Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 111, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

COMPACTNESS OF COMMUTATORS OF TOEPLITZ OPERATORS ON q-PSEUDOCONVEX DOMAINS

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Communicated by Jerome A. Goldstein

Abstract. Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$ and let $1 \leqslant q \leqslant n-1$. If Ω is smooth, we find sufficient conditions for the $\overline{\partial}$-Neumann operator to be compact. If Ω is non-smooth and if $q \leqslant p \leqslant n-1$, we show that compactness of the $\overline{\partial}$-Neumann operator, N_{p+1}, on square integrable $(0,p+1)$-forms is equivalent to compactness of the commutators $[B_p, \overline{z}_j]$, $1 \leqslant j \leqslant n$, on square integrable $\overline{\partial}$-closed $(0,p)$-forms, where B_p is the Bergman projection on $(0,p)$-forms. Moreover, we prove that compactness of the commutator of B_p with bounded functions percolates up in the $\overline{\partial}$-complex on $\overline{\partial}$-closed forms and square integrable holomorphic forms. Furthermore, we find a characterization of compactness of the canonical solution operator, S_{p+1}, of the $\overline{\partial}$-equation restricted on $(0,p+1)$-forms with holomorphic coefficients in terms of compactness of commutators $[T_p^{z_j *}, T_p^{z_j}]$, $1 \leqslant j \leqslant n$, on $(0,p)$-forms with holomorphic coefficients, where $T_p^{z_j}$ is the Bergman-Toeplitz operator acting on $(0,p)$-forms with symbol z_j. This extends to domains which are not necessarily pseudoconvex.

1. Introduction and statement of main results

Since the pioneering work of Lars Hörmander, the $\overline{\partial}$-Neumann problem showed how linear PDE theory could revolutionize the theory of analytic functions of several complex variables and its applications. First in this article, we discuss sufficient conditions for compactness of the $\overline{\partial}$-Neumann problem. Compactness of the $\overline{\partial}$-Neumann operator N_p, $1 \leqslant p \leqslant n$, is a basic property with many useful consequences. In [11], Kohn and Nirenberg showed that N_p is globally regular if it is compact on a smooth bounded pseudoconvex domain. N_p is always compact on a smooth bounded strongly pseudoconvex domain, but on pseudoconvex domains in general not. Krantz [12] showed that N_p is not compact on a certain class of bounded Reinhardt domain. For instance on the bidisc $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$, N_p is not compact. Thus on pseudoconvex domains, conditions for compactness of N_p are very important. However, finding

2010 *Mathematics Subject Classification.* 32F10, 32W05.
Key words and phrases. $\overline{\partial}$ and $\overline{\partial}$-Neumann operator; Bergman-Toeplitz operator; q-convex domains.

Submitted August 4, 2017. Published May 10, 2018.

sufficient conditions for compactness is a significant problem. An important sufficient condition is Catlin's property (P) in [2], which generalized by McNeal [13] to property $(\tilde{P})$.

A domain Ω has property (P) if for every positive number M there exists a smooth plurisubharmonic function λ on $\overline{\Omega}$ such that $0 \leqslant \lambda \leqslant 1$ on $\overline{\Omega}$ and $i\partial\overline{\partial}\lambda \geqslant iM\partial\overline{\partial}|z|^2$ on the boundary $b\Omega$.

A domain Ω has property $(\tilde{P})$ if for every positive number M there exists $\lambda = \lambda_M \in C^2(\overline{\Omega})$ such that $|\partial\lambda|_{i\partial\overline{\partial}\lambda} \leqslant 1$ and the sum of any q eigenvalues of the matrix $\big(\frac{\partial^2\lambda}{\partial z_k \partial \overline{z}_\ell}\big)(z) \geqslant M$, for all $z \in b\Omega$.

Henkin and Iordan [7] showed that N_p, $1 \leqslant p \leqslant n$, is compact on a hyperconvex domain. On locally convex domains, property (P) and property $(\tilde{P})$ are equivalent, and equivalent to compactness of N_p, for $1 \leqslant p \leqslant n$. Moreover, the three properties are equivalent to a simple geometric condition, the absence of p-dimensional varieties from the boundary (see [4]). (Both (P) and $(\tilde{P})$ can also be formulated naturally at the level of $(0,p)$-forms; Thus $(P_p) \Rightarrow (P_{p+1})$, $(\tilde{P}_p) \Rightarrow (\tilde{P}_{p+1})$, and $(P_p) \Rightarrow (\tilde{P}_p)$ for all $1 \leqslant p \leqslant n$, see [4, 13]). In the following theorem we give sufficient conditions for compactness of N_p on a smooth bounded q-pseudoconvex domain for $q \leqslant p \leqslant n$.

Theorem 1.1. *Let Ω be a smooth bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$ and let $1 \leqslant q \leqslant n$. If Ω satisfies property (P), Thus the $\overline{\partial}$-Neumann operator, N_p, is compact for $q \leqslant p \leqslant n$. The same is true if Ω satisfies property $(\tilde{P})$.*

Second, we characterize compactness of the $\overline{\partial}$-Neumann operator on square integrable $(0,p)$-forms. According to a result of Fu and Straube [4], compactness of the restriction to forms with holomorphic coefficients implies compactness of the solution operator S_p to $\overline{\partial}$ on convex domains. Haslinger and Helffer [5] discussed compactness of S_p to $\overline{\partial}$ on weighted L^2 spaces on $\mathbb{C}^n$. On pseudoconvex domains, Haslinger [6] showed that compactness of N_1 restricted to $(0,1)$-forms with holomorphic coefficients is equivalent to compactness of the commutator $[B, \overline{M}]$ defined on $L^2(\Omega)$, where B is the Bergman projection and M is pseudodifferential operator of order 0. He also proved the equivalence of (4), (5), (6), and (7) of Theorem 1.2 when $p = 0$. Çelik and Şahutoğlu [3] proved Theorem 1.2 for any (r,p)-form on pseudoconvex domains. In the following theorem we show that these results are valid for any $(0,p)$-form on bounded q-pseudoconvex domains for $q \leqslant p \leqslant n$.

Theorem 1.2. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$ and let $1 \leqslant q \leqslant n-1$. Thus, for $q \leqslant p \leqslant n-1$, the following are equivalent:*

(1) *N_{p+1} is compact on $L^2_{0,p+1}(\Omega)$,*
(2) *S_{p+1} is compact on $L^2_{0,p+1}(\Omega)$,*
(3) *S_{p+1} is compact on $K^2_{0,p+1}(\Omega)$,*
(4) *$[B_p, \overline{z}_j]$ is compact on $K^2_{0,p}(\Omega)$ for all $1 \leqslant j \leqslant n$,*
(5) *$[B_p, \overline{z}_j]$ is compact on $L^2_{0,p}(\Omega)$ for all $1 \leqslant j \leqslant n$,*
(6) *$[B_p, \phi]$ is compact on $L^2_{0,p}(\Omega)$ for all $\phi \in C(\overline{\Omega})$,*
(7) *$[B_p, \phi]$ is compact on $K^2_{0,p}(\Omega)$ for all $\phi \in C(\overline{\Omega})$.*

Compactness of the $\overline{\partial}$-Neumann operator enjoys several important properties. Among these are the facts that compactness of N_p and those of S_p and the commutators $[B_p, \phi]$ percolate up the complex. That is, if N_p is compact, so is N_{p+1} and similarly for S_p and $[B_p, \phi]$. On a pseudoconvex domain Ω, Çelik and Şahutoğlu [3], proved that the same is true for the commutator of the Bergman projection with a function continuous on the closure of Ω. In the following theorem we show that the same is true for N_p, S_p and the commutator $[B_p, \phi]$ of the Bergman projection B_p with a function continuous on the closure of a q-pseudoconvex domain.

Theorem 1.3. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$ and let $1 \leqslant q \leqslant n-1$ and $\phi \in L^\infty(\Omega)$. Thus, for $q \leqslant p \leqslant n-1$, we have the following:*

1. *compactness of N_p implies compactness of N_{p+1},*
2. *compactness of S_p on $K^2_{0,p}(\Omega)$ implies compactness of S_{p+1} on $K^2_{0,p+1}(\Omega)$,*
3. *compactness of $[B_p, \phi]$ on $K^2_{0,p}(\Omega)$ implies compactness of $[B_{p+1}, \phi]$ on $K^2_{0,p+1}(\Omega)$,*
4. *compactness of $[B_p, \phi]$ on $H^2_{0,p}(\Omega)$ implies compactness of $[B_{p+1}, \phi]$ on $H^2_{0,p+1}(\Omega)$.*

The final purpose of this article is to characterize the connection between the $\overline{\partial}$-Neumann operator and the commutators of the Bergman-Toeplitz operators with multiplication operators. Sheu and Upmeier [16], found a characterization for compactness of N_1 on $(0,1)$-forms with holomorphic coefficients on pseudoconvex Reinhardt domains by the nonexistence of analytic discs in the boundary and also by properties of the Bergman-Toeplitz C^*-algebra. They also showed that compactness of S_1 on $(0,1)$-forms with holomorphic coefficients can be characterized by compactness of commutators of Bergman Toeplitz-operators on pseudoconvex domains. In [15], the structure of Toeplitz operators is studied for the strongly pseudoconvex domains and the more general domains of finite type. Knirsch [10] proved Theorem 1.4 on a pseudoconvex domain. In the following theorem we extend these results to the case of q-pseudoconvex domains.

