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ON q-STEFFENSEN INEQUALITY

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Abstract. In this article, we study an analogue of the classical integral inequality established by Steffensen in q-calculus. The difficulties ensued form differences between the classical and q-integral. We exceed them in two directions: firstly, by restricting the area of parameter q, and another by modifying the expression of the original inequality. We establish the conditions which guarantee their holding on. Finally, we illustrate our considerations by the examples.

1. Introduction

Integral inequalities have important role in the theory of functional analysis, differential equations, and applied sciences. They can be used for studying qualitative and quantitative properties of integrals.

The well-known Steffensen inequality [9, 11] has the form

$$\int_{b-\lambda}^{b} f(x)\,dx \le \int_{a}^{b} f(x)g(x)\,dx \le \int_{a}^{a+\lambda} f(x)\,dx, \tag{1.1}$$

where $\lambda = \int_a^b g(x)\,dx$, and $f(x)$ and $g(x)$ are both integrable functions on $[a,b]$, $f(x)$ is decreasing and $0 \le g(x) \le 1$ for each $x \in (a,b)$. This inequality has attracted the attention of mathematicians since it was established in 1918, because of its unusual and original form. A lot of generalizations and modifications have been presented, such as those in [1, 6] and even applications to other sciences [4].

Although for a few inequalities, for example Chebyshev, Grüss and Hermite-Hadamard inequality, it was a pretty obvious matter, here we confront with some difficulties ensued from differences between the classical and q-integral [8].

The analogous of (1.1) for the q-integral (2.1) was not discussed; so we wish to make a contribution to it through this article.

The paper is organized as follows: the next section deals with the problems with a few basic inequalities for q-integrals. In the Section 3, we present the Steffensen integral inequality in q-calculus with restrictions. Finally, in the las section, we find a modification of the previous inequality which is valid on any interval of the form $(0,b)$ for every $q \in (0,1)$.

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2. Preliminaries

The q-integral of the function f over the interval $[a,b]$ is defined by (see, for example [2, 5, 7])

$$I_q(f;a,b)=\int_a^b f(x)\,d_qx=\int_0^b f(x)\,d_qx-\int_0^a f(x)\,d_qx \quad (0<q<1), \tag{2.1}$$

where

$$\int_0^b f(x)\,d_qx=b(1-q)\sum_{j=0}^{\infty} f(bq^j)q^j. \tag{2.2}$$

We say that $f(x)$ is q-integrable on (a,b) if (2.1) exists. Obviously, if a function $f(x)$ is q-integrable and $f(x)\ge 0$ over $[0,b]$, then

$$\int_0^b f(x)\,d_qx\ge 0 \quad (0<q<1).$$

If f is integrable over $[0,b]$, then

$$\lim_{q\nearrow 1} I_q(f;a,b)=I(f;a,b)=\int_a^b f(t)\,dt \quad (0<a<b). \tag{2.3}$$

Lacks of definition (2.1) was discussed in a few papers especially because of influence of the points outside of the interval $[a,b]$. One way to overcome this problem was suggested in [8] where the definition of the q-integral of the Riemann type was considered. Another way was suggested in [3] by restricting the q-integral over $[a,b]$ to a finite sum with points only inside the interval $[a,b]$. Its number directly depends on a, b and q and the nonnegativity is guaranteed. Namely, if a lower limit of integral has the special form $a=bq^k$ ($k\in\mathbb{N}$), q-integral reduces on the finite sum

$$\int_{bq^k}^b f(x)\,d_qx=b(1-q)\sum_{j=0}^{k-1} f(bq^j)q^j. \tag{2.4}$$

Lemma 2.1. *If a function $f(x)$ is q-integrable, nonnegative and nondecreasing over $[0,b]$, then*

$$\int_a^b f(x)\,d_qx\ge 0 \quad (0\le a\le b;\ 0<q<1). \tag{2.5}$$

Proof. From the definition, we have

$$\int_a^b f(x)\,d_qx=(1-q)\sum_{n=0}^{\infty}\big(bf(bq^n)-af(aq^n)\big)q^n.$$

Since $a<b$, $0<q<1$ and $f(x)\ge 0$, then $af(aq^n)\le bf(aq^n)$. Also, since $aq^n\le bq^n$ and $f(x)$ is nondecreasing, then $bf(aq^n)\le bf(bq^n)$. Hence $bf(bq^n)-af(aq^n)\ge 0$, for every $n\in\mathbb{N}$, wherefrom the nonnegativity of (2.5) follows. □

