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INFINITELY MANY SOLUTIONS FOR A SEMILINEAR PROBLEM ON EXTERIOR DOMAINS WITH NONLINEAR BOUNDARY CONDITION

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Abstract. In this article we prove the existence of an infinite number of radial solutions to $\Delta u + K(r)f(u) = 0$ with a nonlinear boundary condition on the exterior of the ball of radius R centered at the origin in $\mathbb{R}^N$ such that $\lim_{r\to\infty} u(r) = 0$ with any given number of zeros where $f : \mathbb{R} \to \mathbb{R}$ is odd and there exists a $\beta > 0$ with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞) with f superlinear for large u, and $K(r) \sim r^{-\alpha}$ with $0 < \alpha < 2(N - 1)$.

1. Introduction

In this article we study radial solutions to

$$\Delta u + K(|x|)f(u) = 0 \quad \text{for } R < |x| < \infty, \tag{1.1}$$

$$\frac{\partial u}{\partial \eta} + c(u)u = 0 \quad \text{when } |x| = R \text{ and } \lim_{|x|\to\infty} u(x) = 0, \tag{1.2}$$

where $u : \mathbb{R}^N \to \mathbb{R}$ with $N \ge 2$, $R > 0$, $c : [0, \infty) \to (0, \infty)$ is continuous, $\frac{\partial}{\partial \eta}$ is the outward normal derivative, f is odd and locally Lipschitz. We assume:

- (H1) $f'(0) < 0$, there exists $\beta > 0$ such that $f(u) < 0$ on $(0, \beta)$, $f(u) > 0$ on (β, ∞).
- (H2) $f(u) = |u|^{p-1}u + g(u)$ where $p >$ and $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$.
- (H3) Denoting $F(u) \equiv \int_0^u f(t)\,dt$ we assume there exists γ with $0 < \beta < \gamma$ such that $F < 0$ on $(0, \gamma)$ and $F > 0$ on (γ, ∞).
- (H4) K and K' are continuous on $[R, \infty)$ with $K(r) > 0$, $2(N - 1) + \frac{rK'}{K} > 0$ and there exists $\alpha \in (0, 2(N - 1))$ such that $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$.
- (H5) There exist positive d_1, d_2 such that $d_1 r^{-\alpha} \le K(r) \le d_2 r^{-\alpha}$ for $r \ge R$.

Note that (H4) implies $r^{2(N-1)}K$ is increasing. Our main result reads as follows.

Theorem 1.1. *Assume* (H1)–(H5)*,* $N \ge 2$*, and* $0 < \alpha < 2(N - 1)$*. Then for each nonnegative integer* n *there exists a radial solution,* u_n*, of* (1.1)-(1.2) *such that* u_n *has exactly* n *zeros on* (R, ∞)*.*

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The radial solutions of (1.1) on $\mathbb{R}^N$ and $K(r) \equiv 1$ have been well-studied. These include [1, 2, 3, 10, 12, 14]. Recently there has been an interest in studying these problems on $\mathbb{R}^N \backslash B_R(0)$. These include [5, 6, 7, 11, 13]. In [6] positive solutions of a similar problem were studied for $N < \alpha < 2(N-1)$. There the authors use the mountain pass lemma to prove existence of positive solutions. Here we use a scaling argument as in [9, 12] to prove the existence of infinitely many solutions.

2. Preliminaries

Since we are interested in radial solutions of (1.1)-(1.2), we denote $r = |x|$ and write $u(x) = u(|x|)$ where u satisfies

$$u'' + \frac{N-1}{r}u' + K(r)f(u) = 0 \quad \text{for } R < r < \infty, \tag{2.1}$$

$$u(R) = b > 0, \quad u'(R) = bc(b) > 0. \tag{2.2}$$

We will occasionally write $u(r,b)$ to emphasize the dependence of the solution on b. By the standard existence-uniqueness theorem [4] there is a unique solution of (2.1)-(2.2) on $[R, R+\epsilon)$ for some $\epsilon > 0$. We next consider

$$E(r) = \frac{1}{2}\frac{u'^2}{K(r)} + F(u). \tag{2.3}$$

It is straightforward using (2.1) and (H4) to show that

$$E'(r) = -\frac{u'^2}{2rK}[2(N-1) + \frac{rK'}{K}] \le 0. \tag{2.4}$$

Thus E is non-increasing. Therefore

$$\frac{1}{2}\frac{u'^2}{K(r)} + F(u) = E(r) \le E(R) = \frac{1}{2}\frac{b^2c^2(b)}{K(R)} + F(b) \text{ for } r \ge R. \tag{2.5}$$

Since F is bounded from below by (H3), it follows from (2.5) that u and u' are uniformly bounded wherever they are defined from which it follows that the solution of (2.1)-(2.2) is defined on $[R, \infty)$.

