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GLOBAL INTERVAL BIFURCATION AND CONVEX SOLUTIONS FOR THE MONGE-AMPÈRE EQUATIONS

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Abstract. In this article, we establish the global bifurcation result from the trivial solutions axis or from infinity for the Monge-Ampère equations with non-differentiable nonlinearity. By applying the above result, we shall determine the interval of γ, in which there exist radial solutions for the following Monge-Ampère equation

$$\det(D^2u) = \gamma a(x)F(-u), \quad \text{in } B,$$
$$u(x) = 0, \quad \text{on } \partial B,$$

where $D^2u = (\partial^2 u/\partial x_i \partial x_j)$ is the Hessian matrix of u, where B is the unit open ball of $\mathbb{R}^N$, γ is a positive parameter. $a \in C(\overline{B}, [0, +\infty))$ is a radially symmetric weighted function and $a(r) := a(|x|) \not\equiv 0$ on any subinterval of $[0, 1]$ and the nonlinear term $F \in C(\mathbb{R}^+)$ but is not necessarily differentiable at the origin and infinity. We use global interval bifurcation techniques to prove our main results.

1. Introduction

The Monge-Ampère equations are a type of important fully nonlinear elliptic equations [12, 27]. Historically, the study of Monge-Ampère equations is motivated by Minkowski problem [3, 22] and Weyl problem [14, 21]. Existence and regularity results of the Monge-Ampère equations can be found in [4, 5, 13, 15, 19, 22] and the reference therein.

We first consider the real Monge-Ampère equation

$$\begin{gathered} \det(D^2u) = \lambda a(x)(-u)^N + g(x, -u, \lambda), \quad \text{in } B, \\ u(x) = 0, \quad \text{on } \partial B, \end{gathered} \tag{1.1}$$

where $D^2u = (\partial^2 u/\partial x_i \partial x_j)$ is the Hessian matrix of u, B is the unit ball of $\mathbb{R}^N$, $a(x)$ is a weighted function, λ is a positive parameter and $g \in C(\overline{B} \times (\mathbb{R}^+)^2)$. In recent years, the study of the problem (1.1) have attracted the attention of many specialists in differential equations because of their interesting applications. For example, Caffarelli et al. [2] and Gilbarg et al. [12] have investigated problem (1.1) in general domains of $\mathbb{R}^N$. Kutev [17] investigated the existence of strictly convex radial solutions of problem (1.1) with $a \equiv 1$ and $g = 0$. Delano [11] treated the existence of convex radial solutions of problem (1.1).

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In [16, 28], the authors have showed that problem (1.1) can be reduced to the boundary value problem

$$
\begin{gathered}
((u')^N)' = \lambda a(r)(-u)^N + g(r,-u,\lambda), \quad r \in (0,1), \\
u'(0) = u(1) = 0.
\end{gathered} \tag{1.2}
$$

By a solution of problem (1.2) we understand that it is a function which belongs to $C^2[0,1]$ and satisfies (1.2). It has been known that any negative solution of problem (1.2) is strictly convex in (0,1). Hu [16] and Wang [28] (for $a(-u)^N = f(-u), g = 0$) also established several criteria for the existence, multiplicity and nonexistence of strictly convex solutions for problem (1.2) by using fixed index theorem. Lions [18] have proved the existence of the first eigenvalue λ_1 of problem (1.1) with $\lambda a(x) = \lambda^N, g = 0$ via constructive proof. However, there is no information on the bifurcation points and the optimal intervals for the parameter λ so as to ensure existence of single or multiple convex solutions.

Recently, Dai et al. [6, 8] established a global bifurcation result for the Monge-Ampère equations (1.1) with $\lambda a(x)(-u)^N + g(x,-u,\lambda)$ equal $\lambda^N a(x)((-u)^N + g(-u))$ and $\lambda^N((-u)^N + g(-u))$ respectively. Furthermore, the radial solutions of the above problem in [6, 8] of (1.1) is equivalent to the solutions of the corresponding problem (1.2), respectively. Where $g : [0,+\infty) \to [0,+\infty)$ satisfies $\lim_{s\to 0^+} g(s)/s^N = 0$ and

- (H0) $a(x) \in C(\overline{B})$ is radially symmetric and $a(r) \geq 0$, $a(r) \not\equiv 0$ on any subinterval of $[0,1]$, where $r = |x|$ with $x \in \overline{B}$.

However, among the above papers, the nonlinearities are differentiable at the origin. Berestycki [1] established an important global bifurcation theorem from intervals for a class of second-order problems involving non-differentiable nonlinearity. In [26], the result in [1] has been improved partially by Schmitt and Smith. Recently, Ma and Dai [20] improved Berestycki's result in [1] to show a unilateral global bifurcation result for a class of second-order problems involving non-differentiable nonlinearity. Later, Dai [7] considered similar problems with [20], and Dai and Ma [9, 10] considered interval bifurcation problem for a class of p-Laplacian problems involving non-differentiable nonlinearity.

Motivated by above papers, we shall establish a global bifurcation result from interval for the following Monge-Ampère equations with nondifferentiable nonlinearity

$$
\begin{gathered}
\det(D^2u) = \lambda a(x)(-u)^N + F(x,-u,\lambda), \quad \text{in } B, \\
u(x) = 0, \quad \text{on } \partial B,
\end{gathered} \tag{1.3}
$$

where λ is a positive parameter, B is the unit open ball of $\mathbb{R}^N$, and the nonlinear term F has the form $F = f + g$, where $f, g \in C(\overline{B} \times (\mathbb{R}^+)^2)$ are radially symmetric with respect to x, where $\mathbb{R}^+ = [0,\infty)$.

It is clear that the radial solutions of (1.3) is equivalent to the solutions of the problem

$$
\begin{gathered}
((u')^N)' = \lambda N r^{N-1} a(r)(-u)^N + N r^{N-1} F(r,-u,\lambda), \quad r \in (0,1), \\
u'(0) = u(1) = 0,
\end{gathered} \tag{1.4}
$$

where a satisfies (H0), and $F = f + g$, where $f, g \in C([0,1] \times (\mathbb{R}^+)^2)$, satisfying the following conditions:

- (H1) $|\frac{f(r,s,\lambda)}{s^N}| \le M_1$, for any $r \in (0,1)$, $0 < s \le 1$ and $\lambda \in \mathbb{R}$, where M_1 is a positive constant.
- (H2) $g(r,s,\lambda) = o(s^N)$ near $s = 0$ uniformly for $r \in (0,1)$ and λ on bounded sets.
- (H3) $|\frac{f(r,s,\lambda)}{s^N}| \le M_2$ for any $r \in [0,1]$, $C < s$ and $\lambda \in \mathbb{R}^+$, where M_2 is a positive constant, C is a positive constant which is large enough.
- (H4) $g(r,s,\lambda) = o(s^N)$ near $s = +\infty$ uniformly for $r \in [0,1]$ and on bounded λ intervals.