Theorem 1.4. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$ and $1 \leqslant q \leqslant n-1$. Thus, for $q \leqslant p \leqslant n-1$ the following are equivalent:*

1. *N_{p+1} is compact on $H^2_{0,p+1}(\Omega)$,*
2. *S_{p+1} is compact on $H^2_{0,p+1}(\Omega)$,*
3. *$[T_p^{z_j *}, T_p^{z_j}]$ is compact on $H^2_{0,p}(\Omega)$ for all $1 \leqslant j \leqslant n$.*

2. Proof of Theorem 1.1

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $0 \leqslant p \leqslant n$. Let $L^2(\Omega)$ be the space of square integrable functions on Ω with respect to the Lebesgue measure dV in $\mathbb{C}^n$. Let

$$L^2_{0,p}(\Omega) = \Big\{\alpha = \sum_{|K|=p}{}' \alpha_K \, d\overline{z}_K : \alpha_K \in L^2(\Omega), \text{ for all } K\Big\}$$

be the space of $(0,p)$-forms with $L^2(\Omega)$-coefficients. For a real function φ in C^2, the weighted L^2_φ-norm is defined by

$$\|\alpha\|^2_\varphi = \langle \alpha, \alpha\rangle_\varphi := \|\alpha e^{-\varphi/2}\|^2 = \int_\Omega |\alpha|^2 e^{-\varphi} \, dV.$$

The $\overline{\partial}$-operator on $(0,p)$-forms is

$$\overline{\partial}\Big(\sum_{|K|=p}{}' \alpha_K \, d\overline{z}_K\Big) = \sum_{j=1}^n \sum_{|K|=p}{}' \frac{\partial \alpha_K}{\partial \overline{z}_j} \, d\overline{z}_j \wedge d\overline{z}_K.$$

with $\operatorname{dom}\overline{\partial} = \{\alpha \in L^2_{0,p}(\Omega) : \overline{\partial}\alpha \in L^2_{0,p+1}(\Omega)\}$. The derivatives are taken in the sense of distributions. Let $\overline{\partial}^*_\varphi$ be the adjoint operator of $\overline{\partial}$ from $L^2_{0,p+1}(\Omega)$ into $L^2_{0,p}(\Omega)$. Denote by $C^\infty_{0,p}(\mathbb{C}^n)$ the space of complex-valued differential forms of class C^∞ and of type $(0,p)$ on $\mathbb{C}^n$ and $C^\infty_{0,p}(\overline{\Omega}) = \{\alpha\big|_{\overline{\Omega}}; \alpha \in C^\infty_{0,p}(\mathbb{C}^n)\}$ the subspace of $C^\infty_{0,p}(\Omega)$ whose elements can be extended smoothly up to the boundary $b\Omega$.

Proposition 2.1 ([17]). *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary and (C^2) defining function ρ and let $\alpha \in C^\infty_{0,p}(\overline{\Omega}) \cap \operatorname{dom}\overline{\partial}^*_\varphi$, $1 \leqslant p \leqslant n$. Furthermore, assume that $g, \varphi \in C^2(\overline{\Omega})$ with $g \geqslant 0$, thus*

$$
\begin{aligned}
&\|\sqrt{g}\overline{\partial}\alpha\|^2_\varphi + \|\sqrt{g}\overline{\partial}^*_\varphi\alpha\|^2_\varphi \\
&= \sum_{|L|=p-1}{}' \sum_{j,k=1}^n \int_{b\Omega} g \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k} \alpha_{jL}\overline{\alpha}_{kL}\, e^{-\varphi}\, dS \\
&\quad + \sum_{|K|=p}{}' \sum_{k=1}^n \int_\Omega g\Big|\frac{\partial \alpha_K}{\partial \overline{z}_k}\Big|^2 e^{-\varphi}\, dV \\
&\quad + \sum_{|L|=p-1}{}' \sum_{j,k=1}^n \int_\Omega \Big(g\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k} - \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}\Big)\alpha_{jL}\overline{\alpha}_{kL} e^{-\varphi}\, dV \\
&\quad + 2\operatorname{Re}\langle \sum_{|L|=p-1}{}' \sum_{j=1}^n \alpha_{jL}\frac{\partial g}{\partial z_j}\, d\overline{z}_L, \overline{\partial}^*_\varphi \alpha\rangle_\varphi .
\end{aligned}
\tag{2.1}
$$

The case of $g \equiv 1$ and $\varphi \equiv 0$ is the classical Kohn-Morrey formula.

Definition 2.2. Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be an integer with $1 \leqslant q \leqslant n$. A semicontinuous function η defined in Ω is called a q-subharmonic function if for every q-dimension space L in $\mathbb{C}^n$, $\eta|_L$ is a subharmonic function on $L \cap \Omega$. This means that for every compact subset $D \Subset L \cap \Omega$ and every continuous harmonic function h on D satisfies $\eta \leqslant h$ on bD Thus $\eta \leqslant h$ on D.

Proposition 2.3 ([1, 8]). *Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be an integer with $1 \leqslant q \leqslant n$. Let $\rho : \Omega \to [-\infty, \infty)$ be a C^2 smooth function. Thus the following statements are equivalent:*

1. *ρ is a q-subharmonic function.*
2. *For every smooth $(0,p)$-form $\alpha = \sum_{|J|=p} \alpha_J\, d\overline{z}_j$, we have*

$$
\sum_{|K|=p-1}{}' \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j \partial\overline{z}_k}\alpha_{jK}\overline{\alpha}_{kK} \geqslant 0 \quad \textit{for every } p \geqslant q. \tag{2.2}
$$

.

A function $\rho \in C^2(U)$ is called strongly q-subharmonic if ρ satisfies (2.2) with strict inequality. Also Ω is strongly q-pseudoconvex if the boundary of Ω, is of class C^2 and its defining function is strongly q-subharmonic.

Definition 2.4. Ω is said to be q-pseudoconvex if there is a q-subharmonic exhaustion function on Ω.

To prove Theorem 1.1, we need a preliminary estimate, which follows easily, as in [13], from the identity (2.1).

Proposition 2.5. *Let Ω be a smooth bounded q-pseudoconvex domain in $\mathbb{C}^n$ and let $1 \leqslant q \leqslant n$. Assume that $g, \varphi \in C^2(\overline{\Omega})$ with $g \geqslant 0$, Thus, for $q \leqslant p \leqslant n$ and for $\alpha \in C^\infty_{0,p}(\overline{\Omega}) \cap \operatorname{dom} \overline{\partial}^*_\varphi$, one obtains*

$$
\begin{aligned}
&\|\sqrt{g}\overline{\partial}\alpha\|^2_\varphi + \big(1 + \frac{1}{\tau}\big)\|\sqrt{g}\overline{\partial}^*_\varphi\alpha\|^2_\varphi \\
&\geqslant \sum_{|K|=p}{}' \sum_{k=1}^n \int_\Omega g\big|\frac{\partial \alpha_K}{\partial \overline{z}_k}\big|^2 e^{-\varphi}\, dV - \sum_{|L|=p-1}{}' \int_\Omega \tau\Big|\frac{1}{\sqrt{g}}\sum_{j=1}^n \frac{\partial g}{\partial z_j}\alpha_{jL}\Big|e^{-\varphi} \\
&\quad + \sum_{|L|=p-1}{}' \sum_{j,k=1}^n \int_\Omega \Big(g\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} - \frac{\partial^2 g}{\partial z_j\partial\overline{z}_k}\Big)\alpha_{jL}\overline{\alpha}_{kL}e^{-\varphi}\, dV,
\end{aligned}
\tag{2.3}
$$

for any positive number τ.