Note that some obvious integral inequalities in classical mathematical analysis are not valid for q-integrals. Even more, the q-integral of a positive function does not have to be positive.

For instance, for $0<a<b<r$, the function $f(x)=x(r-x)$ is positive on $(0,r)$. However, q-integral

$$\int_a^b x(r-x)\,d_qx=(b-a)\left(\frac{a+b}{1+q}r-\frac{a^2+ab+b^2}{1+q+q^2}\right).$$

is equal to zero for

$$r = r_q = \frac{a^2+ab+b^2}{a+b}\frac{1+q}{1+q+q^2}\,.$$

In special case $a = 9$ and $b = 10$, we obtain $r_q = \frac{271}{19}\frac{1+q}{1+q+q^2}$. For $q \in (0, 0.9)$, the parameter $r_q > 10$, the function $f(x) = x(r_q - x)$ is positive on the interval $(9, 10)$, but

$$\int_9^{10} x(r_q - x)\,d_qx = 0, \quad \int_9^{10} x(c - x)\,d_qx < 0 \quad (10 < c < r_q).$$

As it was illustrated by the previous discussion, the mean value theorem to q-integrals is valid only in a restricted form (see [10]).

Lemma 2.2. *Let $u(x)$ be a continuous function on $[a, b]$ and $v(x)$ be a nonnegative and integrable function such that $I_q(v; a, b) > 0$ for all $q \in (0, 1]$. Then thee exists $\hat{q} \in (0, 1)$ such that for every $q \in (\hat{q}, 1)$ exists $\xi = \xi(q) \in (a, b)$ so that*

$$I_q(uv; a, b) = u(\xi)I_q(v; a, b). \tag{2.6}$$

Proof. Under the assumed conditions, the mean value theorem for the real integrals $(q = 1)$ states that

$$I(uv; a, b) = u(c)I(v; a, b),$$

where $c \in (a, b)$. Using relation (2.3), we can write

$$\lim_{q\to 1} \frac{I_q(uv; a, b)}{I_q(v; a, b)} = u(c).$$

Since $u(x)$ is a continuous function on $[a, b]$, it attains its minimum m_u and maximum M_u. Let $\varepsilon = \min\{M_u - u(c), u(c) - m_u\}$. Then, there exists $\hat{q} = \hat{q}(\varepsilon) \in (0, 1)$ such that for all $q \in (\hat{q}, 1)$ the following implication is true:

$$u(c) - \varepsilon < \frac{I_q(uv; a, b)}{I_q(v; a, b)} < u(c) + \varepsilon \;\Rightarrow\; m_u < \frac{I_q(uv; a, b)}{I_q(v; a, b)} < M_u.$$

Since $u(x)$ takes all values between m_u and M_u, we conclude that there exists $\xi = \xi(q) \in (a, b)$ so that

$$\frac{I_q(uv; a, b)}{I_q(v; a, b)} = u(\xi).$$

□

Although for a few inequalities, for example Chebyshev, Grüss and Hermite-Hadamard inequality, it was pretty obvious matter, here we confront with some difficulties ensued form differences between the classical and q-integral.

Namely, it was easy in [3] to establish q-analog of (1.1) for the q-integrals of the type (2.4) as a relation between finite sums with the values of a function in the points between a and b.

If someone wants to make the analogous of (1.1) for the q-integral (2.1), the infinite sums and to consider the points out of the interval (a, b) need to be considered. Probably, that why no-one has discussed it. In this article, we wish to make a contribution.