Under the above assumptions, we shall establish the global bifurcation results for the problem (1.4), which bifurcates from the trivial solutions axis or from infinity, respectively.

Following the above theory (see Theorem 3.2, 3.5), we shall investigate the existence of radial solutions for the problem

$$\begin{gathered} \det(D^2u) = \gamma a(x)F(-u), \quad \text{in } B, \\ u(x) = 0, \quad \text{on } \partial B, \end{gathered} \tag{1.5}$$

where γ is a positive parameter, the nonlinear term $F \in C(\mathbb{R}^+)$ but is not necessarily differentiable at the origin and infinity.

It is clear that the radial solutions of (1.5) is equivalent to the solutions of the problem

$$\begin{gathered} ((u')^N)' = \gamma N r^{N-1} a(r)F(-u), \quad r \in (0,1), \\ u'(0) = u(1) = 0, \end{gathered} \tag{1.6}$$

where a satisfying condition (H0). We assume that the nonlinear term F has the form $F = f + g$, where f and g are continuous functions on $\mathbb{R}^+$, satisfying the following conditions:

- (H5) $|\frac{f(s)}{s^N}| \le M_3$, $0 < s \le 1$, where M_3 is a positive constant.
- (H6) $|\frac{f(s)}{s^N}| \le M_4$, $C < s$ for some positive constant C large enough, where M_4 is a positive constant.
- (H7) $g : [0,\infty) \to [0,\infty)$ is a continuous function and $g(s) > 0$ for $s \in (0,\infty)$.
- (H8) There exist $g_0, g_\infty \in (0,\infty)$ such that

$$g_0 = \lim_{s\to 0^+} \frac{g(s)}{s^N}, g_\infty = \lim_{s\to+\infty} \frac{g(s)}{s^N}.$$

For the abstract global bifurcation theory, we refer the reader to [6, 10, 20, 24, 25] and the references therein.

Clearly, F is not necessarily differentiable at the origin because of the influence of the term f. So the bifurcation theory of [6, 8] can not be applied directly to obtain our results. Fortunately, using the global interval bifurcation (see Theorems 3.2 and 3.5), we can obtain some results of the existence of negative solutions which extend the corresponding results in [6, 8].

The rest of this article is arranged as follows. In Section 2, we given some Preliminaries. In Section 3, we establish the global bifurcation results which bifurcates from the trivial solutions axis or from infinity for problem (1.4), respectively. In Section 4, on the basis of the interval bifurcation result (see Theorems 3.2, 3.5), we give the intervals for the parameter γ which ensure existence of single or multiple strictly convex solutions for problem (1.6) under the under the assumptions of (H5)–(H8).

2. Preliminaries

Following [6, Section 3-4], we first consider the problem

$$\begin{gathered}((-v')^N)' = h(r), \quad r \in (0,1), \\ v'(0) = v(1) = 0.\end{gathered} \tag{2.1}$$

Let us define the operator $G_N(h) : E \to E$ by

$$G_N(h) = \int_t^1 \Big(\int_0^s (h(\tau))^{\frac{1}{N}} d\tau \Big) ds. \tag{2.2}$$

For a given $h \in Y$, $G_N(h) : Y \to E$ is s completely continuous and (2.2) is equivalent to (2.1).

With a simple transformation $v = -u$, problem (1.2) can be equivalently written as (see [6, Section 4-p.10]).

$$\begin{gathered}((-v')^N)' = \lambda N r^{N-1} a(r) v^N + N r^{N-1} g(r, v, \lambda), \quad r \in (0,1), \\ u'(0) = u(1) = 0,\end{gathered} \tag{2.3}$$

where $g \in C([0,1] \times (\mathbb{R}^+)^2)$ satisfies

$$\lim_{s \to 0^+} \frac{g(r, s, \lambda)}{s^N} = 0 \tag{2.4}$$

uniformly for $r \in (0,1)$ and λ on bounded sets.

Define the Nemitskii operator $H : \mathbb{R} \times E \to Y$ by

$$H(\mu, v)(r) := \mu N r^{N-1} a(r) v^N + N r^{N-1} g(r, v, \mu).$$

Then it is clear that H is continuous (compact) operator and problem (2.3) can be equivalently written as

$$v = G_N \circ H(\mu, v) := F(\mu, v).$$

Here F is completely continuous in $\mathbb{R} \times E \to E$ and $F(\mu, 0) = 0$ for all $\mu \in \mathbb{R}$.

Let $Y = C[0,1]$ with the norm $\|u\|_\infty = \max_{r \in [0,1]} |u(r)|$. Let $E := \{u(r) \in C^1(0,1) | u'(0) = u(1) = 0\}$ with the usual norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$. Let $P^+ = \{u \in E : u(r) > 0, r \in (0,1)\}$. Set $K^+ = \mathbb{R} \times P^+$ under the product topology.

Now, we consider the eigenvalue problem

$$\begin{gathered}((-v')^N)' = \lambda N r^{N-1} a(r) v^N, \quad r \in (0,1), \\ u'(0) = u(1) = 0,\end{gathered} \tag{2.5}$$

By [6, (4.2) Section 4-p.11], the same proof as in [18, Theorem 1.1], we can show that problem (2.5) possesses the first eigenvalue λ_1 which is positive, simple , the unique and the corresponding eigenfunctions are positive in $(0,1)$ and concave on $[0,1]$.

By Rabinowitz [24], using the same method to prove [6, Theorems 4.1 and 4.2] with obvious changes, we may get the following global bifurcation result.

Lemma 2.1 ([6, Theorem 4.2])**.** *) Assume that* (2.4) *and* (H0) *hold. Then* $(\lambda_1, 0)$ *is the unique bifurcation point of problem* (2.3) *and there exists an unbounded continuum* $C \subseteq (K^+ \cup \{(\lambda_1, 0)\})$ *of solutions to problem* (2.3) *emanating from* $(\lambda_1, 0)$.

By [6], to prove our main results, we need the following Sturm type comparison result.

