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RIGOROUS DERIVATION OF A 1D MODEL FROM THE 3D NON-STEADY NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE NONLINEARLY VISCOUS FLUIDS

RICHARD ANDRÁŠIK, ROSTISLAV VODÁK

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Abstract. Problems with three-dimensional models lie very often in their large complexity leading to impossibility to find an analytical solution. Numerical solutions are sometimes an option, but they can be unduly complicated in the case of three-dimensional models. Frequently, researchers investigate models where one or even two dimensions are almost negligible and nothing important is occurring in them. These models can be simplified and turned into one- or two-dimensional models, which is very helpful, because their solutions are easier than solutions of the original three-dimensional models. Since nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids in a three-dimensional domain belongs to the class of models which need a simplification, when possible, to be effectively solved, we performed a dimension reduction for this model. We studied the dynamics of a compressible fluid in thin domains where only one dimension is dominant. We present a rigorous derivation of a one-dimensional model from the three-dimensional Navier-Stokes equations.

1. Introduction

Three-dimensional mathematical models (e. g. partial differential equations) are often used to express real-life problems. Naturally, we focus on finding solutions of mathematical models. However, analytical solutions are frequently not possible to find and numerical solutions can be both theoretically and computationally complicated due to complexity of three-dimensional models. Frequently, researchers investigate models in thin domains where one or two dimensions are dominant. Hence, it is appropriate to derive lower-dimensional models from original three-dimensional models.

The existence of weak solutions for three-dimensional models of fluid dynamics has already been studied. Pierre-Louis Lions proved the global solvability of Navier-Stokes equations for compressible linearly viscous fluids [15]. Further, Eduard Feireisl extensively studied global existence theory for the full Navier-Stokes-Fourier system [9]. An extensive overview on results achieved in the case of Newtonian compressible fluids is given in [20]. Concerning non-Newtonian fluids, [16, 17]

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proved the existence of a global weak solution for compressible Navier-Stokes equations.

An asymptotic analysis was performed in linear elasticity for rods and beams [11, 12, 22], and for plates and shells [3, 5, 6], at first. Subsequently, rigorous derivation of lower-dimensional models was done also for fluids. An asymptotic analysis of three-dimensional steady Navier-Stokes equations based on the asymptotic expansion was presented in [19]. For comparison, the same result was achieved directly in [25] without the need to apply any asymptotic expansion. Regarding nonsteady Navier-Stokes equations for incompressible fluids, they were simplified into a lower-dimensional model in [10]. Further, a three-dimensional system for barotropic Navier-Stokes equations was asymptotically analyzed and the resulting one-dimensional and two-dimensional models were presented in [24] and [18], respectively. It was also shown that weak solutions of both three-dimensional Navier-Stokes equations for barotropic flows and three-dimensional full Navier-Stokes-Fourier equations tend to strong solutions of the respective one-dimensional system as the three-dimensional model tends to the one-dimensional model [2, 4].

New difficulties arise by considering non-Newtonian fluids (nonlinear viscous stress tensor). This problem was tackled for the first time in [23], where a two-dimensional model was derived by a suitable scaling from nonsteady Navier-Stokes equations for compressible fluids. Recently, the current framework was extended in [1] by dealing with the same equations in a deformed three-dimensional domain. Our aim is to follow up on this research and perform a rigorous derivation of a one-dimensional model from nonsteady Navier-Stokes equations for compressible non-Newtonian fluids.

We studied the dynamics of a compressible fluid in a thin pipe $\Omega_\varepsilon \subset \mathbb{R}^3$ which can be transformed to a referential domain $\Omega \subset \mathbb{R}^3$ (see section 3). The motion of a compressible fluid is described by its velocity $\bar{\mathbf{u}}_\varepsilon$ and density $\bar{\rho}_\varepsilon$. The time evolution of $\bar{\mathbf{u}}_\varepsilon$ and $\bar{\rho}_\varepsilon$ is governed by the continuity and momentum equations

$$\partial_t \bar{\rho}_\varepsilon + \bar{\operatorname{div}}\,(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon) = 0, \tag{1.1}$$

$$\partial_t \left(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon\right) + \bar{\operatorname{div}}\,(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \otimes \bar{\mathbf{u}}_\varepsilon) + \bar{\nabla} \bar{p}_\varepsilon = \bar{\operatorname{div}}\, \bar{\mathbb{S}}_\varepsilon + \bar{\rho}_\varepsilon \bar{\mathbf{f}}_\varepsilon \quad \text{in } \Omega_\varepsilon \times (0,T), \tag{1.2}$$

where $T > 0$, $\bar{p}_\varepsilon$ is the pressure, $\bar{\mathbb{S}}_\varepsilon$ stands for the viscous stress tensor and $\bar{\mathbf{f}}_\varepsilon$ represents the external forces (see [16]). The connection of a function to Ω_ε is emphasized by symbol “ $^-$ ” and subscript ε (see section 3).

We focus on derivation of a one-dimensional model from equations (1.1)-(1.2) under Navier boundary conditions. First, we introduce Orlicz spaces and Young functions with an exponential growth in sections 2.2 and 2.3, because this knowledge is necessary to proof our main result. The problem in question is described in detail in section 3. Subsequently, the transformation of governing equations and energy equality is performed in section 4. Finally, section 5 contains the proof of our main result.

2. Preliminaries

The basic notation is summarized in this section. We also pay our attention to Young functions and Orlicz spaces. In particular, we focus on the theory concerning Young functions with an exponential growth, their complementary functions and respective Orlicz spaces, because they are needed in the subsequent sections.

2.1. **Basic notation.** We adopt the notation "$\cdot$" and ":" for the scalar product of vectors and tensors, respectively, and "$\otimes$" for the tensor product. The Cartesian product of two sets is denoted by "$\times$" as well as the cross product of two vectors without danger of confusion. Symbol $|\cdot|$ stands for either the Lebesgue measure of a measurable set or the Euclidean norm defined as $|Z| = \sqrt{Z_{ij}Z_{ij}}$, where $Z \in \mathbb{R}^{m,n}$, m, $n \in \mathbb{N}$. We use Einstein summation convention for notational brevity. Symbols C and C_n, $n \in \mathbb{N}$, stand for unspecified positive constants.

We emphasize the connection of a function to Ω_ε by symbol "$\,^{-}\,$" and subscript ε. On the other hand, objects without symbol "$\,^{-}\,$" are connected to the referential domain Ω (see section 3). Symbols $\bar{D}$ and D represent the symmetric part of the gradient, i. e. $\bar{D}_{ij}\bar{\mathbf{u}}_\varepsilon = \frac{1}{2}(\bar{\partial}_i\bar{u}_j + \bar{\partial}_j\bar{u}_i)$ and $D_{ij}\mathbf{u} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.

Let $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain. We denote by ∂Q the boundary of Q. Bounded domain Q is called a Lipschitz domain if its boundary can be expressed by Lipschitz continuous functions (see [14] for the precise definition). We write $\partial Q \in \mathcal{C}^{0,1}$. All vectors $\mathbf{x} \in \mathbb{R}^3$ in the text are column vectors.

We employ the standard notation of the following function spaces and their norms:

Spaces of continuously differentiable functions up to order m	–	$\mathcal{C}^m(\bar{Q})$, $\\|\cdot\\|_{\mathcal{C}^m(\bar{Q})}$
Lebesgue spaces	–	$L^p(Q)$, $\\|\cdot\\|_p$
Sobolev spaces	–	$W^{1,p}(Q)$, $\\|\cdot\\|_{1,p}$
Duals of $W^{1,p}(Q)$	–	$[W^{1,p}(Q)]^*$, $\\|\cdot\\|_{[W^{1,p}(Q)]^*}$
Orlicz spaces	–	$L_\Phi(Q)$, $\\|\cdot\\|_{L_\Phi(Q)}$
Sobolev-Orlicz spaces	–	$W^1L_\Phi(Q)$, $\\|\cdot\\|_{W^1L_\Phi(Q)}$
Duals of $W^1L_\Phi(Q)$	–	$[W^1L_\Phi(Q)]^*$, $\\|\cdot\\|_{[W^1L_\Phi(Q)]^*}$
Bochner spaces	–	$L^p(0,T;X)$, $\\|\cdot\\|_{L^p(0,T;X)}$, $\mathcal{C}^m(\langle 0,T\rangle;X)$, $\\|\cdot\\|_{\mathcal{C}^m(\langle 0,T\rangle;X)}$

where $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain, $p \in \langle 1,+\infty) \cup \{+\infty\}$, $m \in \mathbb{N} \cup \{0\}$ and X is a Banach space. In addition, $\mathcal{C}_0^m(\bar{Q})$ denotes spaces of continuously differentiable functions up to order m, $m \in \mathbb{N} \cup \{0\}$, with compact support. Naturally, $\mathcal{C}^0(\bar{Q}) = \mathcal{C}(\bar{Q})$ is the space of continuous functions. Next, we denote the space of smooth and compactly supported functions endowed with the inductive limit topology by $\mathcal{D}(Q)$. Its dual space is denoted by $\mathcal{D}'(Q)$.

2.2. **Orlicz spaces.** We pay attention only to the necessary definitions and properties which are used in the subsequent sections. We refer to [13] and [14] for a detailed introduction to the theory of Young functions and Orlicz spaces.

Two special types of ordering can be introduced for Young functions. The first ordering concerns the equivalence property of Young functions and the second ordering is useful for the embedding theorem of Orlicz spaces.

Definition 2.1. Let Φ_1 and Φ_2 be two Young functions. If there exist $C > 0$ and $z_0 > 0$ such that
$$\Phi_1(z) \le \Phi_2(Cz), \quad \forall z \ge z_0,$$
then we write $\Phi_1 \prec \Phi_2$. In addition, if $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$, we say that Φ_1 and Φ_2 are equivalent. If
$$\lim_{z\to+\infty} \frac{\Phi_1(z)}{\Phi_2(\lambda z)} = 0,$$
for any $\lambda > 0$, then we write $\Phi_1 \prec\prec \Phi_2$.

Remark 2.2. Let Φ_1 and Φ_2 be two Young functions, Ψ_1 and Ψ_2 be the respective complementary functions. If $\Phi_1 \prec\prec \Phi_2$, then $\Psi_1 \succ\succ \Psi_2$.