3. The q-Steffensen inequality with restricted parameter

Because of the properties mentioned above, inequality (1.1) is not valid for the q-integrals in its original form for every parameter q. That is why we will examine its feasible region.

Theorem 3.1 (q-Steffensen inequality). *Let $0 < a < b$, $f(x)$ and $g(x)$ are both continuous functions on $[a,b]$, $f(x)$ is decreasing and $0 < g(x) < 1$ on $[a,b]$ and $\int_a^d g(x)\,d_qx > 0$ for every $d \in (a,b)$. If we denote by $\lambda = \int_a^b g(x)\,d_qx$, then there is a $\hat{q} \in (0,1)$ such that*

$$\int_{b-\lambda}^{b} f(x)\,d_qx \le \int_a^b f(x)g(x)\,d_qx \le \int_a^{a+\lambda} f(x)\,d_qx \tag{3.1}$$

for all $q \in (\hat{q},1)$.

Proof. Taking $u \equiv g$ and $v \equiv 1$ in Lemma 2.2, there exists $q_1 \in (0,1)$ such that for every $q \in (q_1,1)$, it exists $\xi_1 = \xi_1(q) \in (a,b)$ so that

$$\lambda = \int_a^b g(x)\,d_qx = g(\xi_1)\int_a^b d_qx = g(\xi_1)(b-a).$$

Since $0 < g(x) < 1$, it is $0 < \lambda < b-a$.

For the same reasons, there exists $q_2 \in (q_1,1)$ such that for every $q \in (q_2,1)$, there exists $\xi_2 = \xi_2(q) \in (a,b)$ so that

$$\int_a^{a+\lambda} g(x)\,d_qx = \lambda\, g(\xi_2) \quad (a < \xi_2 < a+\lambda).$$

Hence

$$\int_a^{a+\lambda} (1-g(x))\ d_qx = \lambda\,(1-g(\xi_2)) > 0.$$

Let us consider the expression

$$RHS = \int_a^{a+\lambda} f(x)\,d_qx - \int_a^b f(x)g(x)\,d_qx,$$

which can be written as

$$\begin{aligned} RHS &= \int_a^{a+\lambda} f(x)\,d_qx - \int_a^{a+\lambda} f(x)g(x)\,d_qx - \int_{a+\lambda}^{b} f(x)g(x)\,d_qx \\ &= \int_a^{a+\lambda} f(x)\,(1-g(x))\ d_qx - \int_{a+\lambda}^{b} f(x)g(x)\,d_qx/ \end{aligned}$$

Let us apply Lemma 2.2 to the first integral with $u \equiv f$ and $v \equiv 1-g$. We conclude that there is $q_3 \in (q_2,1)$ such that, for all $q \in (q_3,1)$, $\xi \in (a, a+\lambda)$ exists such that

$$\int_a^{a+\lambda} f(x)\,(1-g(x))\,d_qx = f(\xi)\int_a^{a+\lambda} (1-g(x))\,d_qx.$$

Since $f(x)$ is decreasing and $\xi < a+\lambda$, it is obvious that $f(\xi) > f(a+\lambda)$. Now, let $q \in (q_3,1)$. Then

$$\int_a^{a+\lambda} f(x)\,(1-g(x))\ d_qx > f(a+\lambda)\int_a^{a+\lambda} (1-g(x))\,d_qx.$$

Since

$$\int_a^{a+\lambda}(1-g(x))\,d_qx=\lambda-\int_a^{a+\lambda}g(x)d_qx=\int_a^b g(x)\,d_qx-\int_a^{a+\lambda}g(x)\,d_qx,$$

we have

$$\int_a^{a+\lambda}f(x)\,(1-g(x))\;d_qx>f(a+\lambda)\int_{a+\lambda}^b g(x)\,d_qx.$$

The previous inequality and the function $f(x)$ begin decreasing, give

$$\begin{aligned}RHS&>f(a+\lambda)\int_{a+\lambda}^b g(x)\,d_qx-\int_{a+\lambda}^b f(x)g(x)\,d_qx\\&=\int_{a+\lambda}^b\big(f(a+\lambda)-f(x)\big)g(x)\,d_qx.\end{aligned}$$

Since the integrand is nonnegative on $[a,b]$, there is $\hat{q}_1\in(q_3,1)\subset(0,1)$ such that $RHS\geq 0$ for all $q\in(\hat{q}_1,1)$.