Lemma 2.2 ([6, Lemma 4.6])**.** *Let $b_i(r) \in C(0,1)$, $i = 1,2$ such that $b_2(r) \ge b_1(r)$ for $r \in (0,1)$ and the inequality is strict on some subset of positive measure in $(0,1)$. Also let v_1, v_2 be solutions of the differential equations*

$$\begin{gathered} ((-v')^N)' = b_i(r)v^N, \quad r \in (0,1),\ i = 1,2, \\ v'(0) = v(1) = 0, \end{gathered} \tag{2.6}$$

respectively. If $v_1 \neq 0$ in $(0,1)$, then v_2 has at least one zero in $(0,1)$.

Next, we give an important lemma which will be used later.

Lemma 2.3. *Let I be an interval and if y and z are functions such that y, z, $\varphi_N(y')$ and $\varphi_N(z')$ are differentiable on I and $y(t) > 0, z(t) > 0, y'(t) < 0, z'(t) < 0$ for $t \in I$. Then we have the identity*

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{y}{\varphi_N(z)}[\varphi_N(y)\varphi_N(-z') - \varphi_N(z)\varphi_N(-y')]\Big\} \\ &= \frac{y}{\varphi_N(z)}[\varphi_N(y)L_N[z] - \varphi_N(z)L_N[y]] \\ &\quad + \Big[(-y')^{N+1} + N\big(\frac{-yz'}{z}\big)^{N+1} + (N+1)y^N y'\big(\frac{-z'}{z}\big)^N\Big], \end{aligned} \tag{2.7}$$

where $\varphi_N(s) = s^N$, $L_N[y] = (\varphi_N(-y'))'$.

Proof. The left-hand side of (2.7) equals

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{y^{N+1}(-z')^N}{z^N} - y(-y')^N\Big\} \\ &= \frac{[(N+1)y^N y'(-z')^N + y^{N+1}((-z')^N)']z^N - y^{N+1}(-z')^N N z^{N-1}z'}{z^{2N}} \\ &\quad - y'(-y')^N - y((-y')^N)' \\ &= \frac{y}{\varphi_N(z)}\,[\varphi_N(y)L_N[z] - \varphi_N(z)L_N[y]] \\ &\quad + \Big[(-y')^{N+1} + N\big(\frac{-yz'}{z}\big)^{N+1} + (N+1)y^N y'\big(\frac{-z'}{z}\big)^N\Big]. \end{aligned}$$

□

Remark 2.4. In (2.7), by Young's inequality, we obtain

$$\Big[(-y')^{N+1} + N\big(\frac{-yz'}{z}\big)^{N+1} + (N+1)y^N y'\big(\frac{-z'}{z}\big)^N\Big] \ge 0 \tag{2.8}$$

and the equality holds if and only if $\operatorname{sgn} y = \operatorname{sgn} z$ and $|\frac{y'}{y}|^{N+1} = |\frac{z'}{z}|^{N+1}$.

We use Young's inequality

$$AB \le \frac{A^\alpha}{\alpha} + \frac{B^\beta}{\beta}, \tag{2.9}$$

where $A, B \in \mathbb{R}^+, \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let $\alpha = N+1$, $\beta = \frac{N+1}{N}$, $A = -(N+1)^{\frac{1}{(N+1)}}y'$, $B = (N+1)^{\frac{N}{(N+1)}}y^N\big(\frac{-z'}{z}\big)^N$ in (2.9). We obtain that inequality (2.8) holds.

By Lemma 2.3 and Remark 2.4, we have the following result.

Lemma 2.5. *In* (2.7), *we have*

$$\int_0^1 \frac{y}{\varphi_N(z)}\left(\varphi_N(y)L[z]-\varphi_N(z)L[y]\right)\,dr \le 0.$$

Proof. By Lemma 2.3, it follows that

$$\begin{aligned}
&\int_0^1 \Big\{\frac{y}{\varphi_N(z)}\left[\varphi_N(y)\varphi_N(-z')-\varphi_N(z)\varphi_N(-y')\right]\Big\}'\,dr\\
&=\int_0^1 \Big[(-y')^{N+1}+N\big(\frac{-yz'}{z}\big)^{N+1}+(N+1)y^N y'\big(\frac{-z'}{z}\big)^N\Big]\,dr \qquad (2.10)\\
&\quad+\int_0^1 \frac{y}{\varphi_N(z)}\left[\varphi_N(y)L_N[z]-\varphi_N(z)L_N[y]\right]\,dr.
\end{aligned}$$

As in the proof of [6, Lemma 4.5], we can show that the left-hand side of (2.10) equals 0. By Remark 2.4, We have the result. ☐

3. Global bifurcation from an interval

With a simple transformation $v=-u$, problem (1.4) can be equivalently written as

$$\begin{gathered}
((-v')^N)'=\lambda N r^{N-1}a(r)v^N+Nr^{N-1}F(r,v,\lambda),\quad r\in(0,1),\\
v'(0)=v(1)=0.
\end{gathered}\qquad (3.1)$$

Let $\mathscr{S}$ denote the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions (λ,v) of (3.1) with $v\in P^+$. By an argument similar to that of [6, Lemma 4.1] with obvious changes, we can show that the following existence and uniqueness theorem is valid for problem (3.1).

Lemma 3.1 ([6, Lemma 4.1])**.** *If* (λ,v) *is a solution of* (3.1) *under the assumptions of* (H0)–(H2) *and* v *has a double zero, then* $u\equiv 0$.

Our first main result for (3.1) is the following theorem.

Theorem 3.2. *Let* (H0)–(H2) *hold. Let* $d_1=M_1/a_0$, *where* $a_0=\min_{r\in[0,1]}a(r)$, *and let* $I_1^0=[\lambda_1-d_1,\lambda_1+d_1]$. *The component* $\mathscr{C}$ *of* $\mathscr{S}\cup(I_1^0\times\{0\})$, *containing* $I_1^0\times\{0\}$ *is unbounded and lies in* $K^+\cup(I_1^0\times\{0\})$.

For the proof we introduce the auxiliary approximate problem

$$\begin{gathered}
((-v')^N)=\lambda N r^{N-1}a(r)v^N+Nr^{N-1}f(r,v|v|^{\epsilon},\lambda)+Nr^{N-1}g(r,v,\lambda),\\
r\in(0,1),\\
v'(0)=v(1)=0.
\end{gathered}\qquad (3.2)$$

The next lemma will play a key role in the proof of Theorem 3.2.