Lemma 2.1. *Assume* (H1)–(H5) *and* $N \ge 2$. *Let* $u(r,b)$ *be the solution of* (2.1)-(2.2) *and suppose* $0 < \alpha < 2(N-1)$. *If* $b > 0$ *and* b *is sufficiently small then* $u(r,b) > 0$ *for all* $r > R$.

Proof. We proceed as in [9]. Since $u(R,b) = b > 0$ and $u'(R,b) = bc(b) > 0$ we see that $u(r,b) > 0$ on $(R, R+\delta)$ for some $\delta > 0$. If $u'(r,b) > 0$ for all $r \ge R$ then $u(r,b) > 0$ for all $r > R$ and so we are done in this case.

If u is not increasing for all $r > R$ then there exists a local maximum at some M_b with $M_b > R$ and $u'(r,b) > 0$ on $[R, M_b)$. If $u(M_b, b) < \gamma$ then $E(r) \le E(M_b) < 0$ for $r > M_b$ since E is non-increasing. It follows then that $u(r,b)$ cannot be zero for any $r > M_b$ for if there were such a $z_b > M_b$ then $0 \le \frac{1}{2}\frac{u'^2(z_b)}{K(z_b)} = E(z_b) \le E(M_b) < 0$ which is impossible. Also, since $u'(r,b) > 0$ on $[R, M_b)$ it follows then that $u(r,b) > 0$ on (R, ∞) if $u(M_b, b) < \gamma$. So if $u(r,b)$ has a local maximum at M_b with $u(M_b, b) < \gamma$ then we are done in this case as well.

In addition, if $E(R) = \frac{1}{2}\frac{b^2c^2(b)}{K(R)} + F(b) \le 0$ then $E(t) < 0$ for $t > R$ and a similar argument shows that $u(r,b)$ cannot be zero for $t > R$.

So for the rest of this proof we assume that $u(r,b)$ has a local maximum at M_b, $u(M_b, b) \ge \gamma$, $u'(r,b) > 0$ on $[R, M_b)$, and $E(R) = \frac{1}{2}\frac{b^2c^2(b)}{K(R)} + F(b) > 0$ for all sufficiently small $b > 0$. From this it then follows from (H1) and (H3) that there exists r_b and r_{b_1} with $R < r_b < r_{b_1} < M_b$ such that $u(r_b, b) = \beta$ and $u(r_{b_1}, b) = \frac{\beta+\gamma}{2}$.

From (H5) and from rewriting (2.5) we see that

$$\frac{|u'|}{\sqrt{\frac{b^2c^2(b)}{K(R)} + 2F(b) - 2F(u)}} \le \sqrt{K} \le \sqrt{d_2} r^{-\frac{\alpha}{2}} \quad \text{for } r \ge R. \tag{2.6}$$

On $[R, r_b]$ we have $u' > 0$ and so integrating (2.6) on $[R, r_b]$ when $\alpha \neq 2$ gives

$$\begin{aligned} \int_0^\beta \frac{dt}{\sqrt{\frac{b^2c^2(b)}{K(R)} + 2F(b) - 2F(t)}} &= \int_R^{r_b} \frac{u'(r)\,dr}{\sqrt{\frac{b^2c^2(b)}{K(R)} + 2F(b) - 2F(u(r))}} \\ &\le \frac{\sqrt{d_2}}{\frac{\alpha}{2} - 1} \left(R^{1-\frac{\alpha}{2}} - {r_b}^{1-\frac{\alpha}{2}} \right). \end{aligned} \tag{2.7}$$

In the case $\alpha = 2$ the right-hand side of (2.7) is replaced by:

$$\sqrt{d_2} \ln(r_b/R). \tag{2.8}$$

As $b \to 0^+$ the left-hand side of (2.7) goes to $+\infty$ since by (H1) and the definition of F,

$$\sqrt{\frac{b^2c^2(b)}{K(R)} + 2F(b) - 2F(t)} \le \sqrt{\frac{b^2c^2(b)}{K(R)} + 2F(b) + 2|f'(0)|t^2}$$

for small positive t thus

$$\int_0^\epsilon \frac{dt}{\sqrt{\frac{b^2c^2(b)}{K(R)} + 2F(b) - 2F(t)}} \ge \int_0^\epsilon \frac{dt}{\sqrt{\frac{b^2c^2(b)}{K(R)} + 2F(b) + 2|f'(0)|t^2}} \to \infty \tag{2.9}$$

as $b \to 0^+$.