Prior to the definition of Orlicz spaces, we define Orlicz classes.

Definition 2.3. Let Φ be a Young function and $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is an open subset. We say that $u \in \tilde{L}_\Phi(Q)$, if

$$\int_Q \Phi(|u(x)|)\,\mathrm{d}x < +\infty.$$

The set $\tilde{L}_\Phi(Q)$ is called an Orlicz class.

An Orlicz class is only a convex subset of $L^1(Q)$ (see [14, page 130]), in general. Therefore, we define Orlicz spaces $L_\Phi(Q)$ and also spaces $E_\Phi(Q)$.

Definition 2.4. Let $u : Q \to \mathbb{R}$, $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a measurable function and let Φ, Ψ be a pair of complementary Young functions. The set $L_\Phi(Q)$ of all u such that $\|u\|_{L_\Phi(Q)} < +\infty$ is called the Orlicz space. Its norm $\|u\|_{L_\Phi(Q)}$ is defined as

$$\|u\|_{L_\Phi(Q)} = \sup_v \int_Q |u(x)v(x)|\,\mathrm{d}x,$$

where the supremum is taken over all functions $v \in \tilde{L}_\Psi(Q)$ satisfying condition $\int_Q \Psi(|v(x)|)\,\mathrm{d}x \le 1$.

Definition 2.5. Let Φ be a Young function. The space $E_\Phi(Q)$ is defined as the closure of the set of all bounded measurable functions defined on Q with respect to the norm $\|\cdot\|_{L_\Phi(Q)}$.

Definition 2.6. A Young function Φ satisfies the $\boldsymbol{\Delta_2}$-condition, if there exists $C > 0$ and $z_0 \ge 0$ such that $\Phi(2z) \le C\Phi(z)$, for all $z \ge z_0$.

Remark 2.7. Let Φ and Ψ be a pair of complementary Young functions. In general, it holds that $L_\Psi(Q) = [E_\Phi(Q)]^*$ and $E_\Phi(Q) \subseteq \tilde{L}_\Phi(Q) \subseteq L_\Phi(Q)$. Furthermore, Φ satisfies the Δ_2-condition if and only if $E_\Phi(Q) = \tilde{L}_\Phi(Q) = L_\Phi(Q)$.

Besides the strong convergence in the Orlicz space $L_\Phi(Q)$ given in terms of the norm $\|\cdot\|_{L_\Phi(Q)}$, we can also define the E_Ψ-weak convergence.

Definition 2.8. A sequence $\{u_n\}_{n=1}^{+\infty} \subset L_\Phi(Q)$ converges E_Ψ-weakly to $u \in L_\Phi(Q)$, if

$$\lim_{n\to+\infty} \int_Q (u_n(x) - u(x))v(x)\,\mathrm{d}x = 0, \quad \forall v \in E_\Psi(Q).$$

We write $u_n \overset{\Psi}{\rightharpoonup} u$.

Remark 2.9. The weak-* convergence in $L_\Phi(Q)$ is equivalent to the E_Ψ-weak convergence. Therefore, the boundedness of $\{u_n\}_{n=1}^{+\infty}$ in $L_\Phi(Q)$ implies the existence of E_Ψ-weakly convergent subsequence of $\{u_n\}_{n=1}^{+\infty}$.

Let us suppose that Φ_1 and Φ_2 are Young functions. Orlicz spaces have the following properties:

- $L_{\Phi_1}(Q) \hookrightarrow L_{\Phi_2}(Q)$ if and only if $\Phi_1 \succ \Phi_2$,
- inclusion $L_{\Phi_1}(Q) \subset L_{\Phi_2}(Q)$ is equivalent to the embedding $L_{\Phi_1}(Q) \hookrightarrow L_{\Phi_2}(Q)$ in case of Orlicz spaces,
- Φ_1 and Φ_2 are equivalent if and only if $L_{\Phi_1}(Q) = L_{\Phi_2}(Q)$,

- if $\Phi_1 \succ\succ \Phi_2$, then $L_{\Phi_1}(Q) \hookrightarrow E_{\Phi_2}(Q)$.

The definition of Sobolev-Orlicz spaces $W^1L_\Phi(Q)$ and spaces $W^1E_\Phi(Q)$ is similar to the definition of Sobolev spaces. Finally, we present inequalities which are necessary for deriving estimates in the following text.

Theorem 2.10 (Hölder's inequality). *Let $u \in L_\Phi(Q)$ and $v \in L_\Psi(Q)$, where Φ, Ψ is a pair of complementary Young functions. Then $uv \in L^1(Q)$ and*

$$\int_Q |u(x)v(x)|\,\mathrm{d}x \le \|u\|_{L_\Phi(Q)}\|v\|_{L_\Psi(Q)}. \tag{2.1}$$

Theorem 2.11 (Young's inequality). *Let a, $b \in \langle 0,+\infty)$ and Φ, Ψ be a complementary Young functions. It holds that*

$$ab \le \Phi(a) + \Psi(b). \tag{2.2}$$

Theorem 2.12 (Jensen's inequality). *Let us assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is a convex function and $\alpha(x)$ is positive almost everywhere in $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Then*

$$\Phi\Big(\frac{\int_Q \alpha(x)u(x)\,\mathrm{d}x}{\int_Q \alpha(x)\,\mathrm{d}x}\Big) \le \frac{\int_Q \alpha(x)\Phi(u(x))\,\mathrm{d}x}{\int_Q \alpha(x)\,\mathrm{d}x} \tag{2.3}$$

for any non-negative function $u : Q \to \mathbb{R}$ supposing that all the integrals in (2.3) *are meaningful.*

2.3. **Special Young functions.** We focus on Young functions with an exponential growth which are used in the following sections to analyze the asymptotic behavior of equations (1.1)-(1.2).

Definition 2.13. Let us define Young functions $\Phi_\gamma(z) = (1+z)\ln^\gamma(1+z)$, $\gamma > 1$, and $\Phi_1(z) = z\ln(z+1)$. Functions Ψ_γ, $\gamma \ge 1$, denote the complementary functions to Φ_γ, $\gamma \ge 1$. Subsequently, we define $M(z) = \mathrm{e}^z - z - 1$ and its complementary function $N(z) = (1+z)\ln(1+z) - z$. Further, we denote $\Phi_{1/\alpha}$, $\alpha \in (1,+\infty)$, the Young functions with growth $z\ln^{1/\alpha} z$, $z \ge z_0 > 0$, and their complementary functions $\Psi_{1/\alpha}$.

It is apparent that:

- $\Phi_\gamma(z) = O(z\ln^\gamma z)$, $\gamma > 0$, and $M(z) = O(\mathrm{e}^z)$,
- $\Psi_\gamma(z) = O(\mathrm{e}^{z^{1/\gamma}})$, $\gamma > 0$, and $N(z) = O(z\ln z)$,
- $L_{\Phi_1}(Q) = L_N(Q)$ and also $L_{\Psi_1}(Q) = L_M(Q)$,
- Young functions Φ_γ, $\gamma \ge 1$, satisfy the global Δ_2-condition,
- if $\gamma_2 > \gamma_1 \ge 1$, then $\Phi_{\gamma_2} \succ\succ \Phi_{\gamma_1}$ and also $\Psi_{\gamma_2} \prec\prec \Psi_{\gamma_1}$,
- if $u \in L_{\Phi_\gamma}(Q)$, $\gamma \ge 1$, then $\int_Q \Phi_\gamma(|u(x)|)\,\mathrm{d}x < +\infty$,
- if $u \in L_{\Psi_\gamma}(Q)$, $\gamma \ge 1$, then $\int_Q \Psi_{\gamma'}(|u(x)|)\,\mathrm{d}x < +\infty$, $\forall\gamma' > \gamma$.

3. Statement of the problem

We focus on the motion of a compressible fluid in a thin pipe. The dynamics of a compressible fluid is governed by equations (1.1)-(1.2). Let us suppose that the fluid is isothermal and non-Newtonian. It means that (without the loss of generality)

$$\bar{\mathbb{S}}_\varepsilon = P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon, \quad \bar{p}_\varepsilon = \bar{\rho}_\varepsilon.$$

Domain $\Omega_\varepsilon \subset \mathbb{R}^3$ is defined by the use of a referential domain $\Omega = (0,1) \times S$ with $S \subset \mathbb{R}^2$ and $\partial S \in C^{0,1}$, and mapping $\mathbf{R}_\varepsilon : \Omega \to \Omega_\varepsilon$ so that

$$\mathbf{R}_\varepsilon : (x_1, x_2, x_3) \mapsto (x_1, \varepsilon x_2, \varepsilon x_3).$$

It means that $\Omega_\varepsilon = (0,1) \times \varepsilon S$. As well as in [20], section 4.17.2.4, we suppose that Ω is not axially symmetric.

Symbols $\mathbf{n}$ and $\bar{\mathbf{n}}_\varepsilon$ stand for unit outward normals to Ω and Ω_ε, respectively. Similarly, $\mathbf{t}$ and $\bar{\mathbf{t}}_\varepsilon$ are vectors from the corresponding tangent planes. We employ the following notation for the borders of domains Ω and Ω_ε:

$$\Gamma_1 = (0,1) \times \partial S, \quad \Gamma_2 = \{0,1\} \times S,$$
$$\Gamma_{1,\varepsilon} = \mathbf{R}_\varepsilon(\Gamma_1), \quad \Gamma_{2,\varepsilon} = \mathbf{R}_\varepsilon(\Gamma_2).$$

To ensure the well-posedness of our problem (see [23]), we prescribe Navier boundary conditions

$$\bar{\mathbf{t}}_\varepsilon \cdot \big(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon\bar{\mathbf{n}}_\varepsilon\big) + h(\varepsilon)\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{t}}_\varepsilon = 0 \quad \text{on } \Gamma_{1,\varepsilon} \times (0,T), \tag{3.1}$$

$$\bar{\mathbf{t}}_\varepsilon \cdot \big(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon\bar{\mathbf{n}}_\varepsilon\big) + q\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{t}}_\varepsilon = 0 \quad \text{on } \Gamma_{2,\varepsilon} \times (0,T), \tag{3.2}$$

$$\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{n}}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \times (0,T). \tag{3.3}$$

We suppose that $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$ and $q > 0$. The asymptotic behavior of $h(\varepsilon)$ will be discussed during derivation of weak convergences of density and velocity field (section 5.2).