Lemma 3.3. *Let* ϵ_n, $0<\epsilon_n<1$, *be a sequence converging to* 0. *If there exists a sequence* $(\lambda_n,v_n)\in K^+$ *such that* (λ_n,v_n) *is a nontrivial solution of problem* (3.2) *corresponding to* $\epsilon=\epsilon_n$, *and* (λ_n,v_n) *converges to* $(\lambda,0)$ *in* $\mathbb{R}\times E$, *then* $\lambda\in I_1^0$.

Proof. Let $w_n = v_n/\|v_n\|$, then w_n satisfies

$$\begin{gathered}((-w_n')^N)' = \lambda_n N r^{N-1} a(r) w_n^N + \frac{N r^{N-1} f(r, u_n|u_n|^\epsilon, \lambda_n)}{\|u_n\|^N} \\ + \frac{N r^{N-1} g(r, u_n, \lambda_n)}{\|u_n\|^N}, \quad r \in (0,1), \\ w_n'(0) = w_n(1) = 0,\end{gathered} \tag{3.3}$$

Let

$$\overline{g}(r, v, \lambda) = \max_{0\le |s| \le v} \big|g(r, s, \lambda)\big| \qquad \text{for all } r \in (0,1) \text{ and } \lambda \text{ on bounded sets,}$$

then $\overline{g}$ is nondecreasing with respect to v and

$$\lim_{v\to 0^+} \frac{\overline{g}(r, v, \lambda)}{v^N} = 0 \tag{3.4}$$

uniformly for $r \in (0,1)$ and λ on bounded sets. Further it follows from (3.4) that

$$\frac{|g(r, v, \lambda)|}{\|v\|^N} \le \frac{\overline{g}(r, |v|, \lambda)}{\|v\|^N} \le \frac{\overline{g}(r, \|v\|_\infty, \lambda)}{\|v\|^N} \le \frac{\overline{g}(r, \|v\|, \lambda)}{\|v\|^N} \to 0 \tag{3.5}$$

as $\|v\| \to 0$, uniformly for $r \in (0,1)$ and λ on bounded sets.

Clearly, (H1) implies

$$\begin{aligned}\frac{|f(r, v_n|v_n|^{\epsilon_n}, \lambda_n)|}{\|v_n\|^N} &= \frac{|f(t, v_n|v_n|^{\epsilon_n}, \lambda_n)|}{v_n^N |v_n|^{N\epsilon_n}} \cdot \frac{v_n^N |v_n|^{N\epsilon_n}}{\|v_n\|^N} \\ &\le M_1 \cdot |v_n|^{N\epsilon_n} \to M_1\end{aligned} \tag{3.6}$$

for all $r \in (0,1)$.

Note that $\|w_n\| = 1$ implies $\|w_n\|_\infty \le 1$. Using this fact with (3.5) and (3.6), we have $\lambda_n N r^{N-1} a(r) w_n^N + N r^{N-1} f(r, v_n|v_n|^{\epsilon_n}, \lambda_n)/\|v_n\|^N + N r^{N-1} g(r, v_n, \lambda_n)/\|v_n\|^N$ is bounded in E for n large enough. The compactness of G_N implies that w_n is convergence in E. Without loss of generality, we may assume that $w_n \to w$ in E with $\|w\| = 1$. Clearly, we have $w \in \overline{P^+}$.

We claim that $w \in P^+$. On the contrary, suppose that $w \in \partial P^+$, by Lemma 3.1, then $w \equiv 0$, which is a contradiction with $\|w\| = 1$.

Now, we deduce the boundedness of λ. Let $\psi \in P^+$ be an eigenfunction of problem (2.5) corresponding to λ_1. We know that w_n satisfies

$$\begin{aligned}((-w_n')^N)' = \lambda_n N r^{N-1} a(r) w_n^N &+ N r^{N-1} f(r, v_n|v_n|^{\epsilon_n}, \lambda_n)/\|v_n\|^N \\ &+ N r^{N-1} g(r, v_n, \lambda_n)/\|v_n\|^N,\end{aligned}$$

$r \in (0,1)$, $w_n'(0) = w_n(1) = 0$ and ψ satisfies $((-\psi')^N)' = \lambda_1 N r^{N-1} a(r) \psi^N, r \in (0,1), \psi'(0) = \psi(1) = 0$.

By Lemma 2.5, it follows that

$$\begin{aligned}&\int_0^1 \frac{w_n}{\varphi_N(\psi)} \left(\varphi_N(w_n) L[\psi] - \varphi_N(\psi) L[w_n]\right) dr \\ &= \int_0^1 \Big[(\lambda_1 - \lambda_n) a(r) - \frac{f(r, v_n|v_n|^{\epsilon_n}, \lambda_n)}{\|v_n\|^N w_n^N} - \frac{g(r, v_n, \lambda_n)}{\|v_n\|^N w_n^N}\Big] r^{N-1} N w_n^{N+1} dr \le 0.\end{aligned} \tag{3.7}$$

Similarly, we can also show that

$$\begin{aligned}\int_0^1 \Big[(\lambda_n-\lambda_1)a(r)+\frac{f(r,v_n|v_n|^{\epsilon_n},\lambda_n)}{\|v_n\|^N w_n^N}\\ +\frac{g(r,v_n,\lambda_n)}{\|v_n\|^N w_n^N}\Big] r^{N-1}N\psi^{N+1}\,dr\le 0.\end{aligned}\tag{3.8}$$

If $\lambda\le\lambda_1$, considering (3.7), (H1) and (H2), we have

$$\begin{aligned}&\int_0^1(\lambda_1-\lambda)a(r)Nr^{N-1}w^{N+1}\,dr\\ &\le \lim_{n\to\infty}\int_0^1\frac{f(r,v_n|v_n|^{\epsilon_n},\lambda_n)}{\|v_n\|^N w_n^N}Nr^{N-1}w_n^{N+1}\,dr\\ &\le \lim_{n\to\infty}\int_0^1\frac{f(r,v_n|v_n|^{\epsilon_n},\lambda_n)}{v_n^N|v_n|^{N\epsilon_n}}|v_n|^{N\epsilon_n}Nr^{N-1}w_n^{N+1}\,dr\\ &\le\int_0^1 M_1Nr^{N-1}w^{N+1}\,dr.\end{aligned}$$

Hence, we obtain

$$\int_0^1(\lambda_1-\lambda)a_0Nr^{N-1}w^{N+1}\,dr\le\int_0^1 M_1Nr^{N-1}w^{N+1}\,dr,$$

which implies $\lambda\ge\lambda_1-d_1$.