Combining (2.7) and (2.9) we see that if $2 < \alpha < 2(N-1)$ then

$$\frac{\sqrt{d_2}}{\frac{\alpha}{2} - 1} R^{1-\frac{\alpha}{2}} \ge \frac{\sqrt{d_2}}{\frac{\alpha}{2} - 1} \left(R^{1-\frac{\alpha}{2}} - {r_b}^{1-\frac{\alpha}{2}} \right) \to \infty \quad \text{as } b \to 0^+$$

which is impossible since R is fixed. Thus it follows that $u(M_b, b) < \gamma$ if $b > 0$ is sufficiently small and as indicated earlier in this lemma it then follows that $u(r, b) > 0$ for $r > R$ if $b > 0$ is sufficiently small.

For the case $0 < \alpha \le 2$ a lengthier argument is required and the details are carried out in [9]. There it is shown that $E(r_{b_1}) < 0$ for sufficiently small $b > 0$ and therefore $u(r, b)$ cannot be zero for any $z_b > r_{b_1}$ as indicated earlier in this lemma. This completes the proof. □

Lemma 2.2. *Assume* (H1)–(H5) *and* $N \ge 2$. *Let* $u(r, b)$ *be the solution of* (2.1)-(2.2) *and suppose* $0 < \alpha < 2(N-1)$. *Given a positive integer* n *then* $u(r, b)$ *has at least* n *zeros on* $(0, \infty)$ *if* $b > 0$ *is chosen sufficiently large.*

Proof. Let $v(r) = u(r + R)$. Then v satisfies,

$$v''(r) + \frac{N-1}{R+r} v'(r) + K(R+r) f(v) = 0, \tag{2.10}$$

$$v(0) = b, v'(0) = bc(b). \tag{2.11}$$

Now let

$$v_\lambda(r) = \lambda^{-\frac{2}{p-1}} v\left(\frac{r}{\lambda}\right) \quad \text{for } \lambda > 0. \tag{2.12}$$

Then

$$v'_\lambda(r) = \lambda^{-\frac{2}{p-1}-1} v'\left(\frac{r}{\lambda}\right),$$
$$v''_\lambda(r) = \lambda^{-\frac{2}{p-1}-2} v''\left(\frac{r}{\lambda}\right).$$

Thus

$$v''\left(\frac{r}{\lambda}\right) + \frac{N-1}{R+\frac{r}{\lambda}} v'\left(\frac{r}{\lambda}\right) + K(\frac{r}{\lambda}+R) f\left(v\left(\frac{r}{\lambda}\right)\right) = 0$$

and so it then follows that

$$v''_\lambda + \frac{N-1}{(R\lambda + r)} v'_\lambda + \frac{K(\frac{r}{\lambda}+R)}{\lambda^{\frac{2p}{p-1}}} f(\lambda^{\frac{2}{p-1}} v_\lambda) = 0. \tag{2.13}$$

From (H2) we have $f(u) = |u|^{p-1}u + g(u)$ and $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$ so rewriting (2.13) gives

$$v''_\lambda + \frac{N-1}{(R\lambda + r)} v'_\lambda + \frac{K(\frac{r}{\lambda}+R)}{\lambda^{\frac{2p}{p-1}}} \big[\lambda^{\frac{2p}{p-1}} |v_\lambda|^{p-1} v_\lambda + g(\lambda^{\frac{2}{p-1}} v_\lambda)\big] = 0. \tag{2.14}$$

Thus

$$v''_\lambda + \frac{N-1}{(R\lambda + r)} v'_\lambda + K(\frac{r}{\lambda}+R)\big[|v_\lambda|^{p-1} v_\lambda + \frac{g(\lambda^{\frac{2}{p-1}} v_\lambda)}{\lambda^{\frac{2p}{p-1}}}\big] = 0, \tag{2.15}$$

$$v_\lambda(0) = \lambda^{\frac{-2}{p-1}} b, \tag{2.16}$$

$$v'_\lambda(0) = \lambda^{\frac{-2}{p-1}-1} b c(b) = \lambda^{-\frac{p+1}{p-1}} b c(b). \tag{2.17}$$

Now let

$$E_\lambda(r) = \frac{v'^2_\lambda}{2K(\frac{r}{\lambda}+R)} + \frac{F(\lambda^{\frac{2}{p-1}} v_\lambda)}{\lambda^{\frac{2p}{p-1}}}. \tag{2.18}$$

A straightforward calculation using (H4) and (2.13) gives

$$E'_\lambda(r) = -\frac{v'^2_\lambda}{2(\frac{r}{\lambda}+R)K(\frac{r}{\lambda}+R)} \Big[\frac{(\frac{r}{\lambda}+R)K'(\frac{r}{\lambda}+R)}{K(\frac{r}{\lambda}+R)} + 2(N-1)\Big] \le 0$$

for $0 < \alpha < 2(N-1)$. Thus for $r \ge 0$,

$$\frac{v'^2_\lambda}{2K(\frac{r}{\lambda}+R)} + \frac{F(v_\lambda)}{\lambda^{\frac{2p}{p-1}}} = E_\lambda(r) \le E_\lambda(0) = \frac{b^2 c^2(b)}{2\lambda^{\frac{2(p+1)}{p-1}} K(R)} + \frac{F(\lambda^{\frac{-2}{p-1}} b)}{\lambda^{\frac{2p}{p-1}}}. \tag{2.19}$$

We now divide the rest of the proof into two cases.