To prove left side inequality we consider $G(x)=1-g(x)$ and $\Lambda=\int_a^b G(x)\,d_qt=b-a-\lambda$. Applying just proven inequality we conclude that there is $\hat{q}_2\in(0,1)$ such that

$$\int_a^b f(x)G(x)\,d_qx\leq\int_a^{a+\Lambda}f(x)\,d_qx,$$

i.e,

$$\int_{b-\lambda}^b f(x)\,d_qx-\int_a^b f(x)g(x)\,d_qx\leq 0,$$

for all $q\in(\hat{q}_2,1)$. If we denote by $\hat{q}=\max\{\hat{q}_1,\hat{q}_2\}$, than the both sides of the inequality hold on for all $q\in(\hat{q},1)$. $\square$

When $q\to 1^-$, this reduces to the well-known Steffensen inequality (1.1). Here, we will present a few examples which include different values for a bound $\hat{q}$ which illustrates the q-Steffensen inequality (3.1).

Example 3.2. The function $f(x)=(9-x^2)/4$ is decreasing and $g(x)=x/4$ is bounded $0\leq g(x)\leq 1$ on $[1,3]$. Thus, $f(x)$ and $g(x)$ fulfill the assumptions of Theorem 3.1 and $\lambda(q)=\frac{2}{1+q}\in[1,2]$. Here,

$$\int_{b-\lambda}^b f(x)\,d_qx\leq\int_a^b f(x)g(x)\,d_qx\quad(0<q<1),$$

but, the right inequality is valid for q on stricter interval, i.e.,

$$\int_a^b f(x)g(x)\,d_qx\leq\int_a^{a+\lambda}f(x)\,d_qx\quad(\hat{q}<q<1;\;\hat{q}\approx 0.1383).$$

This is shown on Figure 1. Notice that, for $q\in(0,0.18)$, all integrals are negative although the functions are positive. Even more, the integral $I_q[fg,a,b]=\int_a^b f(x)g(x)\,d_qx$, for $q\in(0,\hat{q})$, is not between the limit integrals $I_q[f;a,a+\lambda]$ and $I_q[f;b-\lambda,b]$ in any way.

Similarly, if we take both decreasing functions: $f(x)=1-x^2$ and $g(x)=2-x$ on $[1,2]$, shown on the Figure 2, the both sides of inequality (3.1) are true for every $q\in(1/2,1)$.

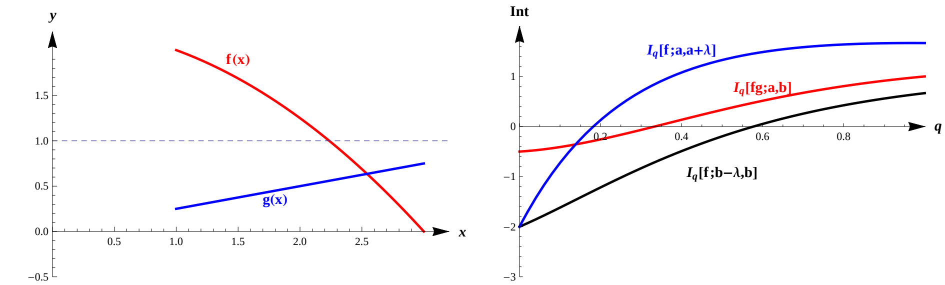


FIGURE 1. The case $f(x) = (9 - x^2)/4$ and $g(x) = x/4$ on $[1, 3]$ and the integrals.

