_0 \in -D_+.$$

Proof. Using (3.1) and the nonlinear regularity theory of Lieberman [14], we have

$$K_{\widehat{\varphi}_+} \subseteq C_+ \quad \text{and} \quad K_{\widehat{\varphi}_-} \subseteq -C_+. \tag{3.31}$$

So, we assume that both sets are finite or otherwise we already have two sequences consisting of distinct positive and negative solutions which in fact belong in D_+ and D_- respectively (see the last part of this proof).

First, we prove the existence of a positive solution. Since $K_{\widehat{\varphi}_+}$ is finite and $u = 0$ is a local minimizer of $\widehat{\varphi}_+$ (see Proposition 3.3), we can find $\rho \in (0,1)$ small such that

$$\widehat{\varphi}_+(0) = 0 < \inf[\widehat{\varphi}_+(u) : \|u\| = \rho] = \widehat{m}_\rho^+ \tag{3.32}$$

(see Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 29]). Also because $m \ge 2$ and $q < p$, we see that

$$\widehat{\varphi}_+(t\widehat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{3.33}$$

Then (3.32), (3.33) and Proposition 3.1, permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$u_0 \in K_{\widehat{\varphi}_+} \subseteq C_+ \text{ (see (3.31))} \quad \text{and} \quad \widehat{m}_\rho^+ \le \widehat{\varphi}_+(u_0),$$

which implies $u_0 \in C_+\backslash\{0\}$ (see (3.32)).

We have

$$\begin{gathered} -\Delta_p u_0(z) - \Delta_q u_0(z) = f(z,u_0(z)) \quad \text{for a.a. } z\in\Omega, \\ \frac{\partial u_0}{\partial n_{pq}} + \beta_1(z)u_0^{p-1} + \beta_2(z)u_0^{q-1} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.34}$$

(see Papageorgiou-Radulescu [23]). Hypotheses (H1)(i),(iv) and (H3) imply that given $\rho > 0$, we can find $\widehat{\xi}_\rho > 0$ such that

$$f(z,x)x + \widehat{\xi}_\rho[|x|^p + |x|^q] \ge 0 \quad \text{for a.a. } z\in\Omega, \text{ all } |x| \le \rho$$

Let $\rho = \|u_0\|_\infty$. Then from (3.34) we have

$$\Delta_p u_0(z) + \Delta_q u_0(z) \le \widehat{\xi}_\rho[u_0(z)^{p-1} + u_0(z)^{q-1}] \quad \text{for a.a. } z\in\Omega,$$

which implies $u_0 \in D_+$ (see Pucci-Serrin [26, Theorem 5.4.1, p.111 and Theorem 5.5.1, p.120]). Similarly, working this time with $\widehat{\varphi}_-$, we produce a negative solution $v_0 \in -D_+$. $\square$

4. Three solutions theorem

In this section, using Morse Theory (critical groups), we produce a third nontrivial smooth solution. For this purpose, using Proposition 2.4, we compute the critical groups of φ and of $\widehat{\varphi}_\pm$ at infinity.

Proposition 4.1. *If hypotheses* (H1)(i),(ii),(iii), (H2) *hold and K_φ is finite, then* $C_m(\varphi,\infty) \neq 0$.

Proof. Let $\lambda \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p)), \lambda \notin \widehat{\sigma}(p)$ and consider the C^1-functional $\psi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\gamma_p(u) - \frac{\lambda}{p}\|u\|_p^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

We consider the homotopy $h_t(u)$ defined by

$$h_t(u) = (1-t)\varphi(u) + t\psi(u) \quad \text{for all } t \in (0,1), \text{ all } u \in W^{1,p}(\Omega).$$

Note that $h_0(\cdot) = \varphi(\cdot)$ and by Proposition 3.2 φ satisfies the C-condition. Also $h_1(\cdot) = \psi(\cdot)$ and since $\lambda \notin \widehat{\sigma}(p)$, we see that ψ satisfies the C-condition.

Claim 3: There exist $\theta_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that

$$h_t(u) \le \theta_0 \Rightarrow (1+\|u\|)\|(h_t)'(u)\|_* \ge \delta_0(\|u\|^q + \|u\|^p) \quad \text{for all } t \in (0,1).$$

As before we argue by contradiction. Since $(t,u) \to h_t(u)$ maps bounded sets to bounded sets, if Claim 3 is not true, we can find $\{t_n\}_{n\ge1} \subseteq [0,1]$ and $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ such that

$$\begin{gathered} t_n \to t, \ \|u_n\| \to +\infty, \ h_{t_n}(u_n) \to -\infty, \text{ and} \\ \|(h_{t_n})'(u_n)\|_* < \frac{\|u_n\|^q + \|u_n\|^p}{n(1+\|u_n\|)} \quad \text{for all } n \in \mathbb{N}. \end{gathered} \tag{4.1}$$