Case 1: $\frac{c(b)}{b^{\frac{p-1}{2}}} \le C_0$ for all sufficiently large b for some constant C_0. In this case we choose $b = \lambda^{\frac{2}{p-1}}$ so that (2.16)-(2.17) become $v_\lambda(0) = 1$ and

$$v'_\lambda(0) = \lambda^{\frac{-2}{p-1}-1} b c(b) = \frac{c(b)}{\lambda} = \frac{c(b)}{b^{\frac{p-1}{2}}} \le C_0.$$

Next using (H2)-(H3) it follows that

$$F(u) = \frac{|u|^{p+1}}{p+1} + G(u) \tag{2.20}$$

where $G(u) = \int_0^u g(s)\,ds$ and from L'Hôpital's rule it follows that $\frac{G(u)}{|u|^{p+1}} \to 0$ as $u \to \infty$.

So from (2.12), (2.19)-(2.20) and since $b = \lambda^{\frac{2}{p-1}}$ we obtain

$$\frac{v_\lambda'^2}{2K(\frac{r}{\lambda}+R)} + \frac{|v_\lambda|^{p+1}}{p+1} + \frac{G(\lambda^{\frac{2}{p-1}}v_\lambda)}{\lambda^{\frac{2(p+1)}{p-1}}} \le \frac{b^2c^2(b)}{2\lambda^{\frac{2(p+1)}{p-1}}K(R)} + \frac{F(1)}{\lambda^{\frac{2p}{p-1}}} \tag{2.21}$$

$$= \frac{1}{2K(R)}\Big(\frac{c(b)}{b^{\frac{p-1}{2}}}\Big)^2 + \frac{F(1)}{\lambda^{\frac{2p}{p-1}}} \le \frac{C_0^2}{2K(R)} + \frac{F(1)}{\lambda^{\frac{2p}{p-1}}}. \tag{2.22}$$

So since $\frac{G(u)}{|u|^{p+1}} \to 0$ as $u \to \infty$ it follows that $\frac{|G(u)|}{|u|^{p+1}} \le \frac{1}{2(p+1)}$ for say $u > T$. Also, $|G(u)| \le G_0$ for $|u| \le T$ since G is continuous on the compact set $[0,T]$ and thus $|G(u)| \le \frac{1}{2(p+1)}|u|^{p+1} + G_0$ for all u. Similarly using (H2) it follows that $|g(u)| \le \frac{1}{2}|u|^p + g_0$ for all u for some constant g_0 where $|g(u)| \le g_0$ on $[0,T]$.

Therefore for $\lambda > 0$ it follows from (2.21)-(2.22) that

$$\frac{v_\lambda'^2}{2K(\frac{r}{\lambda}+R)} + \frac{|v_\lambda|^{p+1}}{2(p+1)} \le \frac{C_0^2}{2K(R)} + \frac{F(1)}{\lambda^{\frac{2p}{p-1}}} + \lambda^{\frac{-2(p+1)}{p-1}}G_0 \le \frac{C_0^2}{2K(R)} + F(1) + G_0 \text{ for } \lambda > 1.$$

It follows from this that $v_\lambda(r)$ and $v_\lambda'(r)$ are uniformly bounded on $[0,\infty)$ for large λ. It then follows that $\big(\frac{N-1}{R\lambda+r}\big)v_\lambda'$ is uniformly bounded on $[0,\infty)$ and also $K(\frac{r}{\lambda} + R)\big[|v_\lambda|^{p-1}v_\lambda + \frac{g(\lambda^{\frac{2}{p-1}}v_\lambda)}{\lambda^{\frac{2p}{p-1}}}\big]$ is uniformly bounded on $[0,\infty)$. Then from (2.15) we see that v_λ'' is uniformly bounded on $[0,\infty)$ for large λ. Therefore by the Arzela-Ascoli theorem it follows that there is a subsequence (still denoted v_λ) and continuous functions v_0 and v_0' such that $v_\lambda \to v_0$ and $v_\lambda' \to v_0'$ uniformly on compact subsets of $[0,\infty)$ to a solution of

$$\begin{gathered} v_0'' + K(R)v_0^p = 0, \\ v_0(0) = 1, \quad v_0'(0) = d_0 = \lim_{b\to\infty} \frac{c(b)}{b^{\frac{p-1}{2}}} \le C_0. \end{gathered} \tag{2.23}$$

It is now straightforward to show that v_0 has infinitely many zeros on $[0,\infty)$. Thus v_λ has at least n zeros for sufficiently large λ and so $u(r,b)$ has at least n zeros for sufficiently large b. This concludes the proof in Case 1.