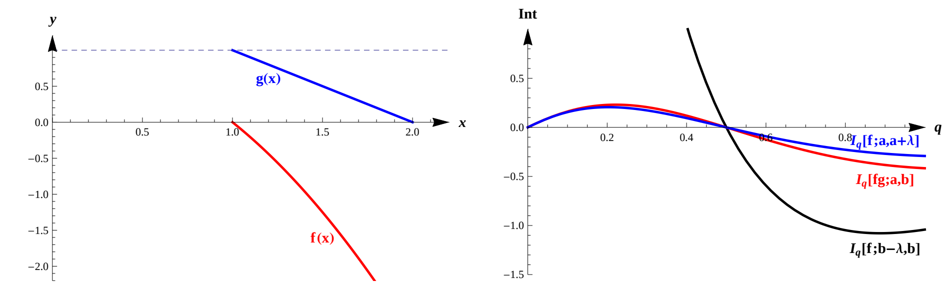


FIGURE 2. The case $f(x) = 1 - x^2$ and $g(x) = 2 - x$ on $[1, 2]$ and the integrals.

4. THE q-STEFFENSEN INEQUALITY ON $(0, b)$

It would be interesting to find modification of the previous inequality which is valid for every $q \in (0, 1)$. We will improve the results from [3] considering the q-integrals $(0, b)$ when they are represented by the infinite sums.

Theorem 4.1. *Let $0 < q < 1$, $b > 0$, $f(x)$ and $g(x)$ are both q-integrable functions on $[0, b]$, $f(x)$ is non-negative and decreasing and $0 \le g(x) \le 1$ for each $x \in [0, b]$ and $\lambda = \int_0^b g(x)\, d_q x$. Let $l, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be such that*

$$l = \lfloor \log_q(1 - \lambda/b) \rfloor, \quad k = \lfloor \log_q(\lambda/b) \rfloor. \tag{4.1}$$

Then

$$L_q(f; 0, b) = \int_{bq^l}^{b} f(x)\, d_q x \le \int_0^b f(x)g(x)\, d_q x \le \int_0^{bq^k} f(x)\, d_q x = U_q(f; 0, b). \tag{4.2}$$

Proof. From condition (4.1), it follows that

$$b(1 - q^l) \le \lambda \le bq^k.$$

Let us consider the right inequality

$$\begin{aligned} RHS &= \int_0^{bq^k} f(x)\, d_q x - \int_0^b f(x)g(x)\, d_q x \\ &= \int_0^{bq^k} f(x)\, d_q x - \int_0^{bq^k} f(x)g(x)\, d_q x - \int_{bq^k}^{b} f(x)g(x)\, d_q x \end{aligned}$$

$$= \int_0^{bq^k} f(x)\big(1-g(x)\big)\,d_qx - \int_{bq^k}^b f(x)g(x)\,d_qx.$$

Using the definition of q-integral, we have

$$\int_0^{bq^k} f(x)\left(1-g(x)\right)\,d_qx = bq^k(1-q)\sum_{j=0}^{\infty} f(bq^{k+j})\left(1-g(bq^{k+j})\right)q^j.$$

Because $f(x)$ is decreasing, $f(bq^{k+j}) \geq f(bq^k)$ is valid for all $j \in \mathbb{N}_0$, so

$$\begin{aligned}\int_0^{bq^k} f(x)\left(1-g(x)\right)\,d_qx &\geq bq^k(1-q)f(bq^k)\sum_{j=0}^{\infty}\left(1-g(bq^{k+j})\right)q^j \\ &= f(bq^k)\int_0^{bq^k}\big(1-g(x)\big)\,d_qx.\end{aligned}$$

Since

$$\int_0^{bq^k} d_qx = bq^k \geq \lambda = \int_0^b g(x)\,d_qx,$$

we can write

$$\begin{aligned}RHS &\geq f(bq^k)\Big(\int_0^b g(x)\,d_qx - \int_0^{bq^k} g(x)\,d_qx\Big) - \int_{bq^k}^b f(x)g(x)\,d_qx \\ &= f(bq^k)\int_{bq^k}^b g(x)\,d_qx - \int_{bq^k}^b f(x)g(x)\,d_qx \\ &= \int_{bq^k}^b \big(f(bq^k)-f(x)\big)g(x)\,d_qx.\end{aligned}$$