Proof. Following (2.2), q-pseudoconvexity of $b\Omega$ implies that the boundary integral in (2.1) is nonnegative for $q \leqslant p \leqslant n$. In the last term on the right-hand side of (2.1), insert $1/\sqrt{g}$ into the first factor of the inner product and $\sqrt{g}$ into the second factor. Using the Cauchy-Schwarz inequality for that term, followed by the simple inequality $2|st| \leqslant \frac{1}{\tau}s^2 + \tau t^2$ with $\tau > 0$, yields

$$
\begin{aligned}
&2\operatorname{Re}\Big\langle \sum_{|L|=p-1}{}' \sum_{j=1}^n \alpha_{jL}\frac{\partial g}{\partial z_j}\, d\overline{z}_L, \overline{\partial}^*_\varphi\alpha\Big\rangle_\varphi \\
&\leqslant 2\Big|\Big\langle \sum_{|L|=p-1}{}' \frac{1}{\sqrt{g}}e^{-\varphi/2}\sum_{j=1}^n \frac{\partial g}{\partial z_j}\alpha_{jL}\, d\overline{z}_j, \sqrt{g}\, e^{-\varphi/2}\, \overline{\partial}^*_\varphi\alpha\Big\rangle\Big| \\
&\leqslant 2\Big\| \sum_{|L|=p-1}{}' \frac{1}{\sqrt{g}}\sum_{j=1}^n \frac{\partial g}{\partial z_j}\alpha_{jL}\, d\overline{z}_j\Big\|_\varphi \|\sqrt{g}\overline{\partial}^*_\varphi\alpha\|_\varphi \\
&\leqslant \sum_{|L|=p-1}{}' \tau\Big\|\frac{1}{\sqrt{g}}\sum_{j=1}^n \frac{\partial g}{\partial z_j}\alpha_{jL}\Big\|^2_\varphi + \frac{1}{\tau}\|\sqrt{g}\overline{\partial}^*_\varphi\alpha\|^2_\varphi.
\end{aligned}
$$

Thus (2.3) follows from (2.1). □

Proposition 2.6. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$ and $1 \leqslant q \leqslant n$. Thus, for $q \leqslant p \leqslant n$ and for $\alpha \in C^\infty_{0,p}(\overline{\Omega}) \cap \operatorname{dom}\overline{\partial}^*$, one obtains*

(1) *$\partial\alpha_K/\partial\overline{z}_k \in L^2(\Omega)$, $1 \leqslant k \leqslant n$, and*

$$
\sum_{|K|=p}{}' \sum_{k=1}^n \int_\Omega \big|\frac{\partial\alpha_K}{\partial\overline{z}_k}\big|^2\, dV \leqslant \|\overline{\partial}\alpha\|^2 + \|\overline{\partial}^*\alpha\|^2. \tag{2.4}
$$

(2) *If $h \in C^2(\overline{\Omega})$, $h \leqslant 0$, Thus*

$$
\sum_{|L|=p-1}{}' \sum_{j,k=1}^n \int_\Omega e^h \frac{\partial^2 h}{\partial z_j\partial\overline{z}_k}\alpha_{jL}\overline{\alpha}_{kL}\, dV \leqslant \|\overline{\partial}\alpha\|^2 + \|\overline{\partial}^*\alpha\|^2. \tag{2.5}
$$

Proof. Since Ω is a bounded q-pseudoconvex domain in $\mathbb{C}^n$, Thus from [1], there exists strongly q-pseudoconvex domains Ω_ν with smooth boundary satisfies

$$
\Omega = \cup_{\nu=1}^\infty \Omega_\nu, \quad \Omega_\nu \Subset \Omega_{\nu+1} \Subset \Omega \quad \text{for all } \nu.
$$

Thus, for every $\alpha \in C^\infty_{0,p}(\overline{\Omega}_\nu) \cap \operatorname{dom} \overline{\partial}^*$ with $q \leqslant p \leqslant n$, one obtains

$$\sum_{|L|=p-1}{}' \sum_{j,k=1}^{n} \int_{b\Omega_\nu} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \alpha_{jL} \overline{\alpha}_{kL}\, dS \geqslant C \int_{b\Omega_\nu} |\alpha|^2\, dS, \tag{2.6}$$

where C is a positive constant. One keep the differentiability assumptions from Proposition 2.1 on Ω_ν and on $\alpha \in \operatorname{dom} \overline{\partial}^*$. Choosing $\varphi \equiv 0$ and $g = 1$ in (2.1) and from (2.6), one obtains

$$\sum_{|K|=p}{}' \sum_{k=1}^{n} \int_{\Omega_\nu} \big|\frac{\partial \alpha_K}{\partial \overline{z}_k}\big|^2\, dV \leqslant \|\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^*\alpha\|^2_{\Omega_\nu}. \tag{2.7}$$

Replace g by $1 - e^h$ with $h \leqslant 0$ an arbitrary twice continuously differentiable function. By applying the Cauchy-Schwarz inequality to the last term on the right-hand side of (2.1), one obtains

$$\begin{aligned} &\sum_{|L|=p-1}{}' \sum_{j,k=1}^{n} \int_{\Omega_\nu} e^h \frac{\partial^2 h}{\partial z_j \partial \overline{z}_k} \alpha_{jL} \overline{\alpha}_{kL}\, dV - \|e^{h/2}\overline{\partial}^*\alpha\|_{\Omega_\nu} \\ &\leqslant \|\sqrt{g}\,\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\sqrt{g}\overline{\partial}^*\alpha\|^2_{\Omega_\nu}. \end{aligned}$$

Since $g + e^h = 1$ and $g \geqslant 1$, it follows that

$$\sum_{|L|=p-1}{}' \sum_{j,k=1}^{n} \int_{\Omega_\nu} e^h \frac{\partial^2 h}{\partial z_j \partial \overline{z}_k} \alpha_{jL} \overline{\alpha}_{kL}\, dV \leqslant \|\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^*\alpha\|^2_{\Omega_\nu}, \tag{2.8}$$

for all $\alpha \in C^\infty_{0,p}(\overline{\Omega}_\nu) \cap \operatorname{dom} \overline{\partial}^*$, $p \geqslant q$. Estimates (2.7) and (2.8) were derived under the assumption that α is continuously differentiable on $\overline{\Omega}_\nu$, it holds by density for all square-integrable forms $\alpha \in \operatorname{dom} \overline{\partial} \cap \operatorname{dom} \overline{\partial}^*$. The latter property carries over to arbitrary bounded q-pseudoconvex domains by exhausting a nonsmooth by smooth ones, and thus so does inequality (2.4) and (2.5). $\square$

The complex Laplacian $\Box_p = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ acts as an unbounded selfadjoint operator on $L^2_{0,p}(\Omega)$, $1 \leqslant p \leqslant n$, it is surjective and thus has a continuous inverse, the $\overline{\partial}$-Neumann operator N_p. The space

$$K^2_{0,p}(\Omega) = \{\alpha \in L^2_{0,p}(\Omega) : \overline{\partial}\alpha = 0\}$$

is a closed subspace of $L^2_{0,p}(\Omega)$ because $\overline{\partial}$ is a closed and densely defined operator. A bounded, linear operator

$$S_{p+1} : L^2_{0,p+1}(\Omega) \cap K^2_{0,p}(\Omega) \to L^2_{0,p}(\Omega)$$

is called a canonical solution operator for $\overline{\partial}$ if $\overline{\partial}S_{p+1}\alpha = \alpha$ for all $\alpha \in L^2_{0,p+1}(\Omega) \cap K^2_{0,p}(\Omega)$ and $S_{p+1}\alpha \perp K^2_{0,p}(\Omega)$. The Bergman projection $B_p : L^2_{0,p}(\Omega) \to K^2_{0,p}(\Omega)$ is the orthogonal projection from $L^2_{0,p}(\Omega)$ onto $K^2_{0,p}(\Omega)$ and B_0 is the classical Bergman projection.