If $\lambda\ge\lambda_1$, considering (3.8), (H1) and (H2), we have

$$\begin{aligned}&\int_0^1(\lambda-\lambda_1)a(r)Nr^{N-1}\psi^{N+1}\,dr\\ &\le \lim_{n\to\infty}\int_0^1\frac{-f(r,v_n|v_n|^{\epsilon_n},\lambda_n)}{\|v_n\|^N w_n^N}Nr^{N-1}\psi^{N+1}\,dr\\ &\le \lim_{n\to\infty}\int_0^1\frac{-f(r,v_n|v_n|^{\epsilon_n},\lambda_n)}{v_n^N|v_n|^{N\epsilon_n}}|v_n|^{N\epsilon_n}Nr^{N-1}\psi^{N+1}\,dr\\ &\le\int_0^1 M_1Nr^{N-1}\psi^{N+1}\,dr.\end{aligned}$$

Hence, we obtain

$$\int_0^1(\lambda-\lambda_1)a_0Nr^{N-1}\psi^{N+1}\le\int_0^1 M_1Nr^{N-1}\psi^{N+1}\,dr,$$

which implies $\lambda\le\lambda_1-d_1$. Therefore, we have that $\lambda\in I_1^0$. □

Proof of Theorem 3.2. We divide the rest of proofs into two steps.

Step 1. We show that $\mathscr{C}\subset(K^+\cup(I_1^0\times\{0\}))$. For any $(\lambda,v)\in\mathscr{C}$, there are two possibilities: (i) $v\in P^+$, or (ii) $v\in\partial P^+$. It is obvious that $(\lambda,v)\in K^+$ in the case of (i). While, the case (ii) implies that v has at least one double zero in $[0,1]$. From Lemma 3.1 it follows that $v\equiv 0$. Hence, there exists a sequence $(\lambda_n,v_n)\in K^+$ such that (λ_n,v_n) is a solution of problem (3.2) corresponding to $\epsilon=0$, and (λ_n,v_n) converges to $(\lambda,0)$ in $\mathbb{R}\times E$. By Lemma 3.3, we have $\lambda\in I_1^0$, i.e., $(\lambda,v)\in(I_1^0\times\{0\})$ in the case of (ii). Hence, $\mathscr{C}\subset(K^+\cup(I_1^0\times\{0\}))$.

Step 2. We prove that $\mathscr{C}$ is unbounded. Suppose on the contrary that $\mathscr{C}$ is bounded. Using the similar method to prove [1, Theorem 1] with obvious changes, we can find a neighborhood $\mathcal{O}$ of $\mathscr{C}$ such that $\partial\mathcal{O}\cap\mathscr{S}=\emptyset$.

In order to complete the proof of this theorem, we consider problem (3.2). For $\epsilon>0$, it is easy to show that nonlinear term $f(r,v|v|^{\epsilon},\lambda)+g(r,v,\lambda)$ satisfies the condition (H2). Let

$$\mathscr{S}_{\epsilon}=\overline{\{(\lambda,v):(\lambda,v)\text{ satisfies (3.2) and } v\not\equiv 0\}}^{\mathbb{R}\times E}.$$

By Lemma 2.1, there exists an unbounded continuum $\mathscr{C}_{\epsilon}$ of $\mathscr{S}_{\epsilon}$ bifurcating from $(\lambda_1,0)$ such that

$$\mathscr{C}_{\epsilon}\subset(K^{+}\cup\{(\lambda_1,0)\}).$$

So there exists $(\lambda_{\epsilon},v_{\epsilon})\in\mathscr{C}_{\epsilon}\cap\partial\mathcal{O}$ for all $\epsilon>0$. Since $\mathcal{O}$ is bounded in K^{+}, Equation (3.2) shows that $(\lambda_{\epsilon},v_{\epsilon})$ is bounded in $\mathbb{R}\times C^{2}$ independently of ϵ. By the compactness of G_N, one can find a sequence $\epsilon_n\to 0$ such that $(\lambda_{\epsilon_n},v_{\epsilon_n})$ converges to a solution (λ,v) of (3.2). So $v\in\overline{P^{+}}$. If $v\in\partial P^{+}$, then from Lemma 3.1 follows that $v\equiv 0$. By Lemma 3.3, $\lambda\in I_1^0$, which contradicts the definition of $\mathcal{O}$. On the other hand, if $v\in P^{+}$, then $(\lambda,v)\in\mathscr{S}\cap\partial\mathcal{O}$ which contradicts $\mathscr{S}\cap\partial\mathcal{O}=\emptyset$. □

From Theorem 3.2 and its proof, we can easily get a corollary.

Corollary 3.4. *There exists a unbounded sub-continua $\mathscr{D}$ of solutions of* (3.1) *in $\mathbb{R}\times E$, bifurcating from $I_1^0\times\{0\}$, and $\mathscr{D}\subset(K^{+}\cup(I_1^0\times\{0\}))$.*

We add the points $\{(\lambda,\infty)|\lambda\in\mathbb{R}\}$ to space $\mathbb{R}\times E$. Let $\mathscr{T}$ denote the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions (λ,v) of (3.1) under conditions (H3) and (H4) with $v\in P^{+}$. Let S_N denote the spectral set of problem (2.5). Let $\overline{I}_{\infty}=[\overline{\lambda}-d_2,\overline{\lambda}+d_2]$, where $\overline{\lambda}\in S_N\setminus\{\lambda_1\}$ and d_2 be given in Theorem 3.5.

By Rabinowitz [25], our second main result for (3.1) is the following theorem.

Theorem 3.5. *Let* (H0), (H3), (H4) *hold. Also let $d_2=M_2/a_0$, where $a_0=\min_{t\in[0,1]}a(t)$, and let $I_1^{\infty}=[\lambda_1-d_2,\lambda_1+d_2]$. There exists a connected component $\mathscr{D}$ of $\mathscr{T}\cup(I_1^{\infty}\times\{\infty\})$, containing $I_1^{\infty}\times\{\infty\}$. Moreover, if $\Lambda\subset\mathbb{R}$ is an interval such that $\Lambda\cap(\cup_{\overline{\lambda}\in S_N\setminus\{\lambda_1\}}(\overline{I}_{\infty}\cup I_1^{\infty}))=I_1^{\infty}$ and $\mathscr{M}$ is a neighborhood of $I_1^{\infty}\times\{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

1. *$\mathscr{D}-\mathscr{M}$ is bounded in $\mathbb{R}\times E$ in which case $\mathscr{D}-\mathscr{M}$ meets $\mathscr{R}=\{(\lambda,0)|\lambda\in\mathbb{R}\}$ or*
2. *$\mathscr{D}-\mathscr{M}$ is unbounded.*

If (2) occurs and $\mathscr{D}-\mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D}-\mathscr{M}$ meets $\overline{I}_{\infty}$. Moreover, there exists a neighborhood $\mathscr{N}\subset\mathscr{M}$ of $I_1^{\infty}\times\{\infty\}$ such that $(\mathscr{D}\cap\mathscr{N})\subset(K^{+}\cup(I_1^{\infty}\times\{\infty\}))$.