According to (2.4), we obtain

$$\int_{bq^k}^b \big(f(bq^k)-f(x)\big)g(x)\,d_qx = b(1-q)\sum_{j=0}^{k-1}\big(f(bq^k)-f(bq^j)\big)g(bq^j) \geq 0,$$

which completes the proof of the right inequality in (4.2). The left inequality can be proved in a similar manner. □

Example 4.2. The functions $f(x) = 2 - \frac{x^2}{4}$ and $g(x) = \frac{x}{2}$ $(x \in [0,2])$, fulfill the conditions of the Theorem 4.1 and $\lambda(q) \in [0,2]$. The integral $I_q[fg;0,b]$ for different values $q \in (0,1)$ and appropriate $L_q(f;0,b)$ and $U_q(f;0,b)$ are shown on the Figure 3.

Corollary 4.3. *Inequality* (4.2) *reduces to well-known Steffensen inequality* (1.1) *when* q *increases to* 1.

Proof. Let us notice that

$$\lim_{q\uparrow 1} q^{\lfloor \log_q x\rfloor} = x \quad (0 < q, x < 1).$$

Really, denoting by $n = \lfloor \log_q x\rfloor$, we can write $n \leq \log_q x < n+1$, i.e., $q^{n+1} < x \leq q^n$. Hence $x \leq q^n < x/q$. Then

$$\lim_{q\uparrow 1} q^n = x \;\Rightarrow\; \lim_{q\uparrow 1} q^{\lfloor \log_q x\rfloor} = x.$$

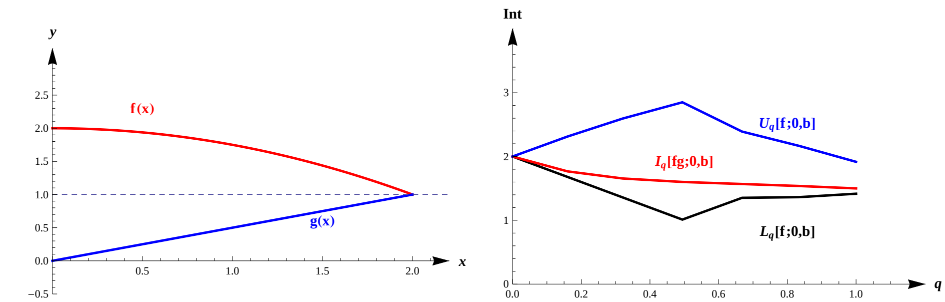


FIGURE 3. The case $f(x) = 2 - \frac{x^2}{4}$ and $g(x) = \frac{x}{2}$ on $[0, 2]$.

Hence
$$\lim_{q\uparrow 1} bq^{\lfloor \log_q(1-\lambda/b)\rfloor} = b - \lambda, \quad \lim_{q\uparrow 1} bq^{\lfloor \log_q(\lambda/b)\rfloor} = \lambda.$$
□

We can formulate and prove another version of q-Steffensen inequality.

Corollary 4.4. *Let $0 < q < 1$, $b > 0$, $f(x)$ and $g(x)$ are both q-integrable functions on $[0, b]$, $f(x)$ is non-negative and decreasing and $0 \le g(x) \le 1$ for each $x \in [0, b]$ and $\lambda = \int_0^b g(x)\, d_qx$. Then*
$$\tilde{L}_q[f; 0, b] \le \int_0^b f(x)g(x)\, d_qx \le \frac{b}{\lambda}\int_0^\lambda f(x)\, d_qx = \tilde{U}_q[f; 0, b], \tag{4.3}$$
where
$$\tilde{L}_q[f; 0, b] = \max\Big\{0, \int_{b-\lambda}^b f(x)\, d_qx - \big(\frac{1}{q} - 1\big)\int_0^b f(x)\, d_qx\Big\}. \tag{4.4}$$