We consider the initial conditions for the density and the momentum

$$\bar{\rho}_\varepsilon(\bar{x}, 0) = \bar{\rho}_{0,\varepsilon}(\bar{x}) \geq 0, \quad \forall \bar{x} \in \Omega_\varepsilon$$
$$(\bar{\rho}_\varepsilon\bar{\mathbf{u}}_\varepsilon)(\bar{x}, 0) = (\bar{\rho}_\varepsilon\bar{\mathbf{u}}_\varepsilon)_0(\bar{x}, 0), \quad \forall \bar{x} \in \Omega_\varepsilon.$$

The variational formulation of our problem is

$$\int_0^T \int_{\Omega_\varepsilon} \big(\bar{\rho}_\varepsilon \partial_t \bar{\varphi} + \bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \cdot \bar{\nabla}\bar{\varphi}\big)\, \mathrm{d}\bar{x}\, \mathrm{d}t = 0, \tag{3.4}$$

$$\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} \big(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \cdot \partial_t \bar{\psi} + \bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \otimes \bar{\mathbf{u}}_\varepsilon : \bar{D}\bar{\psi} + \bar{\rho}_\varepsilon \bar{\operatorname{div}}\bar{\psi}\big)\, \mathrm{d}\bar{x}\, \mathrm{d}t \\
&= \int_0^T \int_{\Omega_\varepsilon} \big(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon : \bar{D}\bar{\psi} - \bar{\rho}_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \bar{\psi}\big)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + h(\varepsilon)\int_0^T \int_{\Gamma_{1,\varepsilon}} \bar{\mathbf{u}}_\varepsilon \cdot \bar{\psi}\, \mathrm{d}\bar{\Gamma}\, \mathrm{d}t + q\int_0^T \int_{\Gamma_{2,\varepsilon}} \bar{\mathbf{u}}_\varepsilon \cdot \bar{\psi}\, \mathrm{d}\bar{\Gamma}\, \mathrm{d}t,
\end{aligned} \tag{3.5}$$

for any $\bar{\varphi} \in \mathcal{D}\big(\mathbb{R}^3 \times (0,T)\big)$ and $\bar{\psi} \in C_0^\infty(0,T; C^\infty(\bar{\Omega}_\varepsilon)^3)$ satisfying condition $\bar{\psi} \cdot \bar{\mathbf{n}}_\varepsilon|_{\partial\Omega_\varepsilon \times (0,T)} = 0$.

Similarly as in [23], we assume that the function P satisfies, for any $\bar{U}$, $\bar{V} \in [\tilde{L}_M(\Omega_\varepsilon)]^9$, the following five conditions

$$\int_{\Omega_\varepsilon} P(|\bar{U}|)|\bar{U}|^2\, \mathrm{d}\bar{x} \geq \int_{\Omega_\varepsilon} M(|\bar{U}|)\, \mathrm{d}\bar{x}, \tag{3.6}$$

$$\int_{\Omega_\varepsilon} \big(P(|\bar{U}|)\bar{U} - P(|\bar{V}|)\bar{V}\big) : (\bar{U} - \bar{V})\, \mathrm{d}\bar{x} \geq 0, \tag{3.7}$$

$$P(z)|z|^2 \text{ is a convex function for } z \geq 0, \tag{3.8}$$

$$\int_{\Omega_\varepsilon} N(P(|\bar U|)|\bar U|)\,\mathrm{d}\bar x \le c\Big(1+\int_{\Omega_\varepsilon} M(|\bar U|)\Big)\,\mathrm{d}\bar x, \tag{3.9}$$

$$P(|\bar U-\lambda\bar V|)(\bar U-\lambda\bar V) \xrightarrow{M} P(|\bar U|)\bar U,\ \text{ for } \lambda\to 0. \tag{3.10}$$

For example, the function

$$P(z)=\begin{cases}\frac{M(z)}{z}, & \text{for } z\neq 0,\\ 0, & \text{for } z=0\end{cases}$$

satisfies all conditions (3.6)-(3.10).

4. Transformation and related results

We transform the governing equations and the energy equality to the referential domain. First, we denote

$$\mathbf{u}_\varepsilon:\Omega\times\langle 0,T\rangle\to\mathbb{R}^3,\quad \rho_\varepsilon:\Omega\times\langle 0,T\rangle\to\mathbb{R},$$

where $\mathbf{u}_\varepsilon(x,t)=\bar{\mathbf{u}}_\varepsilon(\mathbf{R}_\varepsilon(x),t)$ and $\rho_\varepsilon(x,t)=\bar\rho_\varepsilon(\mathbf{R}_\varepsilon(x),t)$, for all $x\in\Omega$. Since $\bar x=\mathbf{R}_\varepsilon(x)$, $\bar x\in\Omega_\varepsilon$, we can write $\mathbf{u}_\varepsilon(x,t)=\bar{\mathbf{u}}_\varepsilon(\bar x,t)$ and $\rho_\varepsilon(x,t)=\bar\rho_\varepsilon(\bar x,t)$.

We express the spatial gradient of a scalar function $\bar\varphi$ according to the chain rule as

$$\bar\nabla\bar\varphi(\bar x,t)=\bar\nabla\varphi(\mathbf{R}_\varepsilon^{-1}(x),t)=\nabla_\varepsilon\varphi,$$

where $\nabla_\varepsilon=(\partial_1,\varepsilon^{-1}\partial_2,\varepsilon^{-1}\partial_3)$. Similarly, we transform the symmetric part of the gradient of a vector function $\bar{\mathbf{u}}_\varepsilon$ and arrive at $\bar D\bar{\mathbf{u}}_\varepsilon(\bar x,t)=\bar D\bar{\mathbf{u}}_\varepsilon(\mathbf{R}_\varepsilon^{-1}(x),t)=\omega_\varepsilon(\mathbf{u}_\varepsilon(x,t))$, where

$$\omega_\varepsilon(\mathbf{u}_\varepsilon)=\begin{pmatrix}\partial_1u_{1,\varepsilon} & \frac12(\partial_1u_{2,\varepsilon}+\varepsilon^{-1}\partial_2u_{1,\varepsilon}) & \frac12(\partial_1u_{3,\varepsilon}+\varepsilon^{-1}\partial_3u_{1,\varepsilon})\\ \cdot & \varepsilon^{-1}\partial_2u_{2,\varepsilon} & \frac12\varepsilon^{-1}(\partial_2u_{3,\varepsilon}+\partial_3u_{2,\varepsilon})\\ \text{sym} & \cdot & \varepsilon^{-1}\partial_3u_{3,\varepsilon}\end{pmatrix}. \tag{4.1}$$

4.1. **Transformation of the governing equations.**

According to [6], we use the following equalities

$$\mathrm{d}\bar x=\varepsilon^2\,\mathrm{d}x,$$
$$\mathrm{d}\bar\Gamma=\varepsilon\,\mathrm{d}\Gamma\quad\text{on }\Gamma_1,$$
$$\mathrm{d}\bar\Gamma=\varepsilon^2\,\mathrm{d}\Gamma\quad\text{on }\Gamma_2$$

to arrive at the transformed equations of the variational formulation (3.4)–(3.5).

Now, we can divide both equations by ε^2 and arrive at transformed governing equations

$$\int_0^T\int_\Omega(\rho_\varepsilon\partial_t\varphi+\rho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_\varepsilon\varphi)\,\mathrm{d}x\,\mathrm{d}t=0, \tag{4.2}$$

$$\begin{aligned}&\int_0^T\int_\Omega[\rho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\psi+\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon:\omega_\varepsilon(\psi)+\rho_\varepsilon\operatorname{div}_\varepsilon\psi]\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_\Omega[P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)\,\omega_\varepsilon(\mathbf{u}_\varepsilon):\omega_\varepsilon(\psi)-\rho_\varepsilon\mathbf{f}_\varepsilon\cdot\psi]\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac{h(\varepsilon)}{\varepsilon}\int_0^T\int_{\Gamma_1}\mathbf{u}_\varepsilon\cdot\psi\,\mathrm{d}\Gamma\,\mathrm{d}t+q\int_0^T\int_{\Gamma_2}\mathbf{u}_\varepsilon\cdot\psi\,\mathrm{d}\Gamma\,\mathrm{d}t,\end{aligned} \tag{4.3}$$

for any $\varphi\in\mathcal{D}\left(\mathbb{R}^3\times(0,T)\right)$ and $\psi\in C_0^\infty(0,T;[C^\infty(\bar\Omega)]^3)$, $\psi\cdot\mathbf{n}|_{\partial\Omega\times(0,T)}=0$, where $\operatorname{div}_\varepsilon\psi=\partial_1\psi+\varepsilon^{-1}\partial_2\psi+\varepsilon^{-1}\partial_3\psi$.

Imposing the same transformation to the renormalized continuity equation (see [15] or [17] for its original form) leads to

$$\int_0^T \int_\Omega b(\rho_\varepsilon)\partial_t\varphi + b(\rho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_\varepsilon\varphi + [(b(\rho_\varepsilon) - \rho_\varepsilon b'(\rho_\varepsilon)) \operatorname{div}_\varepsilon \mathbf{u}_\varepsilon]\,\varphi \,\mathrm{d}x\,\mathrm{d}t = 0, \quad (4.4)$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$.