Proposition 2.7. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$ for $1 \leqslant q \leqslant n$. Thus, for $q \leqslant p \leqslant n$, there exists a bounded linear operator $N_p : L^2_{0,p}(\Omega) \to L^2_{0,p}(\Omega)$ which has the following properties:*

1. *$\mathcal{R}ang(N_p) \subset \operatorname{dom} \Box_p$, $N_p\Box_p = I$ on $\operatorname{dom} \Box_p$,*
2. *for $\alpha \in L^2_{0,p}(\Omega)$, we have $\alpha = \overline{\partial}\overline{\partial}^* N_p\alpha \oplus \overline{\partial}^*\overline{\partial}N_p\alpha$,*

(3) $\overline{\partial}N_p = N_p\overline{\partial}$ *on* $\operatorname{dom}\overline{\partial}$, $q \leqslant p \leqslant n$, $\overline{\partial}^* N_p = N_p\overline{\partial}^*$ *on* $\operatorname{dom}\overline{\partial}^*$, $q+1 \leqslant p \leqslant n$,
(4) *if* $\overline{\partial}\alpha = 0$, *Thus* $u = \overline{\partial}^* N_p\alpha$ *solves the equation* $\overline{\partial}u = \alpha$,
(5) N_p, $\overline{\partial}N_p$ *and* $\overline{\partial}^* N_p$ *are bounded operators with respect to the* L^2*-norms,*
(6) *the Bergmann projection* B_p *is given by*

$$B_p = Id - S_{p+1}\overline{\partial}. \tag{2.9}$$

Proof. If z_0 is a point of Ω_ν, and $h(z) = -1 + |z-z_0|^2/d^2$, where $d = \sup_{z,z'\in\Omega_\nu} |z-z'|$ is the diameter of Ω_ν, Thus (2.8) implies the fundamental estimate

$$\|\alpha\|^2_{\Omega_\nu} \leqslant \Big(\frac{d^2 e}{p}\Big)\Big(\|\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^*\alpha\|^2_{\Omega_\nu}\Big).$$

This estimate was derived under the assumption that α is continuously differentiable on $\overline{\Omega}_\nu$, it holds by density for all square-integrable forms $\alpha \in \operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^*$. Thus, by exhausting as in Proposition 2.6 and for $p \geqslant q$, one obtains

$$\|\alpha\|_\Omega \leqslant \Big(\frac{d^2 e}{p}\Big)\|\Box_p\alpha\|_\Omega. \tag{2.10}$$

Since $\Box_p$ is a linear closed densely defined operator, Thus, from [9, Theorem 1.1.1]; $\operatorname{Rang}(\Box_p)$ is closed. Thus, from [9, (1.1.1)] and the fact that $\Box_p$ is self adjoint, one obtains the Hodge decomposition

$$L^2_{0,p}(\Omega) = \overline{\partial}\overline{\partial}^* \operatorname{dom}\Box_p \oplus \overline{\partial}^*\overline{\partial} \operatorname{dom}\Box_p.$$

Since $\Box_p : \operatorname{dom}\Box_p \to \operatorname{Rang}(\Box_p) = L^2_{0,p}(\Omega)$ is one to one on $\operatorname{dom}\Box_p$ from (2.10), there exists a unique bounded inverse operator $N_p : \operatorname{Rang}(\Box_p) \to \operatorname{dom}\Box_p$ satisfies $N_p\Box_p\alpha = \alpha$ on $\operatorname{dom}\Box_p$ and satisfies $\Box_p N_p = I$ on $L^2_{0,p}(\Omega)$. Thus (1) and (2) follow.

To show that $\overline{\partial}^* N_p = N_p\overline{\partial}^*$ on $\operatorname{dom}\overline{\partial}^*$, by using (2), $\overline{\partial}^*\alpha = \overline{\partial}^*\overline{\partial}\overline{\partial}^* N_p\alpha$, for $\alpha \in \operatorname{dom}\overline{\partial}^*$. Thus

$$N_p\overline{\partial}^*\alpha = N_p\overline{\partial}^*\overline{\partial}\overline{\partial}^* N_p\alpha = N_p(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*)\overline{\partial}^* N_p\alpha = \overline{\partial}^* N_p\alpha.$$

A similar argument shows that $\overline{\partial}N_p = N_p\overline{\partial}$ on $\operatorname{dom}\overline{\partial}$. By using (3) and since $\overline{\partial}\alpha = 0$, one obtains $\overline{\partial}N_p\alpha = N_p\overline{\partial}\alpha = 0$. Thus, by using (2), we obtains $\alpha = \overline{\partial}\overline{\partial}^* N_p\alpha$. Thus $u = \overline{\partial}^* N_p\alpha$ satisfies the equation $\overline{\partial}u = \alpha$. Since $\operatorname{Rang} N_p \subset \operatorname{dom}\Box_p$, Thus by applying (2.10) to $N_p\alpha$ instead of α, (5) follows. To prove (6), we consider the two complementary cases, $\alpha \in K^2_{0,p}(\Omega)$ and $\alpha \perp K^2_{0,p}(\Omega)$, and prove this expression for each. First, if $\alpha \in K^2_{0,p}(\Omega)$, then $(Id - S_{p+1}\overline{\partial})\alpha = \alpha$ as expected. If $\alpha \perp K^2_{0,p}(\Omega)$, then $S_{p+1}\overline{\partial}\alpha = \alpha$ since the ranges of $\overline{\partial}$ and $\overline{\partial}^*$ are closed. Thus, $S_{p+1}\alpha = 0$ and the proof follows. $\square$

Proof of Theorem 1.1. The first part of Theorem 1.1 follows by using (2.5), as in [17, Propositions 4.2 and 4.8]. And by applying Proposition 2.5, the second part of Theorem 1.1 follows as in [13, Theorem 4.1 and Proposition 4.2] and [17] respectively. $\square$

3. Proof of Theorem 1.2

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $0 \leqslant p \leqslant n$ and $\phi \in L^\infty(\Omega)$. Denote by

$$H^2(\Omega) = \{\alpha \in L^2(\Omega) : \alpha \text{ is holomorphic on } \Omega\}$$

the Bergman space which is a closed subspace of $L^2(\Omega)$. The space

$$H^2_{0,p}(\Omega) = \big\{\alpha = \sum_{|K|=p}{}' \alpha_K \, d\overline{z}_K : \alpha_K \in H^2(\Omega), \text{ for all } K\big\}$$

is the space of $(0,p)$-forms with holomorphic coefficients, is equipped with the induced norm from $L^2_{0,p}(\Omega)$ and so $H^2_{0,p}(\Omega)$ is a closed subspace of $L^2_{0,p}(\Omega)$. For $p = 0$, $H^2_{0,0}(\Omega) = K^2_{0,0}(\Omega)$ is called the Bergman space, but for $p > 1$ only one obtains $H^2_{0,p}(\Omega) \subsetneqq K^2_{0,p}(\Omega)$.

Example 3.1 ([10])**.** Let $\alpha = \sum_{j=1}^n \alpha_j \, d\overline{z}_j \in L^2_{0,1}(\Omega)$. Thus

$$\alpha \in K^2_{0,1}(\Omega) \Leftrightarrow \frac{\partial \alpha_j}{\partial \overline{z}_k} = \frac{\partial \alpha_k}{\partial \overline{z}_j}, \quad 1 \leqslant j < k \leqslant n,$$

which can be seen from

$$\overline{\partial}\alpha = \sum_{j,k=1}^n \frac{\partial \alpha_j}{\partial \overline{z}_k} \, d\overline{z}_k \wedge d\overline{z}_j = \sum_{1 \leqslant j < k \leqslant n} \Big[\frac{\partial \alpha_j}{\partial \overline{z}_k} - \frac{\partial \alpha_k}{\partial \overline{z}_j}\Big] \, d\overline{z}_k \wedge d\overline{z}_j.$$

Now let $\alpha = \sum_{j=1}^3 \alpha_j \, d\overline{z}_j \in L^2_{0,1}(\Omega)$ and $g = \sum_{j=1}^3 g_j \, d\overline{z}_j \in L^2_{0,1}(\Omega)$ with $\alpha_1 := z_1 + \overline{z}_2 + \overline{z}_3$, $\alpha_2 := \overline{z}_1 + z_2 + \overline{z}_3$, $\alpha_3 := \overline{z}_1 + \overline{z}_2$ and $g_1 := \alpha_3$, $g_2 = \alpha_1$, $g_3 := \alpha_2$. Using the above equivalence we have $\alpha \in K^2_{0,1}(\Omega)$, but $g \notin K^2_{0,1}(\Omega)$.

Remark 3.2 ([10])**.** The structure of $H^2_{0,p}(\Omega)$ is less complicated than $K^2_{0,p}(\Omega)$. If $\alpha \in H^2_{0,p}(\Omega)$, Thus every form with the same coefficients, but with different indicates is also in $H^2_{0,p}(\Omega)$. But for $\alpha \in K^2_{0,p}(\Omega)$, $p > 0$.