Without loss of generality we assume that $\|u_n\| \ge 1$ for all $n \in \mathbb{N}$. From (4.1) we have

$$\begin{aligned} &\Big|\langle A_p(u_n) + (1-t_n)A_q(u_n), h\rangle + \int_{\partial\Omega} (\beta_1(z)|u_n|^{p-2}u_n + (1-t_n)\beta_2(z)(u_n)^{p-2}u_n)h\, d\sigma \\ &+ (1-t_n)\int_\Omega f(z,u_n)h\, dz - t_n\lambda \int_\Omega |u_n|^{p-2}u_n h\, dz\Big| \\ &\le \frac{\|u_n\|^q + \|u_n\|^p}{n(1+\|u_n\|)} \\ &\le \frac{2}{n}\|u_n\|^{p-1} \end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$ (recall $\|u_n\| \ge 1$ for all $n \in \mathbb{N}$, $q < p$).

Let $y_n = u_n/\|u_n\|$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{4.3}$$

From (4.2) we have

$$\begin{aligned}
&\Big|\langle A_p(y_n) + \frac{1-t_n}{\|u_n\|^{p-q}} A_q(y_n), h\rangle \\
&+ \int_{\partial\Omega} \Big(\beta_1(z)|y_n|^{p-2}y_n + \frac{(1-t_n)\beta_2(z)}{\|u_n\|^{p-q}}|y_n|^{q-2}y_n\Big) h\,d\sigma \\
&- (1-t_n)\int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}} h\,dz - t_n\lambda\int_\Omega |y_n|^{p-2}y_n h\,dz\Big| \\
&\le \frac{2}{n} \quad \text{for all } n\in\mathbb{N}.
\end{aligned} \tag{4.4}$$

From (3.26) we have

$$\Big\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}}\Big\}_{n\ge 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \tag{4.5}$$

On account of (4.5) and by passing to a subsequence if necessary and using hypothesis (H1)(ii) we have

$$\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p)|y|^{p-2}y \quad \text{in } L^{p'}(\Omega). \tag{4.6}$$

If in (4.4) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (4.1), (4.3), (4.5) and the fact that $q < p$, we obtain that $\lim_{n\to\infty}\langle A_p(y_n), y_n - y\rangle = 0$ which implies

$$y_n \to y \quad \text{in } W^{1,p}(\Omega),\ \text{hence } \|y\| = 1 \text{ (see Proposition 2.2).} \tag{4.7}$$

So, if in (4.4) we pass to the limit as $n\to\infty$ and use (4.1), (4.6), (4.7) and the fact that $q < p$, then

$$\langle A_p(y), h\rangle + \int_{\partial\Omega}\beta_1(z)|y|^{p-2}yh\,d\sigma = \int_\Omega [(1-t)\widehat{\lambda}_m(p) + t\lambda]|y|^{p-2}yh\,dz$$

for all $h \in W^{1,p}(\Omega)$, which implies

$$\begin{gathered}
-\Delta_p y(z) = \lambda_t |y(z)|^{p-2}y(z) \quad \text{for a.a. } z\in\Omega, \\
\frac{\partial u}{\partial n_{pq}} + \beta_1(z)|y|^{p-2}y = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.8}$$

where $\lambda_t = (1-t)\widehat{\lambda}_m(p) + t\lambda$.

If $\lambda \notin \widehat{\sigma}(p)$, then from (4.8) it follows that $y = 0$, which contradicts (4.7). So, suppose that $\lambda_t \in \widehat{\sigma}(p)$. Since $y \ne 0$ (see (4.7)), we can find $D \subseteq \Omega$ measurable with $|D|_N > 0$ such that

$$|u_n(z)| \to +\infty \quad \text{for all } z\in\Omega. \tag{4.9}$$

From (2.12) we see that we can find $c_{13} > 0$ and $M > 0$ such that

$$0 < c_{13} \le \frac{f_0(z,x)x - pF_0(z,x)}{|x|^{p-1}} \quad \text{for a.a. } z\in\Omega,\ \text{all } |x|\ge M. \tag{4.10}$$

From (4.9), (4.10) and Fatou's lemma, we have

$$0 < \liminf_{n\to\infty}\int_\Omega \frac{f_0(z,u_n)u_n - pF_0(z,u_n)}{|u_n|^{p-1}}|y_n|^{p-1}\,dz. \tag{4.11}$$

Note that on account of (4.1), we can find $n_0 \in \mathbb{N}$ such that

$$\gamma_p(u_n) + (1-t_n)\frac{p}{q}\gamma_q(u_n) - (1-t_n)\int_\Omega pF(z,u_n)\,dz - t_n\lambda\|u_n\|_p^p \le 0 \tag{4.12}$$

for all $n \ge n_0$.