Case 2: $\frac{c(b)}{b^{\frac{p-1}{2}}} \to \infty$ for some subsequence as $b \to \infty$. Then for these b we let

$$\lambda = (bc(b))^{\frac{p-1}{p+1}} \quad \text{that is } bc(b) = \lambda^{\frac{p+1}{p-1}}. \tag{2.24}$$

From (2.17) and (2.24) we see that

$$v_\lambda(0) = \lambda^{-\frac{2}{p-1}}b = \Big[\frac{b^{\frac{p-1}{2}}}{c(b)}\Big]^{\frac{2}{p+1}} \to 0 \quad \text{as } b \to \infty \text{ and } v_\lambda'(0) = 1.$$

As in case (1) we can show there exist continuous functions v_0 and v_0' such that for some subsequence $v_\lambda \to v_0$ and $v_\lambda' \to v_0'$ as $\lambda \to \infty$ uniformly on compact subsets of $[0,\infty)$ and v_0 is a solution of

$$\begin{gathered} v_0'' + K(R)v_0^p = 0, \\ v_0(0) = 0, \quad v_0'(0) = 1. \end{gathered} \tag{2.25}$$

And again it is easy to show that v_0 has infinitely many zeros on $[0,\infty)$. Thus it follows that $v_\lambda(r)$ and hence $u(r,b)$ has at least n zeros on $[0,\infty)$ when b is sufficiently large. This completes the proof. □

3. Proof of the main theorem

Proof. We proceed as we did in [9]. It follows from Lemma 2.1 that

$$\{b > 0 : u(r, b) > 0 \text{ on } (R, \infty)\}$$

is nonempty and from Lemma 2.2 it follows that this set is bounded from above. Hence we set

$$b_0 = \sup\{b | u(r, b) > 0 \text{ on } (R, \infty)\}.$$

We next show that $u(r, b_0) > 0$ on (R, ∞). This follows because if there is a $z > R$ such that $u(z, b_0) = 0$ then $u'(z, b_0) < 0$ (by uniqueness of solutions of initial value problems) and so $u(r, b_0)$ becomes negative for r slightly larger than z. By continuity with respect to initial conditions it follows that $u(r, b)$ becomes negative for b slightly smaller than b_0 contradicting the definition of b_0. Thus $u(r, b_0) > 0$ on (R, ∞). Next it follows by the definition of b_0 that if $b > b_0$ then $u(r, b)$ must have a zero, z_b, where $z_b > R$. We now show that $z_b \to \infty$ as $b \to b_0^+$. If not then the z_b are uniformly bounded and so a subsequence of them (still denoted z_b) converges to some $z_0 \geq R$. Then since $E' \leq 0$:

$$\frac{1}{2} \frac{u'^2(r, b)}{K(r)} + F(u(r, b)) \leq \frac{1}{2} \frac{b^2 c^2(b)}{K(R)} \quad \text{for } r \geq R \tag{3.1}$$

and since F is bounded from below (by (H3)) it follows that $u(r, b)$ and $u'(r, b)$ are uniformly bounded on $[R, \infty)$ for b near b_0. In addition it follows from (2.1) that $u''(r, b)$ is also uniformly bounded on $[R, \infty)$ for b near b_0. Then by the Arzela-Ascoli theorem a subsequence (still denoted $u(r, b)$ and $u'(r, b)$) converges uniformly to $u(r, b_0)$ and $u'(r, b_0)$ and so we obtain $u(z_0, b_0) = 0$. But we know $u(r, b_0) > 0$ for $r > R$ and so we get a contradiction. Thus $z_b \to \infty$ as $b \to b_0^+$.

We now show that $E(r, b_0) \geq 0$ on $[R, \infty)$. If not then there is an $r_0 > R$ such that $E(r_0, b_0) < 0$. By continuity $E(r_0, b) < 0$ for b slightly larger than b_0. Also for $b > b_0$ the function $u(r, b)$ has a zero, z_b, (by definition of b_0) and $E(z_b) = \frac{1}{2} \frac{u'^2(z_b, b)}{K(z_b)} \geq 0$. But E is non-increasing so $z_b < r_0$ which contradicts $z_b \to \infty$ as $b \to b_0^+$. Thus, $E(r, b_0) \geq 0$ on $[R, \infty)$.