Lemma 3.3. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$.*

(i) *Let $\alpha \in K^2_{0,p+1}(\Omega)$, where $1 \leqslant q \leqslant n-1$. Thus, for $q \leqslant p \leqslant n-1$, there exist $\alpha_j \in K^2_{0,p}(\Omega)$, $1 \leqslant j \leqslant n$, satisfies*

$$\alpha = \sum_{j=1}^n \alpha_j \wedge d\overline{z}_j \quad \text{and} \quad \sum_{j=1}^n \|\alpha_j\| \lesssim \|\alpha\|. \tag{3.1}$$

(ii) *Let $\alpha \in H^2_{0,p+1}(\Omega)$, where $1 \leqslant q \leqslant n-1$. Thus, for $q \leqslant p \leqslant n-1$, there exist $\alpha_j \in H^2_{0,p}(\Omega)$, $1 \leqslant j \leqslant n$, satisfies*

$$\alpha = \sum_{j=1}^n \alpha_j \wedge d\overline{z}_j \quad \text{and} \quad \|\alpha\|^2 = \sum_{j=1}^n \|\alpha_j\|^2. \tag{3.2}$$

(iii) *Let $\alpha \in K^2_{0,p+1}(\Omega)$, where $1 \leqslant p \leqslant n-1$. Thus, for $q \leqslant p \leqslant n-1$, there exist $\alpha_j \in K^2_{0,p}(\Omega)$, $1 \leqslant j \leqslant n$, satisfies*

$$[B_{p+1}, \phi]\alpha = (Id - B_{p+1})\Big(\sum_{j=1}^n ([B_p, \phi]\alpha_j) \wedge d\overline{z}_j\Big). \tag{3.3}$$

Proof. (i) As in [3, Lemma 1], for

$$f = \sum_{|K|=p}{}' f_K \, d\overline{z}_K = S_{p+1}\alpha,$$

one can write

$$f = \sum_{j=1}^n f_j \wedge d\overline{z}_j,$$

where the f_j's are square integrable $(0,p-1)$-forms satisfies there are no common terms between $f_j \wedge d\bar z_j$ and $f_k \wedge d\bar z_k$ if $j \neq k$. This can be done as follows: Let $\vee$ denote the adjoint of the exterior multiplication. That is, if f is a $(0,p)$-form $d\bar z_j \vee f$ is a $(0,p-1)$-form satisfies

$$\langle h \wedge d\bar z_j, f\rangle = \langle h, d\bar z_j \vee f\rangle \quad \text{for all } h \in C^\infty_{0,p-1}(\mathbb{C}^n).$$

Thus, one can define

$$\begin{gathered} f_1 = d\bar z_1 \vee f, \\ f_j = d\bar z_j \vee \Big(f - \sum_{k=1}^{j-1} f_k \wedge d\bar z_k\Big), \quad \text{for } j = 2,3,\dots,n. \end{gathered} \tag{3.4}$$

Namely, f_1 is defined by collecting all terms that contain $d\bar z_1$ and writing their sum as $f_1 \wedge d\bar z_1$. Thus, one defines f_1 by collecting the terms in $f - f_1 \wedge d\bar z_1$ with $d\bar z_1$ and writing their sum as $f_2 \wedge d\bar z_2$ etc. Since $\bar\partial \alpha = 0$ and f is in the range of $\bar\partial^*$, we have $\bar\partial f = \alpha$ and $\bar\partial^* f = 0$. So $f \in \operatorname{dom}\bar\partial \cap \operatorname{dom}\bar\partial^*$. Also since f_j consists of terms f_K for some $|K| = p$, "bar" derivatives of f_j's come from "bar" derivatives of f. Thus

$$\sum_{j,k=1}^n \|\frac{\partial f_j}{\partial \bar z_k}\| \lesssim \sum_{|K|=p}{}' \sum_{k=1}^n \|\frac{\partial f_K}{\partial \bar z_k}\|.$$

And by using (2.4), one obtains

$$\sum_{j,k=1}^n \|\frac{\partial f_j}{\partial \bar z_k}\| \lesssim \|\bar\partial f\| + \|\bar\partial^* f\| = \|\alpha\|.$$

Hence, $\|\bar\partial f_j\| \lesssim \|\alpha\|$ for every j, and

$$\alpha = \bar\partial f = \sum_{j=1}^n \bar\partial f_j \wedge d\bar z_j.$$

Thus (3.1) follows by defining $\alpha_j = \bar\partial f_j$.

(ii) By defining α_1 and α_j as f_1 and f_j in (3.4). Namely, α_1 is defined by collecting all terms that contain $d\bar z_1$ and writing their sum as $\alpha_1 \wedge d\bar z_1$. Thus, one defines α_1 by collecting the terms in $\alpha - \alpha_1 \wedge d\bar z_1$ with $d\bar z_1$ and writing their sum as $\alpha_2 \wedge d\bar z_2$ etc. The proof is completed as in (i).

(iii) Since both sides of (3.3) are orthogonal to $K^2_{0,p+1}(\Omega)$ we need only to show that for any $h \in L^2_{0,p+1}(\Omega)$ that is orthogonal to $K^2_{0,p+1}(\Omega)$, one obtains

$$\big\langle [B_{p+1}, \phi]\alpha - (Id - B_{p+1})\Big(\sum_{j=1}^n ([B_p, \phi]\alpha_j) \wedge d\bar z_j\Big), h >= 0.$$

One can computes that

$$\begin{aligned} &\big\langle [B_{p+1}, \phi]\alpha - (Id - B_{p+1})\Big(\sum_{j=1}^n ([B_p, \phi]\alpha_j) \wedge d\bar z_j\Big), h\big\rangle \\ &= -\langle \phi\alpha, h\rangle - \langle \sum_{j=1}^n B_p(\phi\alpha_j) \wedge d\bar z_j, h\rangle + \langle \sum_{j=1}^n \phi\alpha_j \wedge d\bar z_j, h\rangle \\ &= -\langle \sum_{j=1}^n B_p(\phi\alpha_j) \wedge d\bar z_j, h\rangle. \end{aligned}$$

The fact that $\overline{\partial}(f \wedge d\overline{z}_j) = \overline{\partial} f \wedge d\overline{z}_j$ implies that the $(0, p+1)$-forms $B_p(\phi\alpha_j) \wedge d\overline{z}_j$ are $\overline{\partial}$-closed for $j = 1, \dots, n$. Thus

$$\langle \sum_{j=1}^{n} B_p(\phi\alpha_j) \wedge d\overline{z}_j, h\rangle = 0$$

and (3.3) follows. $\square$

Remark 3.4. Indeed, any $(0, p+1)$-form $\alpha = \sum_{|J|=p+1}{}' \alpha_J \, d\overline{z}_J$ can be written in the form

$$\alpha = \frac{1}{(p+1)} \sum_{j=1}^{n} \sum_{|L|=p-1}{}' \alpha_{jL} d\overline{z}_j \wedge \, d\overline{z}_L,$$

and if α has holomorphic coefficients, the α_{jL} are holomorphic. Let $\alpha \in K^2_{0,p+1}(\Omega)$, one obtains

$$[B_p, \phi]\alpha = B_p\phi\alpha - \phi B_p\alpha = \phi\alpha - S_{p+1}\overline{\partial}(\phi\alpha) - \phi\alpha = -S_{p+1}(\overline{\partial}\phi \wedge \alpha),$$

for all $\phi \in C(\overline{\Omega})$. Thus, by taking $\phi = \overline{z}_j$, one obtains

$$\begin{aligned} -\sum_{j=1}^{n} [B_p, \overline{z}_j]\Big(\sum_{|L|=p-1}{}' \alpha_{jL} \, d\overline{z}_L\Big) &= \sum_{j=1}^{n} S_{p+1}\Big(d\overline{z}_j \wedge \sum_{|L|=p-1}{}' \alpha_{jL} \, d\overline{z}_L\Big) \\ &= S_{p+1}\Big(\sum_{j=1}^{n} \sum_{|L|=p-1}{}' \alpha_{jL} \, d\overline{z}_j \wedge \, d\overline{z}_L\Big) \\ &= (p+1) \, S_{p+1}\alpha. \end{aligned} \tag{3.5}$$

Lemma 3.5. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$ and $1 \leqslant q \leqslant n-1$. Thus, for $q \leqslant p \leqslant n-1$, the following are equivalent:*

1. *S_{p+1} is compact on $K^2_{0,p+1}(\Omega)$,*
2. *$[B_p, \overline{z}_j]$ is compact on $K^2_{0,p}(\Omega)$ for all $1 \leqslant j \leqslant n$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from (3.5). The implication (2) $\Rightarrow$ (1) follows from (3.1) as follows. Assume that $\{\alpha^k\}$ is a bounded sequence in $K^2_{0,p+1}(\Omega)$, Thus for each k there exist $\overline{\partial}$-closed $(0, p)$-forms α^k_j for $1 \leqslant j \leqslant n$ satisfies

$$\alpha^k = \sum_{j=1}^{n} \alpha^k_j \wedge d\overline{z}_j \quad \text{and} \quad \sum_{j=1}^{n} \|\alpha^k_j\| \leqslant \|\alpha^k\|.$$

Thus, from (3.5), one obtains

$$S_{p+1}(\alpha^k) = (-1)^{p+1} \sum_{j=1}^{n} [B_p, \overline{z}_j](\alpha^k_j).$$

Furthermore, if $[B_p, \overline{z}_j]$ is compact on $\overline{\partial}$-closed $(0, p)$-forms for $1 \leqslant j \leqslant n$, the sequences $\{[B_p, \overline{z}_j](\alpha^k_j)\}$ have convergent subsequences for each j. Hence S_{p+1} is compact on $K^2_{0,p+1}(\Omega)$. Thus the proof follows. $\square$