4.2. **Energy equality and its transformation.** For any $t \in \langle 0,T\rangle$, we have the energy equality expressed by the following formula (see [17])

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \Big(\bar\rho_\varepsilon(t)\frac{|\bar{\mathbf{u}}_\varepsilon(t)|^2}{2} + \bar\rho_\varepsilon(t)\ln(\bar\rho_\varepsilon(t))\Big)\,\mathrm{d}\bar x \\
&+ \int_0^t \int_{\Omega_\varepsilon} P(|\bar D\bar{\mathbf{u}}_\varepsilon|)\bar D\bar{\mathbf{u}}_\varepsilon : \bar D\bar{\mathbf{u}}_\varepsilon \,\mathrm{d}\bar x\,\mathrm{d}s + h(\varepsilon)\int_0^t\int_{\Gamma_{1,\varepsilon}} |\bar{\mathbf{u}}_\varepsilon|^2\,\mathrm{d}\bar\Gamma\,\mathrm{d}s \\
&+ q\int_0^t\int_{\Gamma_{2,\varepsilon}} |\bar{\mathbf{u}}_\varepsilon|^2\,\mathrm{d}\bar\Gamma\,\mathrm{d}s \\
&= \int_0^t\int_{\Omega_\varepsilon} \bar\rho_\varepsilon\bar{\mathbf{f}}_\varepsilon\cdot\bar{\mathbf{u}}_\varepsilon\,\mathrm{d}\bar x\,\mathrm{d}s + \int_{\Omega_\varepsilon}\Big(\frac{|(\bar\rho_\varepsilon\bar{\mathbf{u}}_\varepsilon)_0|^2}{2\bar\rho_{0,\varepsilon}} + \bar\rho_{0,\varepsilon}\ln(\bar\rho_{0,\varepsilon})\Big)\,\mathrm{d}\bar x.
\end{aligned} \quad (4.5)$$

By transforming (4.5), we obtain

$$\begin{aligned}
&\int_\Omega \Big(\rho_\varepsilon(t)\frac{|\mathbf{u}_\varepsilon(t)|^2}{2} + \rho_\varepsilon(t)\ln(\rho_\varepsilon(t))\Big)\,\mathrm{d}x \\
&+ \int_0^t\int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&+ \frac{h(\varepsilon)}{\varepsilon}\int_0^t\int_{\Gamma_1} |\mathbf{u}_\varepsilon|^2\,\mathrm{d}\Gamma\,\mathrm{d}s + q\int_0^t\int_{\Gamma_2}|\mathbf{u}_\varepsilon|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&= \int_0^t\int_\Omega \rho_\varepsilon\mathbf{g}_\varepsilon\cdot\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega\Big(\frac{|(\rho_\varepsilon\mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} + \rho_{0,\varepsilon}\ln(\rho_{0,\varepsilon})\Big)\,\mathrm{d}x.
\end{aligned} \quad (4.6)$$

for any $t \in \langle 0,T\rangle$, where

$$\mathbf{g}_\varepsilon = (f_{1,\varepsilon}, \varepsilon^{-1}f_{2,\varepsilon}, \varepsilon^{-1}f_{3,\varepsilon}), \quad \mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon}),$$

It is obvious that $\mathbf{g}_\varepsilon\cdot\mathbf{v}_\varepsilon = \mathbf{f}_\varepsilon\cdot\mathbf{u}_\varepsilon$, but we need to use this notation for making a priori estimates (see inequality 5.4), because a variant of Korn's inequality holds for $\mathbf{v}_\varepsilon$ (see Lemma 5.1).

4.3. **Related results.** It is necessary to mention that equations (4.2)-(4.3) with non-slip boundary conditions have a weak solution in a sufficiently regular domain for any $\varepsilon \in (0,1)$. Moreover, any weak solution satisfies the energy equality (4.6) and it can be constructed as a limit of Rothe approximations (see [17, Theorem 3.5]). The non-slip boundary conditions mean that surface integrals in (4.3) and (4.6) disappears. We remark that $\gamma > 7/2$ in [17], while our result was achieved for a slightly more general $\gamma > 3$.

According to [23], we can treat the case of slip boundary conditions similarly as the barotropic case [20]. In our case, we use the Navier boundary conditions (3.1)-(3.3), because the slip boundary conditions are their special case ($h(\varepsilon) = q = 0$) and the generalization poses no additional technical problems to the existence proof.

The case of non-slip boundary conditions would lead to the zero velocity in the limit. Thus, it was not an interesting choice of boundary conditions for us.

Since we are dealing with a domain which has a shape similar to a cylinder, the assumption on the regularity of the boundary of Ω can be relaxed by simplifying and slightly modifying the approach presented in [8].

5. Derivation of the limiting 1D equations

The first step of the proof concerns a variant of the first Korn's inequality. We need this inequality to perform a priori estimates in section 5.1 and subsequently show boundedness of $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)}$, and perform weak limits. Subsequently, we pass to limits in equations (4.2)-(4.3) in section 5.2. As the last step, we perform the limit passage also for the energy equality (4.6).

From [7], we know that for any $\mathbf{w} \in [W^{1,p}(\Omega)]^3$, $p \geq 2$, the following estimate holds

$$\|\mathbf{w}\|_{1,p} \leq C_1 \left(\|D\mathbf{w}\|_p + \|\mathbf{w}\|_p\right). \tag{5.1}$$

As a consequence, there exists constant $C_2 > 0$ such that

$$\|\mathbf{w}\|_{1,p} \leq C_2 \left(\|D\mathbf{w}\|_p + \|\mathbf{w}\|_{2,\Gamma}\right). \tag{5.2}$$

From inequality (5.2), we can easily deduce the following corollary. Without the loss of generality, we denote $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(t)$. Variable $t \in \langle 0, T\rangle$ is arbitrary but fixed.

Corollary 5.1. *Let $\mathbf{u}_\varepsilon \in [W^{1,p}(\Omega)]^3$, $p > 3$, be such that $\mathbf{u}_\varepsilon \cdot \mathbf{n} = 0$ on $\Gamma = \{0\} \times S$. We define $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon}) \in [W^{1,p}(\Omega)]^3$. Then, there exists $C = C(\Omega, p) > 0$, such that*

$$\|\mathbf{v}_\varepsilon\|_{1,p} \leq C \left(\|\omega_\varepsilon(\mathbf{u}_\varepsilon)\|_p + \|\mathbf{u}_\varepsilon\|_{2,\Gamma}\right), \quad \forall \varepsilon > 0, \tag{5.3}$$

where $\omega_\varepsilon(\mathbf{u}_\varepsilon)$ is defined by (4.1).

5.1. **Boundedness and weak limits.** Now, we make a priori estimates. Equation (4.2) implies the conservation of mass which can be expressed as

$$\int_\Omega \rho_\varepsilon(t)\,\mathrm{d}x = \int_\Omega \rho_{0,\varepsilon}\,\mathrm{d}x, \quad \forall t \in (0,T).$$

Therefore, the first integral on the right-hand side of the energy equality (4.6) can be estimated as follows

$$\begin{aligned} \Big|\int_0^t \int_\Omega \rho_\varepsilon \mathbf{g}_\varepsilon \cdot \mathbf{v}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s\Big| &\leq \int_0^t \|\mathbf{v}_\varepsilon(s)\|_\infty \|\mathbf{g}_\varepsilon(s)\|_\infty \int_\Omega \rho_\varepsilon(s)\,\mathrm{d}x\,\mathrm{d}s \\ &\leq C(\rho_{0,\varepsilon}, \mathbf{g}_\varepsilon) \int_0^t \|\mathbf{v}_\varepsilon(s)\|_{1,p}\,\mathrm{d}s, \quad p > 3. \end{aligned}$$

In the view of inequalities (3.6) and (5.3), and Young's inequality, we arrive at

$$\begin{aligned} \Big|\int_0^t \int_\Omega \rho_\varepsilon \mathbf{g}_\varepsilon \cdot \mathbf{v}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s\Big| &\leq C\Big(C_1 \int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + C_1 \int_0^t \int_{\{0\}\times S} |\mathbf{u}_\varepsilon|^2 \,\mathrm{d}S\,\mathrm{d}s + C_2(C_1)\Big), \end{aligned} \tag{5.4}$$

where $C_1 > 0$ can be made arbitrarily small.

By (3.6) and (5.4), we obtain from (4.6) boundedness

$$\{\sqrt{\rho_\varepsilon}\,|\mathbf{u}_\varepsilon|\}_{\varepsilon\in(0,1)} \quad \text{in } L^\infty\left(0,T; L^2(\Omega)\right), \tag{5.5}$$

$$\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \quad \text{in } L^\infty\left(0,T; L_{\Phi_1}(\Omega)\right), \tag{5.6}$$

$$\{\omega_\varepsilon(\mathbf{u}_\varepsilon)\}_{\varepsilon\in(0,1)} \quad \text{in } [\tilde{L}_M\,(\Omega\times(0,T))]^9, \tag{5.7}$$

$$\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)} \quad \text{in } L^p(0,T;[W^{1,p}(\Omega)]^3)\cap L^2(0,T;[L^2(\partial\Omega)]^3) \tag{5.8}$$

for any $p>3$. From (5.8), we obtain immediately the boundedness

$$\{u_{1,\varepsilon}\}_{\varepsilon\in(0,1)} \quad \text{in } L^p\left(0,T;W^{1,p}(\Omega)\right)\cap L^2\left(0,T;L^2(\partial\Omega)\right). \tag{5.9}$$

Boundedness (5.7) gives us the following convergences

$$\partial_2 u_{3,\varepsilon}+\partial_3 u_{2,\varepsilon}\to 0, \quad \partial_\alpha u_{\alpha,\varepsilon}\to 0, \quad \text{in } L_M(\Omega\times(0,T)),\ \alpha=2,3.$$

Now, we can prove even the boundedness of $\{\varepsilon^{-1}u_{\alpha,\varepsilon}\}_{\varepsilon\in(0,1)}$, $\alpha=2,3$, in $L_M(\Omega\times(0,T))$. Let us denote $\mathbf{w}_\varepsilon=(\varepsilon^{-1}u_{2,\varepsilon},\varepsilon^{-1}u_{3,\varepsilon})$. We begin with the Korn's inequality in a two-dimensional space (see [7]):

$$\|\mathbf{w}_\varepsilon\|_{W^{1,p}(S)}\le C_1\left(\|D\mathbf{w}_\varepsilon\|_{L^p(S)}+\|\mathbf{w}_\varepsilon\|_{L^p(S)}\right), \quad p>2, \tag{5.10}$$

where $x_1\in(0,1)$ and $t\in(0,T)$ are arbitrary but fixed. From (5.10) and axial non-symmetry of Ω, via the standard compactness argument (as in [20] for proving inequality (4.17.19)), we deduce

$$\|\mathbf{w}_\varepsilon\|_{L^p(S)}\le C_2\|D\mathbf{w}_\varepsilon\|_{L^p(S)}. \tag{5.11}$$