Next either: (i) $u(r, b_0)$ has a local maximum at some $M_{b_0} > R$, or (ii) $u'(r, b_0) > 0$ for $r > R$ and since $u(r, b_0)$ is bounded by (3.1) then there is an $L > 0$ such that $u(r, b_0) \to L$ as $r \to \infty$. We show now that (ii) is not possible. Suppose therefore that (ii) occurs. We divide this into three cases.

Case 1: $0 < \alpha < N$. Multiplying (2.1) by r^{N-1} and integrating on (R, r) gives

$$-r^{N-1} u' = -R^{N-1} b_0 + \int_R^r t^{N-1} K(t) f(u) \, dt. \tag{3.2}$$

Dividing (3.2) by $r^N K \to \infty$ as $r \to \infty$ since $0 < \alpha < N$ and taking limits using L'Hôpital's rule and (H4) gives

$$-\frac{u'}{rK} = \lim_{r \to \infty} \frac{\int_R^r t^{N-1} K(t) f(u) \, dt}{r^N K} = \lim_{r \to \infty} \frac{f(u)}{N + \frac{rK'}{K}} = \frac{f(L)}{N - \alpha}. \tag{3.3}$$

Thus since $0 < \alpha < N$ and $u' > 0$, it follows that $f(L) \leq 0$ so that

$$0 < L \leq \beta < \gamma. \tag{3.4}$$

On the other hand integrating the identity

$$(r^{2(N-1}KE)' = (r^{2(N-1}K)'F$$

on (R, r) and using L'Hôpital's rule gives

$$\begin{aligned}\lim_{r\to\infty} E(r,b_0) &= \lim_{r\to\infty} \frac{1}{2}\frac{u'^2}{K} + F(u)\\ &= \lim_{r\to\infty} \frac{1}{2}\frac{R^{2(N-1)}b_0^2}{r^{2(N-1)}K} + \frac{\int_R^r (t^{2(N-1)}K)'F(u(t,b_0))\,dt}{r^{2(N-1)}K} = F(L).\end{aligned}$$

Since we showed earlier that $E(r, b_0) \geq 0$ we see then that

$$0 \leq \lim_{r\to\infty} E(r,b_0) = F(L). \tag{3.5}$$

Thus $L \geq \gamma$ which contradicts (3.4). Therefore it must be the case that $u(r, b_0)$ has a local maximum at some M_{b_0}. This completes Case 1.

Case 2: $\alpha = N$. In this case as well it follows that $f(L) \leq 0$ for suppose $f(L) > 0$. Then by (H5) the integral on the right-hand side of (3.2) grows like $f(L)\ln(r) \to \infty$ as $r \to \infty$ and thus the right-hand side of (3.2) becomes arbitrarily large but the left hand side is negative. Thus it must be that $f(L) \leq 0$ and as in Case 1 we get a contradiction.

Case 3: $N < \alpha < 2(N-1)$. For $b > b_0$ we know that there is an $z_b > R$ such that $u(z_b, b) = 0$ so there is an M_b with $R < M_b < z_b$ such that $u(r, b)$ has a local maximum at M_b. If the M_b are bounded as $b \to b_0^+$ then a subsequence of the M_b converge to some $M_{b_0} < \infty$ and then $u(r, b_0)$ has a local maximum at M_{b_0} contradicting our assumption that $u'(r, b_0) > 0$ for $r > R$. So let us assume that $M_b \to \infty$ as $b \to b_0^+$.

Since E is non-increasing, it follows that $E(r) \leq E(M_b)$ for $r \geq M_b$. Thus

$$\frac{1}{2}\frac{u'^2}{K} + F(u) \leq F(u(M_b)) \text{ for } r \geq M_b. \tag{3.6}$$

Rewriting and integrating (3.6) on $[M_b, z_b]$ (using (H5)) gives

$$\begin{aligned}0 &\leq \int_0^{u(M_b)} \frac{1}{\sqrt{2}\sqrt{F(u(M_b)) - F(t)}}\,dt\\ &= \int_{M_b}^{z_b} \frac{|u'(t)|}{\sqrt{2}\sqrt{F(u(M_b)) - F(u(t))}}\,dt\\ &\leq \int_{M_b}^{z_b} \sqrt{K}\,dt \leq \frac{\sqrt{d_2}(M_b^{1-\frac{\alpha}{2}} - z_b^{1-\frac{\alpha}{2}})}{\frac{\alpha}{2}-1}.\end{aligned} \tag{3.7}$$

Since $\alpha > N \geq 2$ and $M_b \to \infty$ as $b \to b_0^+$ (thus $z_b \to \infty$) we see that the right-hand side of (3.7) goes to 0 as $b \to b_0^+$. On the other hand, since $u(r, b) \to u(r, b_0)$ uniformly on compact subsets of $[R, \infty)$ we see then that $u(M_b) \to L$ as $b \to b_0^+$. Taking limits in (3.7) then gives:

$$\int_0^L \frac{1}{\sqrt{2}\sqrt{F(L) - F(t)}}\,dt = 0$$

which is impossible. Thus the M_b must be bounded as $b \to b_0^+$ which contradicts our assumption that $M_b \to \infty$. Thus $u(r, b_0)$ must have a local maximum M_{b_0}. This completes Case 3.