Lemma 3.6. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$, $n \geqslant 2$ and let $1 \leqslant q \leqslant n-1$. Thus, for $q \leqslant p \leqslant n-1$, the following are equivalent:*

1. *N_{p+1} is compact on $L^2_{0,p+1}(\Omega)$,*
2. *$[B_p, \phi]$ is compact on $K^2_{0,p}(\Omega)$ for all $\phi \in C(\overline{\Omega})$.*

Proof. The implication (1) $\Rightarrow$ (2) follows as [17, Proposition 4.1]. We prove only the implication (2) $\Rightarrow$ (1). Assume that $[B_p, \phi]$ is compact for all $\phi \in C(\overline{\Omega})$ and $f \in K^2_{0,p+2}(\Omega)$. Thus, by Lemma 3.5, S_{p+1} is compact. Moreover, by (3.1), there exist $f_j \in K^2_{0,p+1}(\Omega)$ with

$$f = \sum_{j=1}^{n} f_j \wedge d\overline{z}_j \quad \text{and} \quad \sum_{j=1}^{n} \|f_j\| \lesssim \|f\|.$$

Thus

$$S(f) = \sum_{j=1}^{n} S_{p+1}(f_j) \wedge d\overline{z}_j$$

solves $\overline{\partial} u = f$ and S is compact. Thus $S_{p+2} = (Id - B_{p+1})S$ is compact on $K^2_{0,p+2}(\Omega)$ and N_{p+1} is compact by the Range's formula [14]:

$$N_p = (S_p)^* S_p + S_{p+1}(S_{p+1})^*. \tag{3.6}$$

Thus the proof follows. □

Proof of Theorem 1.2. The equivalence (1) $\Leftrightarrow$ (2) follows from (3.6). The equivalence (2) $\Leftrightarrow$ (3) follows from the compactness of S_{p+1} on $\overline{\partial}$-closed forms is equivalent to compactness of S_{p+1} on $L^2_{0,p+1}(\Omega)$ as S_{p+1} vanishes on the orthogonal complement of $K^2_{0,p+1}(\Omega)$. The equivalence (3) $\Leftrightarrow$ (4) follows from Lemma 3.5. The equivalence (5) $\Leftrightarrow$ (6) follows from [3, Corollary 1]. The equivalence of (1) and (7) follows from Lemma 3.6. The implication (2) $\Rightarrow$ (4) is easy; (2) $\Rightarrow$ (5) follows from [17, Proposition 4.1]. The implications (6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (4) are obvious. □

Corollary 3.7. *For $q \leqslant p \leqslant n-1$, the following are equivalent:*

(1) *$[B_p, \phi]$ is compact on $H^2_{0,p}(\Omega)$, for all $\phi \in C(\overline{\Omega})$,*
(2) *N_{p+1} is compact on $H^2_{0,p+1}(\Omega)$.*

The proof of the above corollary follows by using (3.2) as in Lemma 3.6.

4. Proof of Theorem 1.3

The proof will be based on several steps.

Step 1. (1) follows, by using (2.4) with (3.6), as in [17, Proposition 4.5]. Let $\alpha = \sum_{|J|=p+1}{}' \alpha_J \, d\overline{z}_J \in \operatorname{dom} \overline{\partial} \cap \operatorname{dom} \overline{\partial}^*$. For $k = 1, \dots, n$, one defines $(0,p)$-forms $\alpha_k = \sum_{|K|=p}{}' \alpha_{kK} \, d\overline{z}_K$. Thus $\alpha_k \in \operatorname{dom} \overline{\partial} \cap \operatorname{dom} \overline{\partial}^*$. For $\operatorname{dom} \overline{\partial}$, this holds because the components of $\overline{\partial}\alpha_k$ are linear combinations of terms $\partial \alpha_J / \partial \overline{z}_j$, and their L^2-norm is controlled by $\|\overline{\partial}\alpha\| + \|\overline{\partial}^*\alpha\|$ from (2.4). To see that $\alpha_k \in \operatorname{dom} \overline{\partial}^*$, note first that inner products with α_k are closely related to inner products with u: if $u = \sum_{|K|=p}{}' a_K \, d\overline{z}_K \in L^2_{0,p}(\Omega)$, Thus for k fixed,

$$\begin{aligned} \langle d\overline{z}_k \wedge u, \alpha \rangle &= \langle \sum_{|K|=p}{}' a_K (d\overline{z}_k \wedge d\overline{z}_K), \sum_{|J|=p+1}{}' \alpha_J \, d\overline{z}_J \rangle \\ &= \sum_{|K|=p}{}' a_K \overline{\alpha_{kK}} = \langle u, \alpha_k \rangle. \end{aligned} \tag{4.1}$$

The inner products are in $L^2_{0,p+1}(\Omega)$ and $L^2_{0,p}(\Omega)$, respectively. Thus, for $\beta \in \operatorname{dom}\overline{\partial}$,

$$\begin{aligned} \langle \overline{\partial}\beta, \alpha_k\rangle &= \langle d\overline{z}_k \wedge \overline{\partial}\beta, \alpha\rangle = -\langle \overline{\partial}(d\overline{z}_k \wedge \beta), \alpha\rangle \\ &= -\langle d\overline{z}_k \wedge \beta, \overline{\partial}^*\alpha\rangle \qquad\qquad = -\langle \beta, \gamma_k\rangle, \end{aligned} \tag{4.2}$$

where $\gamma_k = \sum'_{|L|=p-1}(\overline{\partial}^*\alpha)_{kL}\, d\overline{z}_L$. The last equality follows as in (4.1). (4.2) shows that $\alpha_k \in \operatorname{dom}\overline{\partial}^*$, and that

$$\overline{\partial}^*\alpha_k = -\gamma_k. \tag{4.3}$$

Now fix $\varepsilon > 0$. The compactness estimate for the p-forms α_k gives

$$\begin{aligned} \|\alpha\|^2 &= \frac{1}{(p+1)}\sum_{k=1}^n \|\alpha_k\|^2 \\ &\leqslant \frac{1}{(p+1)}\sum_{k=1}^n \Big(\varepsilon(\|\overline{\partial}\alpha_k\|^2 + \|\overline{\partial}^*\alpha_k\|^2) + C_\varepsilon\|\alpha_k\|^2_{W^{-1}(\Omega)}\Big), \end{aligned} \tag{4.4}$$

where the first equality follows from the definition of α_k and the observation that in the sum on the right-hand side of this equality, $\|\alpha_J\|^2$ occurs precisely $(p+1)$ times for each strictly increasing multi-index J of length $p+1$. Both $\|\overline{\partial}\alpha_k\|^2$ and $\|\overline{\partial}^*\alpha_k\|^2$ are dominated by $\|\overline{\partial}\alpha\|^2 + \|\overline{\partial}^*\alpha\|^2$, independently of ε. For $\|\overline{\partial}\alpha_k\|^2$ this was noted at the beginning of the proof, for $\|\overline{\partial}^*\alpha_k\|^2$, this follows from (4.3). Since $\|\alpha_k\|^2_{W^{-1}(\Omega)} \lesssim \|\alpha\|^2_{W^{-1}(\Omega)}$, by the definition of α_k, (4.4) implies a compactness estimate for α.

Step 2. (2) follows by using (3.1), as in [3, Lemma 3]. In fact, we assume that $\{\alpha^k\}$ is a bounded sequence of $\overline{\partial}$-closed $(0,p+1)$-forms. Thus, by (3.1), there exist $\overline{\partial}$-closed $(0,p)$-forms $\{\alpha^k_j\}$'s satisfies

$$\alpha^k = \sum_{j=1}^n \alpha^k_j \wedge d\overline{z}_j \quad\text{and}\quad \sum_{j=1}^n \|\alpha^k_j\| \lesssim \|\alpha^k\|.$$

Let us define $f^k = \sum_{j=1}^n S_p(\alpha^k_j) \wedge d\overline{z}_j$. Thus $\overline{\partial}f^k = \alpha^k$ and compactness of S_p implies that $\{f^k\}$ has a convergent subsequence. Thus, $\overline{\partial}$ has a compact solution operator on $(0,p+1)$-forms. Hence, the canonical solution operator, S_{p+1}, is compact on $K^2_{0,p+1}(\Omega)$.