Also, using (4.2) with $h = u_n \in W^{1,p}(\Omega)$, we obtain

$$-\gamma_p(u_n)-(1-t_n)\gamma_q(u_n)+(1-t_n)\int_\Omega f(z,u_n)u_n\,dz+t_n\lambda\|u_n\|_p^p \le \frac{2}{n}\|u_n\|^{p-1} \tag{4.13}$$

for all$n \in \mathbb{N}$. Adding (4.12) and (4.13) and recalling that $q < p$, we obtain

$$(1-t_n)\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]\,dz \le \frac{2}{n}\|u_n\|^{p-1} \quad \text{for all } n \ge n_0,$$

which implies

$$(1-t_n)\int_\Omega [f_0(z,u_n)u_n - pF_0(z,u_n)]\,dz \le \frac{2}{n}\|u_n\|^{p-1} \quad \text{for all } n \ge n_0.$$

Evidently $t < 1$, otherwise from (4.8) $y = 0$ which contradicts (4.7). So

$$\int_\Omega [f_0(z,u_n)u_n - pF_0(z,u_n)]\,dz \le \frac{2c_{14}}{n}\|u_n\|^{p-1}$$

for all $n \ge n_0$, some $c_{14} > 0$, which implies

$$\begin{gathered}\int_\Omega \frac{f_0(z,u_n)u_n - pF_0(z,u_n)}{|u_n|^{p-1}}|y_n|^{p-1}\,dz \le \frac{2c_{14}}{n},\\ \limsup_{n\to\infty}\int_\Omega \frac{f_0(z,u_n)u_n - pF_0(z,u_n)}{|u_n|^{p-1}}|y_n|^{p-1}\,dz \le 0,\end{gathered}$$

which contradicts (4.11). This proves Claim 3.

On account of Claim 3, we can use Proposition 2.4 and have

$$C_k(\varphi,\infty) = C_k(\psi,\infty) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.14}$$

Next for $r > 0$ we define the following two sets

$$\begin{gathered}C_r = \{u \in W^{1,p}(\Omega) : \gamma_p(u) < \lambda\|u\|_p^p,\ \|u\| = r\},\\ D = \{u \in W^{1,p}(\Omega) : \gamma_p(u) \ge \lambda\|u\|_p^p\}.\end{gathered}$$

The set $\partial B_r = \{u \in W^{1,p}(\Omega) : \|u\| = r\}$ is a C^1-Banach manifold hence it is locally contractible (see Lee [12]). The set C_r is an open subset of ∂B_r, hence it is locally contractible too. Similarly the open set $W^{1,p}(\Omega)\backslash D$ is locally contractible too. Since $\lambda \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p))\backslash\widehat{\sigma}(p)$, we have

$$\operatorname{ind} C_r = \operatorname{ind}(W^{1,p}(\Omega)\backslash D) = m.$$

So, using [8, Theorems 3.2 and 3.6], we have $C_m(\psi, 0) \neq 0$.

However, since $K_\psi = \{0\}$, we have

$$C_k(\psi,0) = C_k(\psi,\infty) \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [17, Proposition 6.61, p.160], which implies $C_m(\psi,\infty) \neq 0$, $\Rightarrow C_m(\varphi,\infty) \neq 0$ (see (4.14)). □

Next we compute the critical groups at infinity for the functionals $\widehat{\varphi}_\pm$.

Proposition 4.2. *If hypotheses* (H1), (H2) *hold, then* $C_k(\widehat{\varphi}_\pm, 0) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. As before let $\lambda \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p)) \backslash \widehat{\sigma}(p)$ and let $\widehat{\psi}_+ : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\widehat{\psi}_+(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta_1(z)|u|^p \, d\sigma - \frac{\lambda}{p}\|u^+\|_p^p \text{ for all } u \in W^{1,p}(\Omega).$$

We consider the homotopy $(h_+)_t(u)$ defined by

$$(h_+)_t(u) = (1-t)\widehat{\varphi}_+(u) + t\widehat{\varphi}_+(u) \quad \text{for all } t \in [0,1],\ u \in W^{1,p}(\Omega).$$

We have $(h_+)_0 = \widehat{\varphi}_+$ which satisfies the C-condition (see Proposition 3.1). Also $(h_+)_1 = \widehat{\psi}_+$ and since $\lambda \notin \widehat{\sigma}(p)$, $\widehat{\psi}_+$ satisfies the C-condition.