Since we know $u(r, b_0) > 0$ for $r > R$ and $u(r, b_0)$ has a local maximum M_{b_0} it follows that $u(r, b_0)$ cannot have a local minimum at m_{b_0} with $m_{b_0} > M_{b_0}$ for at such a point we would have $u(m_{b_0}, b_0) > 0$, $u'(m_{b_0}, b_0) = 0$, and $u''(m_{b_0}) \geq 0$. Thus from (2.1) we see that $f(u(m_{b_0}, b_0)) \leq 0$ which implies $0 < u(m_{b_0}, b_0) \leq \beta$. On the other hand since $E(r, b_0) \geq 0$ for all $r \geq R$ then $E(m_{b_0}, b_0) = F(u(m_{b_0}, b_0)) \geq 0$ and so $\beta \geq u(m_{b_0}, b_0) \geq \gamma > \beta$ which is impossible. Thus it must be that $u'(r, b_0) < 0$ for $r > M_{b_0}$ and hence there is an $L \geq 0$ such that $u(r, b_0) \to L$ as $r \to \infty$. Recalling (3.5) we have $E(r, b_0) \to F(L) \geq 0$ as $r \to \infty$. Thus $L = 0$ or $L \geq \gamma$.

Finally we want to show $L = 0$. There are again three cases to consider.

Case 1: $0 < \alpha < 2$. First suppose $f(L) \neq 0$. Recalling (3.3) it then follows that $\frac{u'}{rK} \to -\frac{f(L)}{N-\alpha}$. Thus for large r we have $u' \sim -\frac{f(L)}{N-\alpha} rK$ and from (H5) we have $rK \sim r^{1-\alpha}$ so

$$|u(r) - u(r_0)| \sim |\frac{f(L)}{N-\alpha}[\frac{r^{2-\alpha} - r_0^{2-\alpha}}{2-\alpha}]| \to \infty \quad \text{as } r \to \infty \text{ since } 0 < \alpha < 2$$

contradicting that u is bounded. Thus $f(L) = 0$ so $L = 0$ or $L = \beta$. But we also know from (3.5) that $F(L) \geq 0$ so $L = 0$ or $L \geq \gamma > \beta$. Thus we see that $L \neq \beta$ and so we must have $L = 0$.

Case 2: $\alpha = 2$. Suppose again $f(L) \neq 0$. This is similar to case 1 but now we have $|u(r) - u(r_0)| \sim |\frac{f(L)}{N-\alpha} \ln(r/r_0)| \to \infty$ contradicting that u is bounded. Thus $f(L) = 0$ so $L = 0$ or $L = \beta$. Since we also know $F(L) \geq 0$ so $L = 0$ or $L \geq \gamma > \beta$. So again we see that $L \neq \beta$ and thus $L = 0$.

Case 3: $2 < \alpha < 2(N-1)$. Here we let

$$u(r) = u_1(r^{2-N}).$$

This transforms (2.1) to

$$u_1''(t) + h(t) f(u_1(t)) = 0 \quad \text{for } 0 < t < R^{2-N} \tag{3.8}$$

where

$$u_1(R^{2-N}) = 0, \; u_1'(R^{2-N}) = -\frac{bR^{N-1}}{N-2} < 0$$

and where $h(t) = \frac{1}{(N-2)^2} t^{\frac{2(N-1)}{2-N}} K(t^{1/(2-N)})$. From (H4) we have $h'(t) < 0$ and we see that for small positive t we have $h(t) \sim \frac{1}{t^q}$ where $q = \frac{2(N-1)-\alpha}{N-2}$. We note also that for $2 < \alpha < 2(N-1)$ we have $0 < q < 2$. Now let