Step 3. (3) follows by using (3.1), as in [3, Theorem 2]. If $p = n-1$ Thus $K^2_{0,p+1}(\Omega) = L^2_{0,n}(\Omega)$ and $[B_{p+1}, \phi]$ is the zero operator, hence compact. So, we need to prove this part for $n \geqslant 3$ and $q \leqslant p \leqslant n-2$ and for $\alpha \in K^2_{0,p+1}(\Omega)$. Thus, from (3.1), there exist $\alpha_j \in K^2_{0,p}(\Omega)$, $1 \leqslant j \leqslant n$, satisfies

$$\alpha = \sum_{j=1}^n \alpha_j \wedge d\overline{z}_j \quad\text{and}\quad \sum_{j=1}^n \|\alpha_j\| \lesssim \|\alpha\|.$$

Let $\{u^k\} \in K^2_{0,p+1}(\Omega)$ be a bounded sequence. Thus (3.1) implies that for each k and $1 \leqslant j \leqslant n$ there exists $\{u^k_j\} \in K^2_{0,p}(\Omega)$ satisfies

$$u^k = \sum_{j=1}^n u^k_j \wedge d\overline{z}_j \quad\text{and}\quad \sum_{j=1}^n \|u^k_j\| \lesssim \|u^k\|.$$

Moreover, compactness of $[B_p,\phi]$ on $K^2_{0,p}(\Omega)$ implies that $\{[B_p,\phi]u^k_j\}$, $1\leqslant j\leqslant n$, has a convergent subsequence. By using (3.3), $[B_{p+1},\phi]$ is compact on $K^2_{0,p+1}(\Omega)$.

Step 4. (4) follows by using (3.2), as in the part (3).

Corollary 4.1. *Compactness of $[B_p,\phi]$ on $K^2_{0,p}(\Omega)$, for a fixed ϕ, does not necessarily imply compactness of $[B_p,\phi]$ on $L^2_{0,p}(\Omega)$.*

The proof of the above corollary follows from [3].

5. Proof of Theorem 1.4

We identify $\phi\in L^\infty(\Omega)$ with the multiplication operator $\phi: L^2_{0,p}(\Omega)\to L^2_{0,p}(\Omega)$ acting by

$$\phi:\alpha=\sum_{|K|=p}{}^{'}\alpha_K\,d\overline{z}_K\mapsto\sum_{|K|=p}{}^{'}(\phi\alpha_K)\,d\overline{z}_K.$$

It follows that ϕ is a bounded operator with $\|\phi\|\leqslant\|\phi\|_\infty$ and $\phi^*=\overline{\phi}$. The composition

$$T^\phi_p=B_p\phi B_p: L^2_{0,p}(\Omega)\to L^2_{0,p}(\Omega)$$

is called the Bergman-Toeplitz operator acting on $(0,p)$-forms with symbol ϕ. The Bergman-Toeplitz operators are bounded operators with $\|T^\phi_p\|\leqslant\|\phi\|_\infty$ and $T^{\phi^*}_p=T^{\overline{\phi}}_p$. Clearly $T^\phi_p\alpha=B_p\phi(\alpha)$, for all $\alpha\in K^2_{0,p}(\Omega)$.

Lemma 5.1 ([10])**.** *The selfcommutator $[T^{z_j*}_p,T^{z_j}_p]$ of $T^{z_j}_p$, is compact if and only if the operator $(Id-B_p)\overline{z}_j$ is compact.*

Proof. Given a Toeplitz operator with symbol $\phi\in H^\infty(\Omega)$, the selfcommutator of T^ϕ_p, is given by

$$[T^{\phi^*}_p,T^\phi_p]f=B_p\overline{\phi}\phi f-B_p\phi B_p\overline{\phi}f=B_p\phi(Id-B_p)\overline{\phi}f,\quad f\in H^2_{0,p}(\Omega). \tag{5.1}$$

Thus, $[T^{\phi^*}_p,T^\phi_p]$ is compact if and only if the operator $f\to(Id-B_p)\overline{\phi}f$ from $H^2_{0,p}(\Omega)$ into $L^2_{0,p}(\Omega)$ is compact. Using the i^{th} coordinate function z_i in place of ϕ in (5.1), one obtains

$$[T^{z_j*}_p,T^{z_j}_p]f=B_p\,z_j(Id-B_p)\overline{z}_jf,\quad f\in H^2_{0,p}(\Omega).$$

Thus, $[T^{z_j*}_p,T^{z_j}_p]$ is compact if and only if the operator $(Id-B_p)\overline{z}_j$ is compact. $\square$

To prove a formula for S_{p+1} restricted on $(0,p+1)$-forms with holomorphic coefficients, the following definitions are needed

$$\mathcal{J}^p:=\{(j_1,\dots,j_p)\in\{1,\dots,n\}^p: j_1<\dots<j_n\},$$
$$\mathcal{M}^p:=\{(j_1,\dots,j_p)\in\{1,\dots,n\}^p: j_1\neq\dots\neq j_n\}.$$

Let $K,M\in\mathcal{M}^p=\cup_{J\in\mathcal{J}^p}\mathcal{M}^p_J$. If K and M have the same components, one can write $K\sim M$ and $\mathcal{M}^p_K:=\{M\in\mathcal{M}^p: K\sim M\}$ is the equivalence class of K. So one can writes

$$\alpha=\sum_{|K|=p}{}^{'}\alpha_K\,d\overline{z}_K=\sum_{J\in\mathcal{J}^p}\alpha_K\,d\overline{z}_K$$

for $(0,p)$-forms α with strongly increasing p-tuple J and

$$\sum_{K\in\mathcal{M}^p}\alpha_K\,d\overline{z}_K=\sum_{J\in\mathcal{J}^p}\sum_{K\in\mathcal{M}^p_J}\alpha_K\,d\overline{z}_K$$

for $(0,p)$-forms with non strongly increasing p-tuple K. It is clear that $\mathcal{M}^p_J \cap \mathcal{J}^p = \{J\}$, for all $J \in \mathcal{J}^p$. Thus the mapping $\mathcal{M}^p \ni M \longmapsto J(M) \in \mathcal{M}^p_M \cap \mathcal{J}^p$ is unique, which we need essentially in the proof of Lemma 5.2 with the facts $|\mathcal{M}^p_M| = p!$ and $|\mathcal{M}^p| = p!\,|\mathcal{J}^p|$. As in [10, Theorem 3.1], we prove the following lemma.

Lemma 5.2. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$ and let $1 \leqslant q \leqslant n-1$, $n \geqslant 2$. Let $\alpha = \sum'_{|K|=p+1} \alpha_K \, d\bar{z}_J \in H^2_{0,p+1}(\Omega)$, Thus, for $q \leqslant p \leqslant n-1$, one obtains*

$$S_{p+1}\alpha = \frac{1}{(p+1)} \sum_{j=1}^{n} [(Id - B_p)\bar{z}_j] \Big(\sideset{}{'}\sum_{|K|=p,\, j\notin K} \alpha_{J(j,K)} \varepsilon^{j,K}_{J(j,K)} d\bar{z}_K \Big), \tag{5.2}$$

where $J(j,K)$ denotes the strongly increasing $(p+1)$-tuple with the same components as the $(p+1)$-tuple (j,K) and $\varepsilon^{j,K}_{J(j,K)}$ is the sign of the permutation $\begin{pmatrix} j,K \\ J(j,K) \end{pmatrix}$.

Proof. First we show that

$$\alpha = \sideset{}{'}\sum_{|J|=p+1} \alpha_J \, d\bar{z}_J = \frac{1}{(p+1)} \sum_{j=1}^{n} \sideset{}{'}\sum_{|K|=p} \alpha_{J(j,K)} \varepsilon^{j,K}_{J(j,K)} d\bar{z}_j \wedge d\bar{z}_K. \tag{5.3}$$

For this we consider the equivalence class $\mathcal{M}^{p+1}_J = \{M_1, \dots, M_{(p+1)!}\}$ and we get

$$\begin{aligned}
(p+1)!\alpha &= \sum_{J\in\mathcal{J}^{p+1}} \alpha_K \big[d\bar{z}_J + \underset{(p+1)!}{\dots} + d\bar{z}_J \big] \\
&= \sum_{J\in\mathcal{J}^{p+1}} \alpha_K \big[\varepsilon^{M_1}_J d\bar{z}_{M_1} + \underset{(p+1)!}{\dots} + \varepsilon^{M_{(p+1)!}}_J d\bar{z}_{M_{(p+1)!}} \big] \\
&= \sum_{J\in\mathcal{J}^{p+1}} \alpha_K \sum_{M\in\mathcal{M}^{p+1}_J} \varepsilon^M_J d\bar{z}_M \\
&= \sum_{M\in\mathcal{M}^{p+1}} \alpha_{J(M)} \varepsilon^M_{J(M)} d\bar{z}_M \\
&= \sum_{j=1}^{n} \sum_{L\in\mathcal{M}^p} \alpha_{J(j,L)} \varepsilon^{j,L}_{J(j,L)} d\bar{z}_j \wedge d\bar{z}_L \\
&= \sum_{j=1}^{n} \sideset{}{'}\sum_{|K|=p} \sum_{L\in\mathcal{M}^p_K} \alpha_{J(j,L)} \varepsilon^{j,L}_{J(j,L)} \varepsilon^K_L d\bar{z}_j \wedge d\bar{z}_K.
\end{aligned}$$