Claim 4: There exist $\theta_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that $(h_+)_t(u) \le \theta_0$ which implies

$$(1 + \|u\|)\|((h_+)_t)'(u)\|_* \ge \delta_0[\|u\|^q + \|u\|^p] \quad \text{for all } t \in [0,1].$$

As in previous occasions, we proceed by a contradiction argument. So, suppose that Claim 4 is not true. Since $(t,u) \to (h_+)_t(u)$ maps bounded sets to bounded sets, we can find two sequences $\{t_n\}_{n\ge1} \subseteq [0,1]$ and $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ such that

$$\begin{gathered} t_n \to t, \quad \|u_n\| \to \infty, \quad (h_+)_{t_n}(u_n) \to -\infty \quad \text{and} \\ \|((h_+)_{t_n})'(u_n)\|_* < \frac{\|u_n\|^q + \|u_n\|^p}{n(1+\|u_n\|)} \quad \text{for all } n \in \mathbb{N}. \end{gathered} \tag{4.15}$$

From inequality (4.15) we have

$$\begin{aligned} &\Big|\langle A_p(u_n), h\rangle + (1-t_n)\langle A_q(u_n), h\rangle \\ &+ \int_{\partial\Omega} (\beta_1(z)|u_n|^{p-2}u_n + (1-t_n)\beta_2(z)|u_n|^{q-2}u_n)h d\sigma - \int_\Omega (u_n^-)^{p-1} h \, dz \\ &- (1-t_n)\int_\Omega f(z, u_n^+) h \, dz - t_n\lambda \int_\Omega (u_n^+)^{p-1} h \, dz\Big| \\ &\le \frac{\varepsilon_n \|h\|}{(1+\|u_n\|} \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+. \end{aligned} \tag{4.16}$$

Choosing $h = -u_n^- \in W^{1,p}(\Omega)$ in the above inequality, we have

$$\gamma_p(u_n^-) + (1-t_n)\gamma_q(u_n^-) + \|u_n^-\|_p^p \le \varepsilon_n \quad \text{for all } n \in \mathbb{N} \tag{4.17}$$

see (3.1), which implies $u_n^- \to 0$ in $W^{1,p}(\Omega)$ (see hypothesis (H2) and recall $\gamma_q \ge 0$). From (4.15) we know that $\|u_n\| \to \infty$. Hence (4.17) implies that

$$\|u_n^+\| \to +\infty \quad \text{as } n \to \infty. \tag{4.18}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{4.19}$$

From (4.16) and (4.17), we infer that

$$\begin{aligned} &\Big|\langle A_p(y_n), h\rangle + \frac{1-t_n}{\|u_p^+\|^{p-q}}\langle A_q(y_n), h\rangle \\ &+ \int_{\partial\Omega} (\beta_1(z)|y_n|^{p-2}y_n + \frac{(1-t_n)\beta_2(z)}{\|u_n^+\|^{p-q}}|y_n|^{q-2}y_n)h \, dz \\ &- (1-t_n)\int_\Omega \frac{f(z,u_n^+)}{\|u_n^+\|^{p-1}} h \, dz - t_n\lambda\int_\Omega y_n^{p-1} h \, dz\Big| \\ &\le \varepsilon_n' \|h\| \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \varepsilon_n' \to 0^+. \end{aligned} \tag{4.20}$$

In this inequality we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.18), (4.19) and the fact that $q < p$. Then

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0$$

which implies

$$y_n \to y \quad \text{in } W^{1,p}(\Omega), \text{ hence } \|y\| = 1,\, y \geq 0 \text{ (see Proposition 2.2)} \tag{4.21}$$

As before (see the proof of Proposition 4.1), hypothesis (H1)(ii) implies that

$$\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p) y^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ as } n \to \infty. \tag{4.22}$$

So, if in (4.20) we pass to the limit as $n \to \infty$ and use (4.18), (4.21), (4.22) and the fact that $q < p$, then

$$\langle A_p(y), h\rangle + \int_{\partial\Omega} \beta(z) y^{p-1} h \, d\sigma = \int_\Omega [(1-t)\widehat{\lambda}_m(p) + t\lambda] y^{p-1} h \, dz$$

which implies

$$\begin{gathered} -\Delta_p y(z) = \lambda_t y(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \frac{\partial y}{\partial n_p} + \beta(z) y^{p-1} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.23}$$

where $\lambda_t = (1-t)\widehat{\lambda}_m(p) + t\lambda$.

If $\lambda_t \notin \widehat{\sigma}(p)$, then from (4.23) we have $y = 0$, contradicting (4.21).

If $\lambda_t \in \widehat{\sigma}(p)$, then since $\lambda_t \geq \widehat{\lambda}_m(p)$ and $m \geq 2$, from (4.23) we infer that y must be nodal, contradicting (4.41). This proves Claim 4. Using Claim 4 and Proposition 2.4 and we have

$$C_k(\widehat{\varphi}_+, \infty) = C_k(\widehat{\psi}_+, \infty) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.24}$$

Now we introduce the homotopy

$$(\widehat{h}_+)_t(u) = \widehat{\psi}_+(u) - t\int_\Omega u \, dz \quad \text{for all } t \in [0,1], \text{ and } u \in W^{1,p}(\Omega).$$

Claim 5: $((\widehat{h}_+)_t)'(u) \neq 0$ for all $t \in [0,1]$, all $u \in W^{1,p}(\Omega)$, $u \neq 0$.