$$E_1 = \frac{1}{2} \frac{u_1'^2}{h(t)} + F(u_1).$$

Then

$$E_1' = -\frac{u_1'^2 h'}{2h^2} \geq 0$$

since $h' < 0$. We see then from (3.8) that when $u_1 > \beta$ then $u_1'' < 0$ and when $0 < u_1 < \beta$ then $u_1'' > 0$. Now for $b > b_0$ we know that $u(r, b)$ has a zero (by definition of b_0) and thus $u_1(t, b)$ has a zero, $z_{1,b}$, with $0 < z_{1,b} < R^{2-N}$ for $b > b_0$. Therefore u_1 has a local maximum at some $M_{1,b}$ and an inflection point at some $t_{1,b}$ with $0 < z_{1,b} < t_{1,b} < M_{1,b} < R^{2-N}$. Since $E_1(z_{1,b}) > 0$ and E_1 is non-decreasing then it follows that $F(u_1(M_{1,b}, b)) = E_1(M_{1,b}) \geq E_1(z_{1,b}) > 0$ and so $u_1(M_{1,b}, b) > \gamma$. Note also that $u_1(t_{1,b}, b) = \beta$. Since $u_1(t, b)$ is concave up on $(z_{1,b}, t_{1,b})$ we see then that $u_1(t, b)$ lies above the line through $(t_{1,b}, \beta)$ with slope $u_1'(t_{1,b}, b) > 0$. Thus:

$$u_1(t, b) \geq \beta + u_1'(t_{1,b}, b)(t - t_{1,b}) \quad \text{on } [z_{1,b}, t_{1,b}].$$

Evaluating this at $t = z_{1,b}$ and rewriting yields

$$t_{1,b} \geq t_{1,b} - z_{1,b} \geq \frac{\beta}{u'(t_{1,b}, b)}. \tag{3.9}$$

In addition, $E_1(t_{1,b}) \leq E_1(M_{1,b})$ so that there is a constant c_1 such that for b close to b_0,

$$\frac{1}{2}\frac{u_1'^2(t_{1,b}, b)}{h(t_{1,b})} + F(\beta) \leq F(u_1(M_{1,b}), b) \leq c_1$$

and thus

$$0 < u_1'(t_{1,b}) \leq c_2\sqrt{h(t_{1,b})} \tag{3.10}$$

where $c_2 = \sqrt{2[c_1 + |F(\beta)|]}$. Combining (3.9)-(3.10) gives

$$\beta \leq t_{1,b}u_1'(t_{1,b}, b) \leq c_2 t_{1,b}\sqrt{h(t_{1,b})} \leq c_3 t_{1,b}^{\frac{2-q}{2}} \tag{3.11}$$

for some constant c_3 for b close to b_0. Since $0 < q < 2$ we see from (3.11) that $t_{1,b}$ is bounded from below by a positive constant. It then follows by continuous dependence on initial conditions that t_{1,b_0} is also bounded from below by a positive constant. In addition, $u_1'(t_{1,b_0}, b_0) \geq 0$ and in fact $u_1'(t_{1,b_0}, b_0) > 0$ for if $u_1'(t_{1,b_0}) = 0$ then since $f(u_1(t_{1,b_0})) = f(\beta) = 0$ then $u_1''(t_{1,b_0}, b_0) = 0$ implying by uniqueness of solutions of initial value problems that $u_1(t, b_0) \equiv \beta$ contradicting that $u_1'(R^{2-N}, b_0) = -\frac{b_0 R^{N-1}}{N-2} > 0$. Thus $u_1'(t_{1,b_0}) > 0$ and this implies $u_1(t, b_0) < \beta$ for $0 < t < t_{1,b_0}$. Thus $L = \lim_{t\to 0^+} u_1(t, b_0) \leq \beta$. But recall from (3.5) that $F(L) \geq 0$ so if $L > 0$ then in fact $\beta \geq L \geq \gamma > \beta$ which is impossible so we see it must be the case that $L = 0$. Thus $\lim_{t\to 0^+} u_1(t, b_0) = 0$ and therefore $\lim_{r\to\infty} u(r, b_0) = 0$.

Next, [12, Lemma 4] states that if $u(r, b_k)$ is a solution of (2.1)-(2.2) with k zeros on $(0, \infty)$ then if b is sufficiently close to b_k then $u(r, b)$ has at most $k + 1$ zeros on $(0, \infty)$. Also [8, Lemma 2.7] proves a similar result on (R, ∞). Applying this lemma with $b = b_0$ we see that $u(r, b)$ has *at most* one zero on (R, ∞) for b close to b_0. On the other hand, by the definition of b_0 if $b > b_0$ then $u(r, b)$ has *at least* one zero on (R, ∞). Therefore: $\{b > b_0 | u(r, b)$ has *exactly one zero* on $(R, \infty)\}$ is nonempty and by Lemma 2.2 this set is bounded above. Then we let:

$$b_1 = \sup\{b > b_0 | u(r, b) \text{ has exactly one zero on } (R, \infty)\}.$$

In a similar fashion we can show that $u(r, b_1)$ has exactly one zero on (R, ∞) and $u(r, b_1) \to 0$ as $r \to \infty$. Similarly we can find $u(r, b_n)$ which has exactly n zeros on (R, ∞) and $u(r, b_n) \to 0$ as $r \to \infty$. This completes the proof. □

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