Proof. The idea is similar to the one in the proof of [25, Theorem 1.6], but we give a rough sketch of the proof for readers convenience. If $(\lambda,v)\in\mathscr{T}$ with $\|v\|\neq 0$, dividing (3.1) by $\|v\|^2$ and setting $w=v/\|v\|^2$ yield

$$\begin{gathered}((-w')^{N})'=\lambda Nr^{N-1}a(r)w^{N}+\frac{Nr^{N-1}F(r,v,\lambda)}{\|v\|^{2N}},\quad r\in(0,1),\\ w'(0)=w(1)=0,\end{gathered}\tag{3.9}$$

Define

$$\widetilde{f}(r,w,\lambda)=\begin{cases}\|w\|^{2N}f(r,\frac{w}{\|w\|^2},\lambda), & \text{if } w\neq 0,\\ 0, & \text{if } w=0;\end{cases}$$

$$\widetilde{g}(r,w,\lambda)=\begin{cases}\|w\|^{2N}g(r,\frac{w}{\|w\|^2},\lambda), & \text{if } w\neq 0,\\ 0, & \text{if } w=0.\end{cases}$$

Clearly, (3.9) is equivalent to

$$\begin{gathered}((-w')^N)'=\lambda Nr^{N-1}a(r)w^N+Nr^{N-1}\widetilde{f}(r,w,\lambda)\\ +Nr^{N-1}\widetilde{g}(r,w,\lambda),\quad r\in(0,1),\\ w'(0)=w(1)=0.\end{gathered}\tag{3.10}$$

It is obvious that $(\lambda,0)$ is always the solution of (3.10). By simple computation, we can show that assumptions (H3) and (H4) imply

- (H9) $|\frac{\widetilde{f}(r,w,\lambda)}{w^N}|\leq M_2$ for all $r\in[0,1]$, $0<w\leq 1$ and $\lambda\in\mathbb{R}^+$, where M_2 is a positive constant.
- (H10) $\widetilde{g}(r,w,\lambda)=o(w^N)$ near $w=0$, uniformly for all $r\in(0,1)$ and on bounded λ intervals.

Now applying Theorem 3.2 to (3.10), we have a connected component $\mathscr{C}$ of $\mathscr{S}\cup(I_1^0\times\{0\})$. Under the inversion $w\to\frac{w}{\|w\|^2}=v$, $\mathscr{C}\to\mathscr{D}$ satisfying problem (3.1). Clearly, $\mathscr{D}$ satisfy the conclusions of this theorem.

Finally, We show that there exists a neighborhood $\mathscr{N}\subset\mathscr{M}$ of $I_1^\infty\times\{\infty\}$ such that $(\mathscr{D}\cap\mathscr{N})\subset(K^+\cup(I_1^\infty\times\{\infty\}))$. Clearly, the inversion $w\to w/\|w\|^2=v$ turns $I_1^0\times\{0\}$ into $I_1^\infty\times\{\infty\}$. Let $\mathscr{O}$ be a bounded neighborhood of $I_1^0\times\{0\}$. Then $(\mathscr{C}\cap(\mathscr{O}\setminus(I_1^0\times\{0\})))\subset K^+$, containing $I_1^0\times\{0\}$ is unbounded and lies in $K^+\cup(I_1^\infty\times\{\infty\})$. While, by the inversion $w\to w/\|w\|^2=v,\mathscr{C}\cap(\mathscr{O}\setminus(I_1^0\times\{0\}))$ is translated to a deleted neighborhood $\mathscr{N}^0$ of $I_1^\infty\times\{\infty\}$. It is obvious that $(\lambda,w)\in\mathscr{C}\cap(\mathscr{O}\setminus(I_1^0\times\{0\}))$ implies that there exists a constant c_0 such that $0<\|w\|\leq c_0$. It follows that $(\lambda,v)\in\mathscr{N}^0$ implies that $1/c_0\leq\|v\|<+\infty$. It follows that $(\mathscr{D}\cap\mathscr{N})\subset(K^+\cup(I_1^\infty\times\{\infty\}))$ by taking $\mathscr{N}:=\mathscr{N}^0\cup(I_1^\infty\times\{\infty\})$. □

4. Applications

In this section, we shall investigate the existence and multiplicity of convex solutions of problem (1.6). With a simple transformation $v=-u$, problem (1.6) can be written as

$$\begin{gathered}((-v')^N)'=\gamma Nr^{N-1}a(r)F(v),r\in(0,1),\\ v'(0)=v(1)=0,\end{gathered}\tag{4.1}$$

The main results of this section are the following theorems.

Theorem 4.1. *Let* $a_0=\min_{r\in[0,1]}a(r)$, $a^0=\max_{r\in[0,1]}a(r)$. *Let* (H0), (H5)–(H8) *hold. If* $g_0a_0>M_3a^0$ *and* $g_\infty a_0>M_4a^0$, *either*

$$\frac{\lambda_1}{g_0-M_3a^0/a_0}<\gamma<\frac{\lambda_1}{g_\infty+M_4a^0/a_0}\tag{4.2}$$

or

$$\frac{\lambda_1}{g_\infty-M_4a^0/a_0}<\gamma<\frac{\lambda_1}{g_0+M_3a^0/a_0},\tag{4.3}$$

then problem (1.6) *has at least one solution* u *such that it is negative, and strictly convex in* $(0,1)$.

Theorem 4.2. *Let* (H0)–(A2), (H7), (H8) *hold. If* $g_0a_0 > M_3a^0$ *but* $g_\infty a_0 \le M_4a^0$, *for*

$$\frac{\lambda_1}{g_0 - M_3a^0/a_0} < \gamma < \frac{\lambda_1}{g_\infty + M_4a^0/a_0},$$

then problem (1.6) *has at least one solution* u *such that it is negative, strictly convex in* $(0,1)$.