Note that for $t \in [0,1]$ we have

$$((\widehat{h}_+)_t)' = \widehat{\psi}_+'(u) - t\eta^* \quad \text{in } W^{1,p}(\Omega)^*, \tag{4.25}$$

with $\eta^* \in W^{1,p}(\Omega)^*$ such that $\langle \eta^*, v\rangle = \int_\Omega v \, dz$ for all $v \in W^{1,p}(\Omega)$. Also $K_{\widehat{\psi}_+} = \{0\}$. Indeed, if $u \in K_{\widehat{\psi}_+}$, then $\widehat{\psi}_+'(u) = 0$ which implies

$$\langle A_p(u), h\rangle - \int_\Omega (u^-)^{p-1} h \, dz + \int_{\partial\Omega} \beta_1(z)|u|^{p-2} u h \, dz = \lambda \int_\Omega (u^+)^{p-1} h \, dz$$

for all $h \in W^{1,p}(\Omega)$. Choosing $h = -u^- \in W^{1,p}(\Omega)$, we obtain

$$\|Du^-\|_p^p + \|u^-\|_p^p \leq 0$$

(see hypothesis (H2)), which implies $u \geq 0$. Then we have

$$\begin{gathered} -\Delta_p u(z) = \lambda u(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

which implies $u = 0$ since $\lambda \notin \widehat{\sigma}(p)$. Therefore $K_{\widehat{\psi}_+} = \{0\}$ and from this and (4.25), Claim 5 follows.

The homotopy invariance property of singular homology groups implies that for $\rho > 0$ small,

$$\begin{aligned} &H_k((\widehat{h}_+)_0(\cdot)^0 \cap B_\rho, (\widehat{h}_+)_0(\cdot)^0 \cap B_\rho \setminus \{0\}) \\ &= H_k((\widehat{h}_+)_1(\cdot)^0 \cap B_\rho, (\widehat{h}_+)_1(\cdot)^0 \cap B_\rho \setminus \{0\}) \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{4.26}$$

Using Claim 5 and Motreanu-Motreanu-Papageorgiou [17, Corollary 5.35, p.115 and Corollary 6.15, p.145], we have

$$H_k((\widehat{h}_+)_1(\cdot)^0 \cap B_\rho\ ,\ (\widehat{h}_+)_1(\cdot)^0 \cap B_\rho \setminus \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.27}$$

On the other hand from the definition of critical groups, we have

$$H_k((\widehat{h}_+)_0(\cdot)^0 \cap B_\rho,\ (\widehat{h}_+)_0(\cdot)^0 \cap B_\rho \setminus \{0\}) = C_k(\widehat{\psi}_+, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.28}$$

From (4.26), (4.27) and (4.28), we have

$$C_k(\widehat{\psi}_+, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.29}$$

But recall that $K_{\widehat{\psi}_+} = \{0\}$. Therefore

$$C_k(\widehat{\psi}_+, \infty) = C_k(\widehat{\psi}_+, 0) \quad \text{for all } k \in \mathbb{N}_0$$

which implies $C_n(\widehat{\psi}_+, \infty) = 0$ for all $k \in \mathbb{N}_0$ (see (4.29)); this in turn implies $C_k(\widehat{\varphi}_+, \infty) = 0$ for all $k \in \mathbb{N}_0$ (see (4.24)). Similarly we show that $C_k(\widehat{\varphi}_-, \infty) = 0$ for all $k \in \mathbb{N}_0$. $\square$

From the proof of Proposition we know that the positive solution $u_0 \in D_+$ (resp. the negative solution $v_0 \in -D_+$) is a critical point of mountain pass type for the functional $\widehat{\varphi}_+$ (resp. The functional $\widehat{\varphi}_-$). So, we have

$$C_1(\widehat{\varphi}_+, u_0) \neq 0 \quad \text{and} \quad C_1(\widehat{\varphi}_-, v_0) \neq 0, \tag{4.30}$$

(see Motreanu-Motreanu-Papageorgiou [17, Corollary 6.81, p.168]). In general to describe more precisely these critical groups, we need a Hilbert space setting and C^2-regularity of the functionals. Nevertheless, here using Propositions 4.1 and 4.2 and some tools from Algebraic Topology (Homological Algebra), we are able to compute exactly the critical groups of the energy functional φ at u_0 and at v_0. Note that since $u_0 \in D_+$, $v_0 \in -D_+$ and $\varphi'|_{C_+} = \widehat{\varphi}'_+|_{C_+}$, $\varphi'|_{-C_+} = \widehat{\varphi}'_-|_{-C_+}$, (see (3.1)) we have $u_0, v_0 \in K_\varphi$. We assume that $K_{\widehat{\varphi}_+} = \{0, u_0\}$, $K_{\widehat{\varphi}_-} = \{0, v_0\}$ or otherwise we are done.