Proof. Since $0 < \lambda < b$ and $0 < q < 1$, there exist $l, k \in \mathbb{N}_0$ such that
$$l = \lfloor \log_q(1 - \lambda/b)\rfloor, \quad k = \lfloor \log_q(\lambda/b)\rfloor.$$
Now, the following inequalities are valid:
$$bq^{l+1} < b - \lambda \le bq^l, \quad bq^{k+1} < \lambda \le bq^k, \tag{4.5}$$
$$\frac{1}{q} > \frac{bq^l}{b-\lambda} \ge 1, \quad \frac{1}{q} > \frac{bq^k}{\lambda} \ge 1. \tag{4.6}$$
Using Theorem 4.1, the right side in (4.3) can be written in the form
$$RHS = \frac{b}{\lambda}\int_0^\lambda f(x)\, d_qx - \int_0^b f(x)g(x)\, d_qx \ge \frac{b}{\lambda}\int_0^\lambda f(x)\, d_qx - \int_0^{bq^k} f(x)\, d_qx.$$
Since $f(x)$ is a decreasing function on $(0, b)$ and in accordance to (4.5), we have $f(bq^{k+j}) \le f(\lambda q^j)$, wherefrom
$$\begin{aligned}\int_0^{bq^k} f(x)\, d_qx - \int_0^\lambda f(x)\, d_qx &= (1-q)\Big(bq^k\sum_{j=0}^\infty f(bq^{k+j})q^j - \lambda\sum_{j=0}^\infty f(\lambda q^j)q^j\Big)\\ &\le (1-q)\Big(bq^k\sum_{j=0}^\infty f(\lambda q^j)q^j - \lambda\sum_{j=0}^\infty f(\lambda q^j)q^j\Big)\end{aligned}$$

$$= (1-q)\big(bq^k-\lambda\big)\sum_{j=0}^{\infty} f(\lambda q^j)q^j,$$

where

$$\int_0^{bq^k} f(x)\,d_qx - \int_0^{\lambda} f(x)\,d_qx \le \frac{bq^k-\lambda}{\lambda}\int_0^{\lambda} f(x)\,d_qx.$$

Using (4.6), we obtain

$$\begin{aligned}
\int_0^{bq^k} f(x)\,d_qx &\le \int_0^{\lambda} f(x)\,d_qx + \frac{bq^k-\lambda}{\lambda}\int_0^{\lambda} f(x)\,d_qx \\
&= \frac{bq^k}{\lambda}\int_0^{\lambda} f(x)\,d_qx \\
&\le \frac{b}{\lambda}\int_0^{\lambda} f(x)\,d_qx,
\end{aligned}$$

where the right inequality in (4.3) follows.

Let us consider the left side inequality. If $\int_{b-\lambda}^{b} f(x)\,d_qx \le 0$, since the other integrals are nonnegative, the inequality (4.3) is immediately fulfilled.

Let $\int_{b-\lambda}^{b} f(x)\,d_qx > 0$. By using the Theorem 4.1, we have

$$\begin{aligned}
LHS &\ge \int_{bq^l}^{b} f(x)\,d_qx - \Big(\int_{b-\lambda}^{b} f(x)\,d_qx - \Big(\frac{1}{q}-1\Big)\int_0^{b} f(x)\,d_qx\Big) \\
&= -\int_{b-\lambda}^{bq^l} f(x)\,d_qx + \Big(\frac{1}{q}-1\Big)\int_0^{b} f(x)\,d_qx.
\end{aligned}$$

By definition of q-integral, we have

$$-\int_{b-\lambda}^{bq^l} f(x)\,d_qx = (1-q)\sum_{j=0}^{\infty}\big((b-\lambda)f\big((b-\lambda)q^j\big) - bq^l f(bq^{l+j})\big)\,q^j.$$

Since $f(x)$ is decreasing on $[0,b]$, from (4.5) we have $f\big((b-\lambda)q^j\big) > f(bq^{l+j})$, i.e., $-f(bq^{l+j}) > -f\big((b-\lambda)q^j\big)$; therefore