From the compact embedding of $W^{1,p}(S)$ in $L^\infty(S)$ and inequality (5.11), we can deduce from (5.10) the inequality

$$\|C_p\mathbf{w}_\varepsilon\|^p_{L^\infty(S)}\le\|D\mathbf{w}_\varepsilon\|^p_{L^p(S)}, \quad p>2,$$

where $C_p=C_p(\Omega,p)>0$. Applying Young function Ψ_p and Jensen's inequality gives us

$$\int_S\Psi_p(|C_p\mathbf{w}_\varepsilon|^p)\,\mathrm{d}x_2\,\mathrm{d}x_3\le C\int_S\Psi_p(|D\mathbf{w}_\varepsilon|^p)\,\mathrm{d}x_2\,\mathrm{d}x_3, \quad p>2.$$

Since $\Psi_p(z^p)$ behaves like $M(z)$, we arrive at

$$\int_S M(|C_p\mathbf{w}_\varepsilon|)\,\mathrm{d}x_2\,\mathrm{d}x_3\le C\Big(\int_S M(|D\mathbf{w}_\varepsilon|)\,\mathrm{d}x_2\,\mathrm{d}x_3+1\Big).$$

After integrating over $x_1\in(0,1)$ and $t\in(0,T)$, we obtain

$$\int_0^T\int_\Omega M(|C_p\mathbf{w}_\varepsilon|)\,\mathrm{d}x\,\mathrm{d}t\le C\Big(\int_0^T\int_\Omega M(|D\mathbf{w}_\varepsilon|)\,\mathrm{d}x\,\mathrm{d}t+T\Big). \tag{5.12}$$

Let us remark that (see [14, page 145])

$$C_p\|\mathbf{w}_\varepsilon\|_{L_M(\Omega\times(0,T))}\le\int_0^T\int_\Omega M(|C_p\mathbf{w}_\varepsilon|)\,\mathrm{d}x\,\mathrm{d}t+1. \tag{5.13}$$

Inequalities (5.12) and (5.13) give us

$$C_p\|\mathbf{w}_\varepsilon\|_{L_M(\Omega\times(0,T))}\le C\Big(\int_0^T\int_\Omega M(|D\mathbf{w}_\varepsilon|)\,\mathrm{d}x\,\mathrm{d}t+T\Big)+1. \tag{5.14}$$

The right-hand side of inequality (5.14) is bounded for any $\varepsilon\in(0,1)$ due to (5.7). Thus, it ensures the boundedness

$$\{\varepsilon^{-1}u_{\alpha,\varepsilon}\}_{\varepsilon\in(0,1)} \text{ in } L_M(\Omega\times(0,T)),\ \alpha=2,3. \tag{5.15}$$

Boundedness of $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^\infty(0,T;L_{\Phi_1}(\Omega))$ can be extended to the space $L^\infty(0,T;L_{\Phi_\gamma}(\Omega))$. We remind that $\gamma>3$ (see Theorem 6.1). We proceed in the following way. First, we test the equation (4.4) by function $\varphi = \varphi(t) \in \mathcal{C}_0^\infty(0,T)$ with $b(z) = \Phi_\gamma(z)$. We obtain

$$\int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon)\varphi'(t) + \left[\left(\Phi_\gamma(\rho_\varepsilon) - \rho_\varepsilon \Phi'_\gamma(\rho_\varepsilon)\right) \operatorname{div}_\varepsilon \mathbf{u}_\varepsilon\right] \varphi(t)\, \mathrm{d}x\, \mathrm{d}t = 0, \tag{5.16}$$

Function $\Phi_\gamma(z) - z\Phi'_\gamma(z)$ behaves asymptotically like $\Phi_{\gamma-1}(z)$. Furthermore, there exists $C_1 > 0$ such that $\Phi_1(\Phi_{\gamma-1}(z)) \le C_1(\Phi_\gamma(z) + 1)$ for $z \ge 0$ [23]. Due to equivalence of Young functions M and Ψ_1, relations (3.6), (5.7) and the Young's inequality, we deduce the estimate

$$\begin{aligned} &\Big| \int_0^T \int_\Omega \left(\Phi_\gamma(\rho_\varepsilon) - \rho_\varepsilon \Phi'_\gamma(\rho_\varepsilon)\right) \operatorname{div}_\varepsilon \mathbf{u}_\varepsilon \,\mathrm{d}x\, \mathrm{d}t\Big| \\ &\le C(T)\Big(\int_0^T \int_\Omega \left(\Phi_\gamma(\rho_\varepsilon) + P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2\right) \,\mathrm{d}x\, \mathrm{d}t + 1\Big), \end{aligned} \tag{5.17}$$

where $C(T) > 0$. With respect to (5.16), (5.17), (6.5) and the Gronwall's lemma, we obtain the boundedness of

$$\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^\infty(0,T; L_{\Phi_\gamma}(\Omega)). \tag{5.18}$$

In the following step, we focus on boundedness of $\{\partial_t \rho_\varepsilon\}_{\varepsilon\in(0,1)}$. Let us test equation (4.2) by function $\varphi(x,t) = \varphi_1(t)\psi(x)$, where $\varphi_1 \in L^{p'}(0,T)$, $1/p + 1/p' = 1$, $p > 3$, and $\psi \in [W^1 L_{\Psi_{\gamma-1}}(\Omega)]^3$, $\gamma > 3$. We can write

$$\begin{aligned} &\Big| \int_0^T \varphi'_1 \int_\Omega \rho_\varepsilon \psi \,\mathrm{d}x\, \mathrm{d}t\Big| \\ &= \Big| \int_0^T \varphi_1 \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_\varepsilon \psi \,\mathrm{d}x\, \mathrm{d}t\Big| \\ &= \Big| \int_0^T \varphi_1 \int_\Omega \rho_\varepsilon (u_{1,\varepsilon}\partial_1\psi + \varepsilon^{-1} u_{2,\varepsilon}\partial_2\psi + \varepsilon^{-1} u_{3,\varepsilon}\partial_3\psi) \,\mathrm{d}x\, \mathrm{d}t\Big|. \end{aligned} \tag{5.19}$$

From (5.15) and (5.19), we obtain the boundedness of

$$\{\partial_t \rho_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^p\left(0,T; [W^1 L_{\Psi_{\gamma-1}}(\Omega)]^*\right). \tag{5.20}$$

By using (5.6)-(5.8), (5.18), (5.20), compact embedding of $W^{1,p}(\Omega)$ in $E_{\Psi_1}(\Omega)$, continuous embedding of $[W^{1,p}(\Omega)]^*$ in $[W^1 L_{\Psi_\gamma}(\Omega)]^*$ and theorem on compact embedding (see [21, Lemma 9]), we obtain (passing to subsequences if necessary)

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right), \tag{5.21}$$

$$\rho_\varepsilon \to \rho \quad \text{in } \mathcal{C}(\langle 0,T\rangle; [W^{1,p}(\Omega)]^*), \tag{5.22}$$

$$\omega_\varepsilon(\mathbf{u}_\varepsilon) \overset{N}{\rightharpoonup} \zeta \tag{5.23}$$

$$u_{1,\varepsilon} \rightharpoonup u_1 \quad \text{in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right). \tag{5.24}$$

Let us recall that we already have (from (5.15))

$$u_{\alpha,\varepsilon} \to u_\alpha = 0 \quad \text{in } L_M(\Omega \times (0,T)),\ \alpha = 2,3. \tag{5.25}$$

Recalling the fact that Ω is not axially symmetric and that $\mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$, we arrive at (see [20, section 4.17.2.4]) $u_2 = u_3 = 0$ almost everywhere in $\Omega \times (0,T)$.

We prove that the limiting function $\mathbf{u}$ does not depend on the second and the third spatial variables. Boundedness (5.7) implies the following convergences

$$\varepsilon\partial_1 u_{\alpha,\varepsilon} + \partial_\alpha u_{1,\varepsilon} \to 0, \quad \text{in } L_M(\Omega \times (0,T)),\ \alpha = 2,3.$$

With respect to (5.25), we arrive at $\partial_\alpha u_1 = 0$ almost everywhere in $\Omega \times (0,T)$, $\alpha = 2,3$. Hence, we obtain $u_1 = u_1(x_1,t) \in L^p\left(0,T;W^{1,p}(0,1)\right)$ with $u_1(0,t) = u_1(1,t) = 0$, $t \in (0,T)$.

Now, we pay attention to convergences of nonlinear terms in equation (4.3). Convergences (passing to subsequences if necessary)

$$\rho_\varepsilon u_{1,\varepsilon} \rightharpoonup \rho u_1 \quad \text{in } L^p(0,T;L_{\Phi_\gamma}(\Omega)) \tag{5.26}$$

$$\rho_\varepsilon u_{\alpha,\varepsilon} \to \rho u_\alpha = 0 \quad \text{in } L_{\Phi_{\gamma-1}}(\Omega \times (0,T)),\ \alpha = 2,3, \tag{5.27}$$

where $\gamma > 3$ (see Theorem 6.1), are immediate consequences of (5.22), (5.24), (5.25) and theorem concerning compact embedding (see [21, Lemma 9]). For instance, we prove convergences (5.27). According to the Hölder's inequality, it holds that

$$\begin{aligned}\|\rho_\varepsilon u_{\alpha,\varepsilon}\|_{L_{\Phi_{\gamma-1}}(\Omega\times(0,T))} &= \sup_\varphi \int_0^T \int_\Omega |\rho_\varepsilon u_{\alpha,\varepsilon}\varphi| \,\mathrm{d}x\,\mathrm{d}t \\ &\le C\|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))} \sup_\varphi \|\rho_\varepsilon\varphi\|_{L_N(\Omega\times(0,T))}\end{aligned}$$

where the supremum is taken over all functions $\varphi \in \tilde{L}_{\Psi_{\gamma-1}}(\Omega \times (0,T))$ such that

$$\int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|) \,\mathrm{d}x\,\mathrm{d}t \le 1.$$

From (5.25), we already know that $\|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))} \to 0$. Therefore, it is sufficient to show the boundedness of $\|\rho_\varepsilon\varphi\|_{L_N(\Omega\times(0,T))}$ for proving (5.27). The equivalence of Orlicz spaces L_N and L_{Φ_1}, and the Young's inequality give us

$$\begin{aligned}\|\rho_\varepsilon\varphi\|_{L_N(\Omega\times(0,T))} &\le \int_0^T \int_\Omega \Phi_1(\rho_\varepsilon|\varphi|) \,\mathrm{d}x\,\mathrm{d}t + C \\ &\le \int_0^T \int_\Omega \rho_\varepsilon \Phi_1(|\varphi|) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega |\varphi|\Phi_1(\rho_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t + C.\end{aligned} \tag{5.28}$$

The second integral on the right-hand side of (5.28) is "the worst" and it is less or equal than

$$\begin{aligned}&\int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \Phi_{\gamma-1}(\Phi_1(\rho_\varepsilon)) \,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|) \,\mathrm{d}x\,\mathrm{d}t + C\int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Hence, we conclude that convergences (5.27) hold.