Proposition 4.3. *If hypotheses* (H1)–(H3) *hold and $u_0 \in D_+$, $v_0 \in -D_+$ are the two constant sign solutions of* (1.1) *produced in Propositon 3.5, then $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$. Here $\delta_{k,m}$ denotes the Kronecker symbol defined by*

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m\,. \end{cases}$$

Proof. We will do the proof for the pair $\{\varphi, u_0\}$, the proof for the pair $\{\varphi, v_0\}$ being similar.

Let $\eta \le 0 < \alpha < \widehat{m}_\rho^+$ (see (3.32)). We consider the following triple of sets

$$\widehat{\varphi}_+^\eta \subseteq \widehat{\varphi}_+^\alpha \subseteq W^{1,p}(\Omega).$$

To this triple corresponds a long exact sequence of singular homology groups (see Motreanu-Motreanu-Papageorgiou [17, Proposition 6.14, p.143]). So, we have

$$\cdots \to H_k(W^{1,p}(\Omega), \widehat{\varphi}_+^{\eta}) \xrightarrow{i_*} H_k(W^{1,p}(\Omega), \widehat{\varphi}_+^{\alpha}) \xrightarrow{\widehat{\theta}_*} H_k(\widehat{\varphi}_+^{\alpha}, \widehat{\varphi}_+^{\eta}) \to \cdots \tag{4.31}$$

with i_* being the homomorphism induced by the inclusion $i : (W^{1,p}(\Omega), \widehat{\varphi}_+^{\eta}) \to (W^{1,p}(\Omega), \widehat{\varphi}_+^{\alpha})$ and $\widehat{\theta}_*$ is the composed boundary homomorphism. From the rank theorem we have

$$\begin{aligned} \operatorname{rank} H_k(W^{1,p}(\Omega), \widehat{\varphi}_+^{\alpha}) &= \operatorname{rank} \ker \widehat{\theta}_* + \operatorname{rank} \operatorname{im} \widehat{\theta}_* \\ &= \operatorname{rank} \operatorname{im} i_* + \operatorname{rank} \operatorname{im} \widehat{\theta}_* \quad \text{(since (4.31) is exact)} \end{aligned} \tag{4.32}$$

Since $\alpha \in (0, \widehat{m}_\rho^+)$ and $K_{\widehat{\varphi}_+} = \{0, u_0\}$, we have

$$H_k(W^{1,p}(\Omega)\widehat{\varphi}_+^{\alpha}) = C_k(\widehat{\varphi}_+, u_0) \quad \text{for all } k \in \mathbb{N}_0, \tag{4.33}$$

(see Motreanu-Motreanu-Papageorgiou [17, Lemma 6.55, p.157]).

Similarly since $\eta < 0 = \widehat{\varphi}_+(0)$, we have

$$H_k(W^{1,p}(\Omega), \widehat{\varphi}_+^{\eta}) = C_k(\widehat{\varphi}_+, \infty) \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{recall } K_{\widehat{\varphi}_+} = \{0, u_0\})$$

which implies $H_k(W^{1,p}(\Omega), \widehat{\varphi}_+^{\eta}) = 0$ for all $k \in \mathbb{N}_0$ (see Proposition 4.2). This in turn implies

$$\operatorname{im} i_* = \{0\} \quad \text{(see (4.31))}. \tag{4.34}$$

In a similar fashion, we see that $H_{k-1}(\widehat{\varphi}_+^{\alpha}, \widehat{\varphi}_+^{\eta}) = C_{k-1}(\widehat{\varphi}_+, 0)$ for all $k \in \mathbb{N}_0$, which implies

$$H_{k-1}(\widehat{\varphi}_+^{\alpha}, \widehat{\varphi}_+^{\eta}) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see Proposition 3.5)}. \tag{4.35}$$

We return to (4.32) and use (4.33), (4.34), (4.35). Then

$$\operatorname{rank} C_k(\widehat{\varphi}_+, u_0) \le 1. \tag{4.36}$$

Note that on account of (4.35) only the tail of (4.31) is nontrivial (that is, the terms for $k \ge 2$ are all zero). So, from (4.36) and (4.30) it follows that

$$C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.37}$$

Next consider the homotopy $\tilde{h}_t(u)$ defined by

$$\tilde{h}_t(u) = (1-t)\varphi(u) + t\widehat{\varphi}_t(u) \quad \text{for all } t \in [0,1], \text{ all } u \in W^{1,p}(\Omega).$$