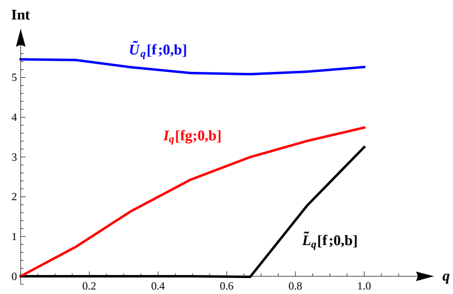
$$\begin{aligned}
-\int_{b-\lambda}^{bq^l} f(x)\,d_qx &\ge (1-q)\sum_{j=0}^{\infty}\big((b-\lambda)f\big((b-\lambda)q^j\big) - bq^l f\big((b-\lambda)q^j\big)\big)\,q^j \\
&= (1-q)\big(b-\lambda-bq^l\big)\sum_{j=0}^{\infty} f\big((b-\lambda)q^j\big)q^j \\
&= \frac{b-\lambda-bq^l}{b-\lambda}\int_0^{b-\lambda} f(x)\,d_qx.
\end{aligned}$$

Hence,

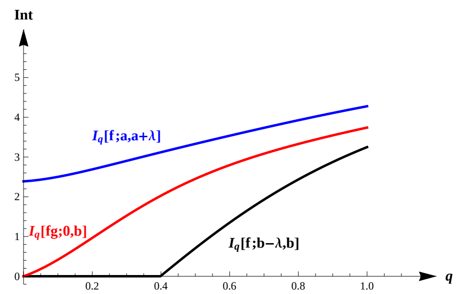
$$\begin{aligned}
LHS &\ge -\int_{b-\lambda}^{bq^l} f(x)\,d_qx + \Big(\frac{1}{q}-1\Big)\int_0^{b} f(x)\,d_qx \\
&\ge \Big(1-\frac{bq^l}{b-\lambda}\Big)\Big(\int_0^{b} f(x)d_qx - \int_{b-\lambda}^{b} f(x)d_qx\Big) + \Big(\frac{1}{q}-1\Big)\int_0^{b} f(x)d_qx \\
&= \Big(\frac{bq^l}{b-\lambda}-1\Big)\int_{b-\lambda}^{b} f(x)\,d_qx + \Big(\frac{1}{q}-\frac{bq^l}{b-\lambda}\Big)\int_0^{b} f(x)\,d_qx \ge 0.
\end{aligned}$$

□

Example 4.5. The functions $f(x) = \frac{9-x^2}{4}$ and $g(x) = 1 - \frac{1}{4}(x - \frac{3}{2})^2$, on $[0,3]$ fulfill the conditions of the Corollary 4.4 and $\lambda(q) \in [0,3]$. The integral $I_q[fg;0,b]$ for different values $q \in (0,1)$ and appropriate $\tilde{L}_q(f;0,b)$ and $\tilde{U}_q(f;0,b)$ are shown on the Figure 4(a).



(a) proved



(b) conjectured

FIGURE 4. Estimations of $I_q[fg;0,3]$ for $f(x) = \frac{9-x^2}{4}$ and $g(x) = 1 - \frac{1}{4}\left(x - \frac{3}{2}\right)^2$, on $[0,3]$

Furthermore, we believe that the sharper estimation is valid as it is shown on the Figure 4b) and supposed in the following conjecture. Using the Theorem 3.1 with $a = 0$, we can easy see that it is true on some interval $(\hat{q}, 1) \subset (0,1)$, but it is not sufficient.

Conjecture. Let $0 < q < 1$ and $b > 0$. Suppose $f(x)$ and $g(x)$ are both nonnegative and q-integrable functions on $[0,b]$ and $\lambda = \int_0^b g(x)\,d_qx$. If $f(x)$ is decreasing and $g(x) \le 1$ on $[0,b]$, then

$$\max\left\{0, \int_{b-\lambda}^{b} f(x)\,d_qx\right\} \le \int_0^b f(x)g(x)\,d_qx \le \int_0^\lambda f(x)\,d_qx.$$

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