To overcome the second term on the left-hand side in equation (4.3), we consider "the worst integrals" in (4.3) and prove their boundedness for any $\varepsilon \in (0,1)$. First, we show that (5.9), (5.18) and (5.25) lead to boundedness of

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) \,\mathrm{d}x\,\mathrm{d}t \tag{5.29}$$

for any $\varepsilon \in (0,1)$ and ψ such that $[\omega_\varepsilon(\psi(x,t))]_{\alpha\beta} = \varepsilon^{-1}\varphi(t)\bar{\psi}(x)$ with $\varphi \in L^q(0,T)$ and $\bar{\psi} \in [E_{\Psi_\gamma}(\Omega)]^9$, where $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$, $2/p + 1/q = 1$ and $\gamma > 3$ (see Theorem 6.1).

There are three types of terms in (5.29), but we analyze only "the worst one": $\rho_\varepsilon u_{\alpha,\varepsilon} u_{\beta,\varepsilon}[\omega_\varepsilon(\psi)]_{\alpha\beta}$, $\alpha,\beta = 2,3$. By using Hölder's inequality, we obtain

$$\begin{aligned}
&\Big|\int_0^T \int_\Omega \rho_\varepsilon u_{\alpha,\varepsilon} u_{\beta,\varepsilon} \varepsilon^{-1} \varphi \bar{\psi}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \|\varepsilon^{-1} u_{\alpha,\varepsilon} u_{\beta,\varepsilon}\|_{L_{\Psi_2}(\Omega\times(0,T))} \|\rho_\varepsilon \varphi \bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))}
\end{aligned} \tag{5.30}$$

Both norms on the right-hand side of inequality (5.30) are bounded. Regarding $\|\varepsilon^{-1} u_{\alpha,\varepsilon} u_{\beta,\varepsilon}\|_{L_{\Psi_2}(\Omega\times(0,T))}$, it holds that

$$\begin{aligned}
\|\varepsilon^{-1} u_{\alpha,\varepsilon} u_{\beta,\varepsilon}\|_{L_{\Psi_2}(\Omega\times(0,T))} &\le \sup_{\varphi_1} \int_0^T \int_\Omega |\varepsilon^{-1} u_{\alpha,\varepsilon} u_{\beta,\varepsilon} \varphi_1|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))} \sup_{\varphi_1} \|\varepsilon^{-1} u_{\beta,\varepsilon} \varphi_1\|_{L_N(\Omega\times(0,T))},
\end{aligned}$$

where $\varphi_1 \in \tilde{L}_{\Phi_2}(\Omega\times(0,T))$ such that $\int_0^T \int_\Omega \Phi_2(|\varphi_1|)\,\mathrm{d}x\,\mathrm{d}t \le 1$. From (5.25), we know that $\|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))} \to 0$. Further, we can write

$$\begin{aligned}
\|\varepsilon^{-1} u_{\beta,\varepsilon} \varphi_1\|_{L_N(\Omega\times(0,T))} &= \sup_{\varphi_2} \int_0^T \int_\Omega |\varepsilon^{-1} u_{\beta,\varepsilon} \varphi_1 \varphi_2|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \|\varepsilon^{-1} u_{\beta,\varepsilon}\|_{L_M(\Omega\times(0,T))} \sup_{\varphi_2} \|\varphi_1\varphi_2\|_{L_N(\Omega\times(0,T))},
\end{aligned}$$

where the first norm is bounded (see 5.15) and $\varphi_2 \in \tilde{L}_M(\Omega\times(0,T))$ such that $\int_0^T \int_\Omega M(|\varphi_2|)\,\mathrm{d}x\,\mathrm{d}t \le 1$. Finally,

$$\|\varphi_1\varphi_2\|_{L_N(\Omega\times(0,T))} \le C_1 \int_0^T \int_\Omega \Phi_1(|\varphi_1\varphi_2|)\,\mathrm{d}x\,\mathrm{d}t + 1,$$

where the integral on the right-hand side is less than or equal to

$$\begin{aligned}
&C_2\Big(\int_0^T \int_\Omega |\varphi_2|\Phi_1(|\varphi_1|) + |\varphi_1|\Phi_1(|\varphi_2|)\,\mathrm{d}x\,\mathrm{d}t\Big) \\
&\le C_3\Big(\int_0^T \int_\Omega M(|\varphi_2|)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \Phi_2(|\varphi_1|)\,\mathrm{d}x\,\mathrm{d}t\Big) \le 2C_3
\end{aligned}$$

Concerning the second norm on the right-hand side of inequality (5.30), we justify its boundedness in the following way

$$\begin{aligned}
\|\rho_\varepsilon \varphi \bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))} &\le \int_0^T \int_\Omega \Phi_2(\rho_\varepsilon |\varphi||\bar{\psi}|)\,\mathrm{d}x\,\mathrm{d}t + C_1 \\
&\le \int_0^T \int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \rho_\varepsilon \Phi_2(|\varphi||\bar{\psi}|)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_0^T \int_\Omega \Phi_1(\rho_\varepsilon)\Phi_1(|\varphi||\bar{\psi}|)\,\mathrm{d}x\,\mathrm{d}t + C_2,
\end{aligned}$$

where "the worst term" can be estimated as

$$\begin{aligned}
&\int_0^T \int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\|\varphi\|_{L^1(0,T)}\Big(\Big\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\,\mathrm{d}x\Big\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma-2}(|\bar{\psi}|)\,\mathrm{d}x\Big).
\end{aligned}$$

We conclude that integral (5.29) is bounded for any $\varepsilon \in (0,1)$ and test function ψ such that $\varphi \in L^q(0,T)$ and $\bar{\psi} \in [E_{\Psi_\gamma}(\Omega)]^9$. Subsequently, we show that also

$$\int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi) \,\mathrm{d}x \,\mathrm{d}t \tag{5.31}$$

is bounded for any $\varepsilon \in (0,1)$ and $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, where $\varphi \in E_{\Psi_{1/\alpha}}(0,T)$, $\alpha > 2$, and $\bar{\psi} \in [W^1 E_{\Psi_{1/2}}(\Omega)]^3$, $\partial_2\bar{\psi} = \partial_3\bar{\psi} = 0$. We remark that

$$\omega_\varepsilon(\bar{\psi}) = \omega(\bar{\psi}) = \begin{pmatrix} \partial_1\bar{\psi}_1 & \frac{1}{2}\partial_1\bar{\psi}_2 & \frac{1}{2}\partial_1\bar{\psi}_3 \\ \cdot & 0 & 0 \\ \mathrm{sym} & \cdot & 0 \end{pmatrix},$$

which is not longer dependent on ε. By Young's inequality, it holds that

$$\begin{aligned} &\Big| \int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega(\bar{\psi})\varphi \,\mathrm{d}x \,\mathrm{d}t \Big| \\ &\le \Big(|\Omega| \int_0^T \Psi_{1/\alpha}(|\varphi|) \,\mathrm{d}t + \int_0^T \int_\Omega \Phi_{1/\alpha}\big(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)||\omega(\bar{\psi})|\big) \,\mathrm{d}x \,\mathrm{d}t \Big), \end{aligned} \tag{5.32}$$

where $\alpha > 2$. For brevity, let us denote $w_\varepsilon = P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|$. Since $w_\varepsilon \in L_{\Phi_1}(\Omega \times (0,T))$ implies $w_\varepsilon \in L_{\Phi_{(\alpha-1)/\alpha}}(0,T; L_{\Phi_{1/\alpha}}(\Omega))$, which follows from Jensen's inequality and estimate

$$\Phi_{(\alpha-1)/\alpha}(\Phi_{1/\alpha}(z)) \le 2\Phi_1(z) + C, \quad z \ge 0,$$

the second term on the right-hand side of (5.32) is less than or equal to

$$\begin{aligned} &\int_0^T \int_\Omega |\omega(\bar{\psi})|\Phi_{1/\alpha}(w_\varepsilon) + w_\varepsilon \Phi_{1/\alpha}(|\omega(\bar{\psi})|) \,\mathrm{d}x \,\mathrm{d}t \\ &\le \int_0^T \int_\Omega \Phi_{(\alpha-1)/\alpha}(\Phi_{1/\alpha}(w_\varepsilon)) + \Psi_{(\alpha-1)/\alpha}(|\omega(\bar{\psi})|) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \Phi_1(w_\varepsilon) + \Psi_1(\Phi_{1/\alpha}(|\omega(\bar{\psi})|)) \,\mathrm{d}x \,\mathrm{d}t \\ &\le 3 \int_0^T \int_\Omega \Phi_1\left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \Psi_{(\alpha-1)/\alpha}\left(|\omega(\bar{\psi})|\right) \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega \Psi_{1/2}\left(|\omega(\bar{\psi})|\right) \,\mathrm{d}x \,\mathrm{d}t + C, \end{aligned}$$

where $\alpha > 2$. By property (3.9), we conclude integral (5.31) is bounded.