Suppose that we could find $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \to t \text{ in } [0,1], \quad u_n \to u_0 \text{ in } W^{1,p}(\Omega) \text{ and } (\tilde{h}_{t_n})(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.38}$$

From equality (4.38) we have

$$\begin{aligned} &\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_{\partial\Omega} (\beta_1(z)|u_n|^{p-2}u_n + \beta_2(z)|u_n|^{q-2}u_n)h\,d\sigma \\ &- t_n \int_\Omega (u_n^-)^{p-1} h\,dz \\ &= (1-t_n)\int_\Omega f(z,u_n)h\,dz + t_n \int_\Omega f(z,u_n^+)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega), \end{aligned}$$

which implies

$$\begin{gathered} -\Delta_p u_n(z) - \Delta_q u_n(z) = (1-t_n)f(z,u_n(z)) + t_n[f(z,u_n^+(z)) + u_n^-(z)] \\ \text{for a.a. } z \in \Omega, \end{gathered}$$

$$\frac{\partial u}{\partial n_{pq}} + \beta_1(z)|u_n|^{p-2}u_n + \beta_2(z)|u_n|^{q-2}u_n = 0 \quad \text{on } \partial\Omega.$$

From Papageorgiou-Radulescu [24], we know that we can find $M_4 > 0$ such that

$$\|u_n\|_\infty \le M_4 \quad \text{for all } n \in \mathbb{N}.$$

Then the nonlinear regularity theory of Lieberman [14] implies that we can find $\alpha \in (0,1)$ and $M_5 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le M_5 \quad \text{for all } n \in \mathbb{N}. \tag{4.39}$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ onto $C^1(\overline{\Omega})$ and (4.18) imply that $u_n \to u_0$ in $C^1(\overline{\Omega})$, thus

$$u_n \in D_+ \quad \text{for all } n \ge n_0 \quad (\text{recall that } u_0 \in D_+).$$

But as we already pointed out $\varphi'|_{C_+} = \widehat{\varphi}'_+|_{C_+}$ (see (3.1)) implies that $\{u_n\}_{n\ge n_0}$ are distinct positive solutions of (1.1), a contradiction (recall $K_{\widehat{\varphi}_+} = \{0, u_0\}$). So (4.38) can not occur and from the homotopy invariance of critical groups (see for example Gasinski-Papageorgiou [11, Theorem 5.125, p.836]), we have

$$C_k(\varphi, u_0) = C_k(\widehat{\varphi}_+, u_0) \quad \text{for all } k \in \mathbb{N}_0,$$
$$\Rightarrow C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{see } (4.37)).$$

Similarly, using this time the functional $\widehat{\varphi}_-$, we show that

$$C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

□

Now we can produce a third nontrivial smooth solution for problem (1.1) and formulate our "three solutions theorem".

Theorem 4.4. *If hypotheses* (H1)–(H3) *hold, then problem* (1.1) *admits at least three nontrivial smooth solutions*

$$u_0 \in D_+, \quad v_0 \in -D_+, \quad y_0 \in C^1(\overline{\Omega}).$$

Proof. From Proposition 3.5, we already have two constant sign solutions

$$u_0 \in D_+, \quad v_0 \in -D_+.$$

From Proposition 4.3 we know that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.40}$$

Also, from Proposition 3.3, we know that $u = 0$ is a local minimizer of φ. Hence

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.41}$$

From Proposition 4.1, we have that $C_m(\varphi, \infty) \ne 0$ $(m \ge 2)$. So, we can find $y_0 \in K_\varphi$ such that $C_m(\varphi, y_0) \ne 0$ and $y_0 \notin \{0, u_0, v_0\}$ (see Motreanu-Motreanu-Papageorgiou [17, Proposition 6.89, p.172]). Then y_0 is a third nontrivial solution of (1.1) and the nonlinear regularity theory of Lieberman [14] implies that $y_0 \in C^1(\overline{\Omega})$. □

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References

[1] S. Aizicovici, N. S. Papageorgiou, V. Staicu; *Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints*, Memoirs of the AMS, Vol. 196, No **915** (2008), pp. 70.

[2] S. Aizicovici, N. S. Papageorgiou, V.Staicu; *Nodal solutions for $(p, 2)$-equations*, Trans AMS, **367** (2015), 7343-7372.

[3] A. Ambrosetti, P. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Functional Anal., **14** (1973), 349-381.

[4] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani; *Solitons in several space dimensions: Derrick'sproblem and infinitely many solutions*, Arch. Rat. Mech. Anal. **154** (2000), 297-324.