Consider the inner sum

$$\begin{aligned}
\sum_{L\in\mathcal{M}^p_K} \alpha_{J(j,L)} \varepsilon^{j,L}_{J(j,L)} \varepsilon^K_L &= \sum_{L\in\mathcal{M}^p_K} \alpha_{J(j,K)} \varepsilon^{j,L}_{J(j,K)} \varepsilon^K_L \\
&= \sum_{L\in\mathcal{M}^p_K} \alpha_{J(j,K)} \varepsilon^{j,L}_{J(j,K)} \varepsilon^{j,K}_{j,L} \\
&= \sum_{L\in\mathcal{M}^p_K} \alpha_{J(j,K)} \varepsilon^{j,K}_{J(j,K)} \\
&= p!\alpha_{J(j,K)} \varepsilon^{j,K}_{J(j,K)}
\end{aligned}$$

and so we get

$$(p+1)!\alpha = p!\sum_{j=1}^{n}\sum_{|K|=p}{}^{'}\alpha_{J(j,K)}\varepsilon^{j,K}_{J(j,K)} d\overline{z}_j \wedge d\overline{z}_K.$$

Thus (5.3) follows. Let $h = \sum_{|K|=p}{}' h_K\, d\overline{z}_K \in K^2_{0,p}(\Omega)$, fix j and consider

$$\overline{z}_j\, h = \sum_{|K|=p}{}^{'}\overline{z}_j\, h_K\, d\overline{z}_K.$$

Thus

$$\overline{\partial}(\overline{z}_j h) = \sum_{|K|=p}{}^{'} h_K\, d\overline{z}_j \wedge d\overline{z}_K$$

and with the projection formula (2.9) one obtains

$$[(Id - B_p)\overline{z}_j](h) = [S_{p+1}\overline{\partial}\overline{z}_j](h) = S_{p+1}\Big(\sum_{|K|=p}{}^{'} h_K\, d\overline{z}_j \wedge d\overline{z}_K\Big). \tag{5.4}$$

Using the assumption that $\alpha \in H^2_{0,p+1}(\Omega)$ and fix j, one defines

$$h_j = \sum_{|K|=p,\, j\notin K}{}^{'}\alpha_{J(j,K)}\varepsilon^{j,K}_{J(j,K)} d\overline{z}_K$$

it follows that $h_j \in H^2_{0,p}(\Omega) \subset K^2_{0,p}(\Omega)$. Thus, one can use (5.4) for h_j and obtains

$$\begin{aligned}
&[(Id - B_p)\overline{z}_j]\Big(\sum_{|K|=p,\, j\notin K}{}^{'}\alpha_{J(j,K)}\varepsilon^{j,K}_{J(j,K)} d\overline{z}_K\Big)\\
&= S_{p+1}\Big(\sum_{|K|=p}{}^{'}\alpha_{J(j,K)}\varepsilon^{j,K}_{J(j,K)}\, d\overline{z}_j \wedge d\overline{z}_K\Big).
\end{aligned}$$

From (5.3), one obtains

$$\begin{aligned}
S_{p+1}(\alpha) &= \frac{1}{(p+1)}\sum_{j=1}^{n} S_{p+1}\Big(\sum_{|K|=p}{}^{'}\alpha_{J(j,K)}\,\varepsilon^{j,K}_{J(j,K)}\, d\overline{z}_j \wedge d\overline{z}_K\Big)\\
&= \frac{1}{(p+1)}\sum_{j=1}^{n}[(Id - B_p)\overline{z}_j]\Big(\sum_{|K|=p,\, j\notin K}{}^{'}\alpha_{J(j,K)}\varepsilon^{j,K}_{J(j,K)} d\overline{z}_K\Big).
\end{aligned}$$

Thus (5.2) follows. $\square$

Proof of Theorem 1.4. The equivalence (1) $\Leftrightarrow$ (2) follows from (3.6). Here, we prove only the equivalence of (2) and (3) as in [10, Theorem 4.4].

First we prove (2) $\Rightarrow$ (3). Let S_{p+1} be a compact on $H^2_{0,p}(\Omega)$. With Lemma 5.1 it is enough to show compactness of $(Id - B_p)\overline{z}_j$ for all $j = 1, \dots, n$. Let $f^m = \sum_{|K|=p}{}' f^m_K\, d\overline{z}_K$ be a bounded sequence in $H^2_{0,p}(\Omega)$. It is clear that for every j

$$u^m_j = \overline{\partial}(\overline{z}_j f^m) = \sum_{|K|=p}{}^{'} f^m_K\, d\overline{z}_j \wedge d\overline{z}_K$$

is a bounded sequence in $H^2_{0,p+1}(\Omega)$. By our assumption there exists a subsequence $u^{m_k}_j$ satisfies $S_{p+1}(u^{m_k}_k)$ converges in $L^2_{0,p}(\Omega)$. Thus, from (5.1) and (2.9), one obtains convergence of $[(Id-B_p)\overline{z}_j](f^{m_k}) = S_{p+1}\overline{\partial}(\overline{z}_j f^{m_k}) = S_{p+1}(u^{m_k}_j)$ in $L^2_{0,p}(\Omega)$ for every j.

Second, we prove (3) $\Rightarrow$ (2). Let $f^m = \sum_{|J|=p+1}{}' f^m_J \, d\bar{z}_J$ be a bounded sequence in $H^2_{0,p+1}(\Omega)$. We have to show the existence of a subsequence f^{m_l} satisfies $S_{p+1}(f^{m_l})$ converges in $L^2_{0,p}(\Omega)$. With the equivalence of Lemma 5.1, one can assume that $(Id - B_p)\bar{z}_j$ is compact on $H^2_{0,p}(\Omega)$ for all $j = 1, \dots, n$. For a fix j, one defines

$$h^m_j = \sum_{|K|=p \, j\notin K}{}' f^m_{J(j,K)} \varepsilon^{j,K}_{J(j,K)} \, d\bar{z}_K .$$

By using formula (5.2), one obtains

$$\begin{aligned} S_{p+1}(f^m) &= \frac{1}{(p+1)} \sum_{j=1}^{n} [(Id - B_p)\bar{z}_j] \Big(\sum_{|K|=p \, j\notin K}{}' f^m_{J(j,K)} \varepsilon^{j,K}_{J(j,K)} \, d\bar{z}_K \Big) \\ &= \frac{1}{(p+1)} \sum_{j=1}^{n} [(Id - B_p)\bar{z}_j](h^m_j). \end{aligned}$$

Clearly h^m_j is a bounded sequence in $H^2_{0,p}(\Omega)$ for every j. Since $(Id - B_p)\bar{z}_j$ is compact on $H^2_{0,p}(\Omega)$ for every j, there exists a subsequence $(h_1^{m_{l(1)}})$ satisfies $((Id - B_p)\bar{z}_1)(h_1^{m_{l(1)}})$ converges in $L^2_{0,p}(\Omega)$. Clearly $(h_2^{m_{l(1)}})$ is also a bounded sequence in $H^2_{0,p}(\Omega)$. Thus it exists a further subsequence $(h_2^{m_{l(2)}})$ satisfies the sequences $((Id - B_p)\bar{z}_1)(h_2^{m_{l(2)}})$ converge in $L^2_{0,p}(\Omega)$. By continuing this process, one obtains a subsequence $(h_j^{m_{l(j)}})$ satisfies $((Id - B_p)\bar{z}_1)(h_j^{m_{l(j)}})$ converges in $L^2_{0,p}(\Omega)$ for all $j = 1, \dots, n$. By defining $m_l = m_{l(n)}$ and with formula (5.2) one obtains convergence of

$$S_{p+1}(f^{m_l}) = \frac{1}{(p+1)} \sum_{j=1}^{n} ((Id - B_p)\bar{z}_j)\, (h^{m_l}_j).$$

in $L^2_{0,p}(\Omega)$ for every j. ☐

Corollary 5.3. *For $q \leqslant p \leqslant n-1$, compactness of S_{p+1} on $K^2_{0,p+1}(\Omega)$ implies compactness of $[T_p^{z_j *}, T_p^{z_j}]$ on $K^2_{0,p}(\Omega)$ for all j with $1 \leqslant j \leqslant n$.*

The proof of the above corollary follows by repeating the implication (2) $\Rightarrow$ (3) of Theorem 1.4 for $f \in K^2_{0,p}(\Omega)$.

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