Theorem 4.3. *Let* (H0), (H5)–(H8) *hold. If* $g_0a_0 \le M_3a^0$ *but* $g_\infty a_0 > M_4a^0$, *for*

$$\frac{\lambda_1}{g_\infty - M_4a^0/a_0} < \gamma < \frac{\lambda_1}{g_0 + M_3a^0/a_0},$$

then problem (1.6) *has at least one solution* u *such that it is negative, strictly convex in* $(0,1)$.

Remark 4.4. From (H8), we can see that there exists a positive constant M_5 such that $g(s)/s^N \ge M_5$ for all $s \ne 0$.

Remark 4.5. Note that if $M_i \equiv 0$ $(i = 3,4)$, then the cases of Theorems 4.2 and 4.3 do not occur and Theorem 4.1 is equivalent to [8, Theorem 4.1] or [6, Theorem 5.1]. In this sense, Theorem 4.1 is also a generalization of [8, Theorem 4.1] or [6, Theorem 5.1].

To prove Theorem 4.1, we need the following results.

Lemma 4.6. *Let* (H0), (H5)–(H8) *hold. If* $g_0a_0 > M_3a^0$ *and* $g_\infty a_0 > M_4a^0$, *either* (4.2) *or* (4.3) *hold, then*

- (i) *There is a distinct unbounded component* $\mathscr{D}_0$ *of* $\mathscr{S} \cup (I_1^0 \times \{0\})$, *containing* $I_1^0 \times \{0\}$ *and lying in* $K^+ \cup (I_1^0 \times \{0\})$.
- (ii) *There is a distinct unbounded component* $\mathscr{D}_\infty$ *of* $\mathscr{T} \cup (I_1^\infty \times \{\infty\})$, *which satisfy the alternates of Theorem 3.5. Moreover, there exists a neighborhood* $\mathscr{N} \subset \mathscr{M}$ *of* $I_1^\infty \times \{\infty\}$ *such that* $(\mathscr{D}_\infty \cap \mathscr{N}) \subset (K^+ \cup (I_1^\infty \times \{\infty\}))$.

Proof. Firstly, we study the bifurcation phenomena for the following eigenvalue problem

$$\begin{gathered}((-v')^N)' = \lambda\gamma N r^{N-1}a(r)g(v) + \gamma N r^{N-1}a(r)f(v), \quad r \in (0,1),\\ v'(0) = v(1) = 0,\end{gathered} \tag{4.4}$$

where $\lambda > 0$ is a parameter.

(i) Clearly, condition (H5) implies

$$\Big|\frac{a(r)f(s)}{s^N}\Big| \le M_3a^0, \quad 0 < s \le 1. \tag{4.5}$$

Let $\zeta \in C(\mathbb{R}\setminus\mathbb{R}^-, \mathbb{R}\setminus\mathbb{R}^-)$ be such that

$$g(s) = g_0s^N + \zeta(s) \tag{4.6}$$

with $\lim_{s\to0^+}\zeta(s)/s^N = 0$. Let $\overline{\zeta}(v) = \max_{0\le|s|\le v}|\zeta(s)|$, then $\overline{\zeta}(v)$ is nondecreasing and

$$\lim_{s\to0^+}\frac{\overline{\zeta}(s)}{s^N} = 0. \tag{4.7}$$

Further it follows from (4.7) that

$$\frac{|\zeta(v(r))|}{\|v\|^N} \le \frac{\overline{\zeta}(|v(r)|)}{\|v\|^N} \le \frac{\overline{\zeta}(\|r\|_\infty)}{\|v\|^N} \le \frac{\overline{\zeta}(\|r\|)}{\|v\|^N} \quad \text{as } \|v\| \to 0. \tag{4.8}$$

Hence, (4.5), (4.6) and (4.8) imply that conditions (H1) and (H2) hold. Moreover, let $d_3 = M_3 a^0/g_0 a_0$ and $I_1^0 = [\frac{\lambda_1}{\gamma g_0} - d_3, \frac{\lambda_1}{\gamma g_0} + d_3]$. By Theorem 3.2, the result follows.

(ii) Clearly, condition (H6) implies

$$|\frac{a(t)f(s)}{s^N}| \le M_4 a^0, \quad C < s. \tag{4.9}$$

Let $\xi \in C(\mathbb{R} \setminus \mathbb{R}^-, \mathbb{R} \setminus \mathbb{R}^-)$ be such that

$$g(s) = g_\infty s^N + \xi(s) \tag{4.10}$$

with $\lim_{s\to+\infty} \xi(s)/s^N = 0$. Let $\overline{v} = \max_{0\le|s|\le v} |\xi(s)|$, then $\overline{\xi}$ is nondecreasing and

$$\lim_{s\to+\infty} \frac{\overline{\xi}(s)}{s^N} = 0. \tag{4.11}$$

Further it follows from (4.11) that

$$\frac{|\xi(v(r))|}{\|v\|^N} \le \frac{\overline{\xi}(|v(r)|)}{\|v\|^N} \le \frac{\overline{\xi}(\|v\|_\infty)}{\|v\|^N} \le \frac{\overline{\xi}(\|v\|)}{\|v\|^N} \quad \text{as } \|v\| \to +\infty. \tag{4.12}$$

Hence, (4.9), (4.10) and (4.12) imply that conditions (H3) and (H4) hold. Moreover, let $d_4 = M_4 a^0/g_\infty a_0$ and $I_1^\infty = [\frac{\lambda_1}{\gamma g_\infty} - d_4, \frac{\lambda_1}{\gamma g_\infty} + d_4]$. Using Theorem 3.5, the result follows. □

Lemma 4.7. *If $\mathscr{D}_0$ and $\mathscr{D}_\infty$ are defined as in Lemma 4.6, then $\mathscr{D}_0 = \mathscr{D}_\infty$.*

Proof. (i) We shall prove that (1) of Theorem 3.5 occurs. It suffices to show that $\mathscr{D}_\infty$ meets some point $(\lambda_*, 0)$ of $\mathscr{R}$. In fact, if this occurs, we can show that $\lambda_* \in I_1^0$. Suppose on the contrary that $\lambda_* \notin I_1^0$, hence $\lambda_* \in \overline{I}_0$. So $(\mathscr{D}_\infty \cap \mathscr{N}) \subset \mathscr{D}_\infty \subset \overline{\mathscr{D}}_0 \subset ((\mathbb{R} \times \overline{P}) \cup (\overline{I}_0 \times \{0\}))$, noting $(\mathscr{D}_\infty \cap \mathscr{N}) \cap (\mathbb{R} \times \{0\}) = \emptyset$, which contradicts $(\mathscr{D}_\infty \cap \mathscr{N}) \subset (K^+ \cup (I_\infty \times \{\infty\}))$. Where $\overline{\mathscr{T}}_0$ denote the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions (λ, v) of (3.1) under conditions (H5), (H7) and (H8) with $v \in \overline{P}$, where $\overline{P} = \{v | (\overline{\lambda}, v) \in (S_N \setminus \{\lambda_1\}) \times E\}$ and $\overline{I}_0 = [\frac{\overline{\lambda}}{\gamma g_0} - d_3, \frac{\overline{\lambda}}{\gamma g_0} + d_3]$. $\overline{\mathscr{D}}_0$ is a connected component of $\overline{\mathscr{T}}_0 \cup (\overline{I}_0 \times \{0\})$, containing $\overline{I}_0 \times \{0\}$. Hence $\lambda_* \in I_1^0$, it follows that $\mathscr{D}_0 = \mathscr{D}_\infty$.