Terms (5.29) and (5.31) represent "the worst integrals" in (4.3). Thus, we omit the estimates of the others and take $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, where $\varphi \in E_{\Psi_{1/\alpha}}(0,T)$ with $\alpha > 2$, and $\bar{\psi} \in [W^1 E_{\Psi_{1/2}}(\Omega)]^3$ such that $\bar{\psi} = (\bar{\psi}_1(x_1), 0, 0)$ and complies with $\bar{\psi}_1(0) = \bar{\psi}_1(1) = 0$, as a test function. By using estimates (5.29) and (5.31), we demonstrate how to perform a limit passage in the second term on the left-hand side of equation (4.3). Let us test the equation (4.3) by function $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, where $\varphi \in \mathcal{C}_0^\infty(0,T)$ and $\bar{\psi} \in [W^1 E_{\Psi_{1/2}}(\Omega)]^3$, $\bar{\psi} = (\bar{\psi}_1(x_1), 0, 0)$ and

$\bar\psi_1(0) = \bar\psi_1(1) = 0$. We obtain

$$
\begin{aligned}
&\Big|\int_0^T \varphi' \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \bar\psi \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \int_0^T |\varphi| \int_\Omega \Big(|\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega(\bar\psi)| + |\rho_\varepsilon \partial_1 \bar\psi| \\
&\quad + |P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega(\bar\psi)| + |\rho_\varepsilon \mathbf{f}_\varepsilon \cdot \bar\psi|\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{h(\varepsilon)}{\varepsilon}\int_0^T |\varphi| \int_{\Gamma_1} |\mathbf{u}_\varepsilon \cdot \bar\psi| \,\mathrm{d}\Gamma\,\mathrm{d}t + q\int_0^T |\varphi| \int_{\Gamma_2} |\mathbf{u}_\varepsilon \cdot \bar\psi| \,\mathrm{d}\Gamma\,\mathrm{d}t,
\end{aligned}
\tag{5.33}
$$

Using the denseness of $\mathcal{C}_0^\infty(0,T)$ in $E_{\Psi_{1/2}}(0,T)$, the embedding $L_{\Psi_{1/\alpha}}(0,T) \hookrightarrow E_{\Psi_{1/2}}(0,T) \subset \tilde L_{\Psi_{1/2}}(0,T)$, $\alpha > 2$, and the boundedness of all terms on the right-hand side of inequality (5.33) –see (5.29) and (5.31), we deduce the boundedness

$$
\{\partial_t \int_S \rho_\varepsilon \mathbf{u}_\varepsilon \,\mathrm{d}x_2 x_3\}_{\varepsilon\in(0,1)} \quad \text{in } L_{\Phi_{1/\alpha}}(0,T;([W^1 L_{\Psi_{1/2}}(0,1)]^*)^3). \tag{5.34}
$$

By using (5.26), (5.34), compact embedding of $W^{1,p}(0,1)$ in $E_{\Psi_1}(0,1)$, continuous embedding of $[W^{1,p}(0,1)]^*$ in $[W^1 L_{\Psi_1}(0,1)]^*$ and theorem concerning compact embedding (see [21, Lemma 9]), we obtain (passing to subsequences if necessary)

$$
\int_S \rho_\varepsilon \mathbf{u}_\varepsilon \,\mathrm{d}x_2\,\mathrm{d}x_3 \to \int_S \rho\mathbf{u} \,\mathrm{d}x_2\,\mathrm{d}x_3 \quad \text{in } \mathcal{C}(\langle 0,T\rangle; ([W^{1,p}(0,1)]^*)^3). \tag{5.35}
$$

To perform a limit passage in the second term on the left-hand side of equation (4.3), we need the following lemma which can be proven in a similar way as [24, Proposition 3.2] and [1, Lemma 6.2].

Lemma 5.2. *Assume that $\{\mathbf{u}_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfies condition (5.7) and $\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)}$, where $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon})$, satisfies condition (5.8). Then for any $p > 3$ (passing to a subsequence if necessary), it holds that*

$$
\|u_{1,\varepsilon} - \frac{1}{|S|}\int_S u_{1,\varepsilon} \,\mathrm{d}x_2\,\mathrm{d}x_3\|_{L^p(0,T;L^\infty(\Omega))} \to 0, \text{ for } \varepsilon \to 0. \tag{5.36}
$$

This lemma can be applied the following way. It holds that

$$
\begin{aligned}
&\int_0^T \int_0^1 \int_S \rho_\varepsilon u_{1,\varepsilon} u_{1,\varepsilon} \psi \,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}t \\
&= \int_0^T \int_0^1 \int_S \rho_\varepsilon u_{1,\varepsilon}(u_{1,\varepsilon} - \frac{1}{|S|}\int_S u_{1,\varepsilon} \,\mathrm{d}x_2\,\mathrm{d}x_3)\psi \,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}t \\
&\quad + \int_0^T \int_0^1 \Big(\int_S \rho_\varepsilon u_{1,\varepsilon} \,\mathrm{d}x_2\,\mathrm{d}x_3\Big)\Big(\frac{1}{|S|}\int_S u_{1,\varepsilon} \,\mathrm{d}x_2\,\mathrm{d}x_3\Big)\psi \,\mathrm{d}x_1\,\mathrm{d}t,
\end{aligned}
$$

where $\psi \in \mathcal{C}_0^\infty(0,T;\mathcal{C}^\infty(\bar\Omega))$, $\partial_2\psi = \partial_3\psi = 0$. The first integral on the right-hand side tends to zero for $\varepsilon \to 0$ due to Lemma 5.2. Concerning the second integral, we apply strong convergence (5.35) and weak convergence

$$
\int_S u_{1,\varepsilon} \,\mathrm{d}x_2\,\mathrm{d}x_3 \rightharpoonup \int_S u_1 \,\mathrm{d}x_2\,\mathrm{d}x_3 \text{ in } L^p(0,T;W^{1,p}(0,1)),
$$

which follows from (5.24). In addition, it holds that

$$
\int_S \rho u_1 \,\mathrm{d}x_2\,\mathrm{d}x_3 = \hat\rho u_1, \quad \int_S u_1 \,\mathrm{d}x_2\,\mathrm{d}x_3 = u_1,
$$

where $\hat{\rho} = \int_S \rho \, \mathrm{d}x_2 \, \mathrm{d}x_3$, because $\mathbf{u}$ is independent of x_2 and x_3. Hence, the convergence

$$\int_0^T \int_\Omega \rho_\varepsilon u_{1,\varepsilon} u_{1,\varepsilon} \psi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} u_1 u_1 \psi \, \mathrm{d}x_1 \, \mathrm{d}t \tag{5.37}$$

is an immediate consequence of (5.24), (5.35) and (5.36). Convergence (5.37) is applied in the next section to overcome the nonlinearity in the second term on the left-hand side of (4.3)

5.2. **Limit of the governing equations.** Now, we can perform limit passages in (4.2) and (4.3). Throughout this section, we denote an integral of a function in the second and third spatial variable over set S by symbol " $\hat{}$ " over the function. Obviously, these integrals depend only on x_1. For example, we write $\hat{\rho} = \int_S \rho \, \mathrm{d}x_2 \, \mathrm{d}x_3$.

We remark that prescribed behavior (6.5) enables us to use the Gronwall's lemma in the proof of boundedness (5.18). Further, we assume that $h(\varepsilon) > 0$ in (4.3) satisfies the condition $h(\varepsilon) \sim O(\varepsilon)$ to ensure the convergence of $\frac{h(\varepsilon)}{\varepsilon}$ to a real positive number.

First, we test the equation (4.2) by function $\varphi \in \mathcal{D}(\mathbb{R} \times \langle 0, T \rangle)$. We arrive at

$$\int_0^T \int_\Omega \rho_\varepsilon \partial_t \varphi + \rho_\varepsilon u_{1,\varepsilon} \partial_1 \varphi \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Subsequently, we perform the limit passage for $\varepsilon \to 0$, apply convergences (5.22) and (5.26), and get

$$\int_0^T \int_0^1 \hat{\rho} \partial_t \varphi + \hat{\rho} u_1 \partial_1 \varphi \, \mathrm{d}x_1 \, \mathrm{d}t = 0, \tag{5.38}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R} \times \langle 0, T \rangle\right)$.

Second, we test the equation (4.3) by function $\psi = (\psi_1(x_1, t), 0, 0)$, where $\psi_1 \in \mathcal{C}_0^\infty(0, T; \mathcal{C}^\infty(\langle 0, 1 \rangle))$ complies with $\psi_1(0, t) = \psi_1(1, t) = 0$, for all $t \in (0, T)$. We will show the limit passage for each term in (4.3) separately.

(a) $\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi$. Since convergence (5.26) holds, we obtain

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} u_1 \partial_t \psi_1 \, \mathrm{d}x_1 \, \mathrm{d}t,$$

as $\varepsilon \to 0$.

(b) $\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi)$. From the definition of the test function ψ, we know that

$$\omega_\varepsilon(\psi) = \omega(\psi) = \begin{pmatrix} \partial_1 \psi_1 & 0 & 0 \\ \cdot & 0 & 0 \\ \mathrm{sym} & \cdot & 0 \end{pmatrix}. \tag{5.39}$$

After applying convergence (5.37), we conclude that

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} u_1 u_1 \partial_1 \psi_1 \, \mathrm{d}x_1 \, \mathrm{d}t,$$

as $\varepsilon \to 0$.

(c) $\rho_\varepsilon \operatorname{div}_\varepsilon \psi$. Since $\operatorname{div}_\varepsilon \psi = \partial_1 \psi_1$, we have (see convergence (5.22))

$$\int_0^T \int_\Omega \rho_\varepsilon \operatorname{div}_\varepsilon \psi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} \partial_1 \psi_1 \, \mathrm{d}x_1 \, \mathrm{d}t,$$

as $\varepsilon \to 0$.