[5] N. Benouhiba, Z. Belyacine; *On the solutions of the (p, q)-Laplacian problem at resonance*, Nonlin. Anal., **77** (2013), 74-81.

[6] V. Bobkov, M. Tanaka; *On positive soluitons for (p, q)-Laplace equations with two parameters*, Calc. Var., **54** (2015), 1959-1972.

[7] L. Cherfils, Y. Ilyason; *On the stationary solutions of generalized reaction-diffusion equations with (p, q)-Laplacian*, Comm. Pure Appl. Anal., **4** (2005), 9-22.

[8] S. Cingolani, M. Degiovanni; *Nontrivial solutions for the p-Laplace equations with right hand side having p-linear growth at infinity*, Comm. Partial Diff. Equations, **30** (2005), 1191-1203.

[9] L. Gasinski, N. S. Papageorgiou; *Nonlinear Analysis*, Chapman and Hall/CRC Press, Boca Raton, (2006).

[10] L. Gasinski, N. S. Papageorgiou; *Dirichlet (p, q)-equations at resonance*, Discr. Cont. Dynam. Systems A, **34** (2014), 2037-2060.

[11] L. Gasinski, N. S. Papageorgiou; *Exercises in Analysis Part 2:Nonlinear Analysis*, Springer Cham, Heidelberg, (2016)

[12] J. M. Lee; *Introduction to Topological Manifolds*, Springer-Verlag, New York (2000).

[13] Z. Liang-J. Su; *Multiple solutions for semilinear elliptic boundary value problems with double resonance*, J. Math. Anal. Appl. **354** (2009), 147-158.

[14] G. M. Lieberman; *The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations*, Comm. Partial Diff. Equations 16 (1991), 311-361.

[15] S. Marano-S. Mosconi, N. S. Papageorgiou, *Multiple solutions to (p, q)-Laplacian problem with resonant concave nonlinearity*, Adv. Nonlin. Studies **16** (2016), 51-65.

[16] S. Marano-N. S. Papageorgiou; *Constant sign and nodal solutions of concave (p, q)-Laplacian problems*, Nonlin. Anal., **77** (2013), 118-129.

[17] D. Motreanu, V. Motreanu-N. S. Papageorgiou; *Topological and Variationsl Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York (2014).

[18] D. Mugnai, N. S. Papageorgiou; *Resonant nonlinear Neumann problems with indefinite weight*, Annali Scuola Normale Super. Pisa CI. Sci II, **5** (2012), 729-788.

[19] D. Mugnai, N. S. Papageorgiou; *Wang's multiplicity result for superlinear (p, q)-equations without the Ambrosetti-Rabinowitz condition*, Trans. AMS, **366** (2014), 4919-4937.

[20] N. S. Papageorgiou, V. D. Radulescu; *Quatitative phenomena for some clasees of quasilinear elliptic equations with multiple resonance*, Appl. Math. Optim., **69** (2014), 393-430.

[21] N. S. Papageorgiou, V. D. Radulescu; *Solutions with sign information for nonlinear nonhomogeneous elliptic equations*, Topol, Metho. Nonlin. Anal., **45** (2015), 575-600.

[22] N. S. Papageorgiou, V. D. Radulescu; *Multiplicity theorem for nonlinear nonhomogeneous Robin problem*, Revista Mat. Iberoamericana, **33** (2017), 251-289.

[23] N. S. Papageorgiou, V. D. Radulescu; *Multiple solutions with precise sign for parametric Robin problems*, J. Differential Equations, **256** (2014), 2449–2479.

[24] N. S. Papageorgiou, V. D. Radulescu; *Nonlinear nonhomogeneous Robin problems with superlinear reaction term*, Adv. Nonlin. Studies, **16** (2016), 737-764.

[25] N. S. Papageorgiou, P. Winkert, *Resonant $(p, 2)$-equations with concave terms*, Appl. Anal. **94** (2015), 341-359.

[26] P. Pucci, J. Serrin; *The maximum Principle*, Birkhäuser, Basel, 2007.

[27] M. Sun; *Multiplicity of solutions for a class of quasilinear elliptic equations with resonance*, J. Math. Anal. Appl., **386** (2012), 661-668.

[28] M. Sun, M. Zhang, J. Su; *Critical groups of zero and multiple solutions for a quasilinear elliptic equation*, J. Math. Anal. Appl., **428** (2015), 712.

[29] M. Tanaka; *Uniqueness of a positive solution and existence of a sign-changing solution for (p, q)-Laplacian equation*, J. Nonlin. Funct. Anal., No. **14** (2014), pp.15.

[30] Z. Yang,-H. Yin; *A class of (p, q)-Laplacian type equation with concave-convex nonlinearities in bounded domain*, J. Math. Anal. Appl., **382** (2011), 843-855.

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