(ii) We shall show that (2) of Theorem 3.5 does not occur. Suppose on the contrary that (2) of Theorem 3.5 occurs, then we shall deduce a contradiction. We divide the proof into two steps.

Step 1. We show that $\mathscr{D}_\infty - \mathscr{M}$ has a bounded projection on $\mathbb{R}$. Firstly, we show that $\mathscr{D}_\infty \subset K^+$. If $(\mathscr{D}_\infty - (\mathscr{D}_\infty \cap \mathscr{N})) \not\subset K^+$, then there exists $(\mu, v) \in (\mathscr{D}_\infty - (\mathscr{D}_\infty \cap \mathscr{N})) \cap (\mathbb{R} \times \partial P^+)$. Since $v \in \partial P^+$, by Lemma 3.1, $v \equiv 0$, i.e. (1) of Theorem 3.5 occurs, which is a contradiction.

On the contrary, we suppose that $(\mu_n, v_n) \in \mathscr{D}_\infty - \mathscr{M}$ such that

$$\lim_{n\to\infty} \mu_n = +\infty.$$

It follows that

$$\begin{gathered} ((-v_n')^N)' = \mu_n \gamma N r^{N-1} a(r) g(v_n) + \gamma N r^{N-1} a(r) f(v_n), \quad r \in (0,1), \\ v_n'(0) = v_n(1) = 0, \end{gathered} \tag{4.13}$$

In view of Remark 4.4, (H5) and (H6),we have that

$$\lim_{n\to\infty}(\mu_n Nr^{N-1}a(r)\frac{g(v_n)}{v_n^N}+Nr^{N-1}a(r)\frac{f(v_n)}{v_n^N})=+\infty$$

for any $r\in(0,1)$. By the Sturm Comparison Lemma 2.2, we get that v_n has more change its sign in $(0,1)$ for n large enough, and this contradicts the fact that $(\mu_n,v_n)\in\mathscr{D}_\infty^+-\mathscr{M}$.

Step 2. We show that the case of $\mathscr{D}_\infty-\mathscr{M}$ meeting $\overline{I}_\infty\times\{\infty\}$ is impossible. Assume on the contrary that $\mathscr{D}_\infty-\mathscr{M}$ meets $\overline{I}_\infty\times\{\infty\}$. So there exists a neighborhood $\widetilde{\mathscr{N}}\subset\widetilde{\mathscr{M}}$ of $\overline{I}_\infty\times\{\infty\}$ such that $(\mathscr{D}_\infty-\mathscr{M})\cap(\widetilde{\mathscr{N}}\setminus(\overline{I}_\infty\times\{\infty\}))\subset(\mathbb{R}\times\overline{P})$, where $\widetilde{\mathscr{M}}$ is a neighborhood of $\overline{I}_\infty\times\{\infty\}$ which satisfies the assumptions of Theorem 3.5, which contradicts $\mathscr{D}_\infty\subset\overline{P}$, where $\overline{P}=\{v|(\overline{\lambda},v)\in(S_N\setminus\{\lambda_1\})\times E\}$ and $\overline{I}_\infty=[\frac{\overline{\lambda}}{\gamma g_\infty}-d_4,\frac{\overline{\lambda}}{\gamma g_\infty}+d_4]$. □

Proof of Theorem 4.1. It suffices to prove that problem (4.1) has at least one solution v such that it is positive, strictly concave in $(0,1)$.

By Lemmas 4.6 and 4.7, we write $\mathscr{D}=\mathscr{D}_0=\mathscr{D}_\infty$ for simplicity. It is clear that any solution of (4.4) of the form $(1,v)$ yields a solution v of (4.1). In this case, $d_3<1,d_4<1$. By (4.2), we obtain

$$\frac{\lambda_1}{\gamma g_0}+d_3<1,\quad \frac{\lambda_1}{\gamma g_\infty}-d_4>1. \tag{4.14}$$

By (4.3), we have

$$\frac{\lambda_1}{\gamma g_\infty}+d_4<1,\quad \frac{\lambda_1}{\gamma g_0}-d_3>1 \tag{4.15}$$

From $I_1^0=[\frac{\lambda_1}{\gamma g_0}-d_3,\frac{\lambda_1}{\gamma g_0}+d_3]$ and $I_1^\infty=[\frac{\lambda_1}{\gamma g_\infty}-d_4,\frac{\lambda_1}{\gamma g_\infty}+d_4]$, it follows that the subsets $I_1^0\times E$ and $I_1^\infty\times E$ of $\mathbb{R}\times E$ can be separated by the hyperplane $\{1\}\times E$. Furthermore, we have $\mathscr{D}$ cross the hyperplane $\{1\}\times E$ in $\mathbb{R}\times E$. □

Proof of Theorem 4.2. The proof is similar to that of Theorem 4.1. In the case, $d_3<1,d_4\ge 1$, which follows that (4.14) hold. By $d_4\ge 1$, it follows that (4.15) is impossible. □

Proof of Theorem 4.3. The proof is similar to that of Theorem 4.1. In the case, $d_3\ge 1,d_4<1$, which follows that (4.15) hold. By $d_3\ge 1$, it follows that (4.14) is impossible. □

Remark 4.8. Note that if $d_3\ge 1,d_4\ge 1$, (4.14) and (4.15) are impossible, it follows that the subsets $I_1^0\times E$ and $I_1^\infty\times E$ of $\mathbb{R}\times E$ can not be separated by the hyperplane $\{1\}\times E$. In this case, we cannot give a suitable interval of γ in which there exist positive solutions for (4.1). It would be interesting to have more information about this case.

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