(d) $P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon):\omega_\varepsilon(\psi)$. It holds that

$$\int_0^T\int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon):\omega_\varepsilon(\psi)\,\mathrm{d}x\,\mathrm{d}t\to\int_0^T\int_\Omega\overline{P(|\zeta|)\zeta}:\omega(\psi)\,\mathrm{d}x\,\mathrm{d}t,$$

as $\varepsilon\to 0$, where $\omega(\psi)$ is defined by (5.39) and

$$\zeta=\begin{pmatrix}\partial_1 u_1 & \zeta_{12} & \zeta_{13}\\ \cdot & \zeta_{22} & \zeta_{23}\\ \mathrm{sym} & \cdot & \zeta_{33}\end{pmatrix}. \tag{5.40}$$

Later, we will show that

$$\int_0^t\int_\Omega\overline{P(|\zeta|)\zeta}:\omega(\psi)\,\mathrm{d}x\,\mathrm{d}s=|S|\int_0^t\int_0^1 P(|\partial_1 u_1|)\partial_1 u_1\partial_1\psi_1\,\mathrm{d}x_1\,\mathrm{d}s,$$

for any $t\in(0,T)$.

(e) $\rho_\varepsilon\mathbf{f}_\varepsilon\cdot\psi$. Since $\mathbf{f}_\varepsilon\cdot\psi=f_{1,\varepsilon}\psi_1$, we obtain

$$\int_0^T\int_\Omega\rho_\varepsilon\mathbf{f}_\varepsilon\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\to\int_0^T\int_0^1\widehat{\rho f_1}\psi_1\,\mathrm{d}x_1\,\mathrm{d}t,$$

as $\varepsilon\to 0$, where f_1 denotes the limit of $f_{1,\varepsilon}$.

(f) $\frac{h(\varepsilon)}{\varepsilon}\mathbf{u}_\varepsilon\cdot\psi$. According to the supposed behavior of $h(\varepsilon)$, i. e. $h(\varepsilon)\sim O(\varepsilon)$, we can use convergence (5.24) to derive

$$\varepsilon^{-1}\int_0^T\int_{\Gamma_1}h(\varepsilon)\mathbf{u}_\varepsilon\cdot\psi\,\mathrm{d}\Gamma\,\mathrm{d}t\to|\partial S|h\int_0^T\int_0^1 u_1\psi_1\,\mathrm{d}x_1\,\mathrm{d}t$$

as $\varepsilon\to 0$, where h is a positive constant.

(g) $\mathbf{u}_\varepsilon\cdot\psi$. Using (5.24), we arrive at

$$\int_0^T\int_{\Gamma_2}\mathbf{u}_\varepsilon\cdot\psi\,\mathrm{d}\Gamma\,\mathrm{d}t=\int_0^T\int_{\Gamma_2}u_{1,\varepsilon}\psi_1\,\mathrm{d}\Gamma\,\mathrm{d}t\to 0,$$

as ε tends to zero, because $u_1(0,t)=u(1,t)=0$, for all $t\in(0,T)$.

Finally, we arrive at

$$\begin{aligned}&\int_0^T\int_0^1\hat\rho u_1\partial_t\psi_1+\hat\rho u_1^2\partial_1\psi_1+\hat\rho\partial_1\psi_1\,\mathrm{d}x_1\,\mathrm{d}t\\&=\int_0^T\int_\Omega\overline{P(|\zeta|)\zeta}:\omega(\psi)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_0^1\widehat{\rho f_1}\psi_1\,\mathrm{d}x_1\,\mathrm{d}t\\&\quad+|\partial S|h\int_0^T\int_0^1 u_1\psi_1\,\mathrm{d}x_1\,\mathrm{d}t.\end{aligned}\tag{5.41}$$

5.3. **Limit of the energy equality.** Applying similar approach as in section 5.2, we perform the limit passage for $\varepsilon\to 0$ also in the energy equality (4.6). We arrive at the inequality

$$\begin{aligned}&\int_0^1\hat\rho\frac{|u_1|^2}{2}+\hat\rho\ln(\hat\rho)\,\mathrm{d}x_1+\int_0^t\int_\Omega\overline{P\left(|\zeta|\right)|\zeta|^2}\,\mathrm{d}x\,\mathrm{d}s\\&+|\partial S|h\int_0^t\int_0^1|u_1|^2\,\mathrm{d}x_1\,\mathrm{d}s\\&\le\int_0^t\int_0^1\widehat{\rho f_1}u_1\,\mathrm{d}x_1\,\mathrm{d}s+\int_0^1\frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\,\mathrm{d}x_1+\int_0^1\rho_0\ln(\rho_0)\,\mathrm{d}x_1.\end{aligned}\tag{5.42}$$

By using a similar procedure as in [17, Lemmas 3.2 and 3.3], based on the renormalized continuity equation and the Steklov function, we derive the energy equality

$$
\begin{aligned}
&\int_0^1 \hat{\rho}\frac{|u_1|^2}{2} + \hat{\rho}\ln(\hat{\rho})\,\mathrm{d}x_1 + \int_0^t \int_\Omega \overline{P\left(|\zeta|\right)\zeta} : \omega(\mathbf{u})\,\mathrm{d}x_1\,\mathrm{d}s \\
&+ |\partial S| h \int_0^t \int_0^1 |u_1|^2\,\mathrm{d}x_1\,\mathrm{d}s \\
&= \int_0^t \int_0^1 \widehat{\rho f_1} u_1\,\mathrm{d}x_1\,\mathrm{d}s + \int_0^1 \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\,\mathrm{d}x_1 + \int_0^1 \rho_0\ln(\rho_0)\,\mathrm{d}x_1.
\end{aligned}
\tag{5.43}
$$

from (5.38) and (5.41), where $\omega(\mathbf{u})$ is defined in the same way as $\omega(\psi)$ in relation (5.39). It means that its only nonzero term is $[\omega(\mathbf{u})]_{11} = \partial_1 u_1$.

Since the function $P(|z|)z$ is monotone, we obtain

$$
\begin{aligned}
0 &\le \lim_{\varepsilon\to 0}\int_0^t \int_\Omega \left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)\omega_\varepsilon(\mathbf{u}_\varepsilon) - P(|T|)T\right) : \left(\omega_\varepsilon(\mathbf{u}_\varepsilon) - T\right)\,\mathrm{d}x\,\mathrm{d}s \\
&= \lim_{\varepsilon\to 0}\int_0^t \int_\Omega P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : T + P(|T|)T : \zeta + P(|T|)|T|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{5.44}
$$

for any symmetric $T \in [\tilde{L}_M(\Omega\times(0,T))]^9$. As a consequence of (4.6), (5.43), convexity and Jensen's inequality, we arrive at

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&= \lim_{\varepsilon\to 0}\Big(- \int_\Omega \rho_\varepsilon\frac{|\mathbf{u}_\varepsilon|^2}{2} + \rho_\varepsilon\ln(\rho_\varepsilon)\,\mathrm{d}x \\
&\quad - \frac{h(\varepsilon)}{\varepsilon}\int_0^t \int_{\Gamma_1} |\mathbf{u}_\varepsilon|^2\,\mathrm{d}\Gamma\,\mathrm{d}s - q\int_0^t \int_{\Gamma_2} |\mathbf{u}_\varepsilon|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad + \int_0^t \int_\Omega \rho_\varepsilon\mathbf{f}_\varepsilon\cdot\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \frac{|(\rho_\varepsilon\mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\,\mathrm{d}x + \int_\Omega \rho_{0,\varepsilon}\ln(\rho_{0,\varepsilon})\,\mathrm{d}x\Big) \\
&\le - \int_0^1 \hat{\rho}\frac{|u_1|^2}{2} + \hat{\rho}\ln(\hat{\rho})\,\mathrm{d}\hat{x} - |\partial S| h\int_0^t \int_0^1 |u_1|^2\,\mathrm{d}x_1\,\mathrm{d}s \\
&\quad + \int_0^t \int_0^1 \widehat{\rho f_1} u_1\,\mathrm{d}x_1\,\mathrm{d}s + \int_0^1 \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\,\mathrm{d}x_1 + \int_0^1 \rho_0\ln(\rho_0)\,\mathrm{d}x_1 \\
&= \int_0^t \int_\Omega \overline{P\left(|\zeta|\right)\zeta} : \omega(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{5.45}
$$

Hence, from (5.44), we obtain

$$
0 \le \int_0^t \int_\Omega \left(\overline{P(|\zeta|)\zeta} - P(|T|)T\right) : \left(\omega(\mathbf{u}) - T\right)\,\mathrm{d}x\,\mathrm{d}s.
$$

Taking $T = \zeta + \lambda\omega(\psi)$ and $T = \zeta - \lambda\omega(\psi)$, for $\lambda > 0$, $\psi = (\psi_1, 0, 0)$, where $\psi_1 \in \mathcal{C}_0^\infty(0,T;\mathcal{C}^\infty(\bar{\Omega}))$ is such that $\partial_2\psi_1 = \partial_3\psi_1 = 0$ and $\psi_1(0,t) = \psi_1(1,t) = 0$, for all $t \in (0,T)$, we conclude that

$$\begin{aligned}\int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\,\mathrm{d}x\,\mathrm{d}s &= |S| \int_0^t \int_0^1 P(|\omega(\mathbf{u})|)\omega(\mathbf{u}) : \omega(\psi)\,\mathrm{d}x_1\,\mathrm{d}s \\ &= |S| \int_0^t \int_0^1 P(|\partial_1 u_1|)\partial_1 u_1 \partial_1 \psi_1\,\mathrm{d}x_1\,\mathrm{d}s.\end{aligned} \tag{5.46}$$

6. Conclusions

To sum it up, the limit equations together with the energy equality are given by the following formulas

$$\int_0^T \int_0^1 \hat\rho \partial_t \varphi + \hat\rho u_1 \partial_1 \varphi \,\mathrm{d}x_1\,\mathrm{d}t = 0, \tag{6.1}$$

for any $\varphi \in \mathcal{D}(\mathbb{R} \times \langle 0,T\rangle)$;

$$\begin{aligned}&\int_0^T \int_0^1 \hat\rho u_1 \partial_t \psi_1 + \hat\rho u_1^2 \partial_1 \psi_1 + \hat\rho \partial_1 \psi_1 \,\mathrm{d}x_1\,\mathrm{d}t \\ &= |S| \int_0^T \int_0^1 P(|\partial_1 u_1|)\partial_1 u_1 \partial_1 \psi_1\,\mathrm{d}x_1\,\mathrm{d}t - \int_0^T \int_0^1 \widehat{\rho f_1}\psi_1\,\mathrm{d}x_1\,\mathrm{d}t \\ &\quad + |\partial S| h \int_0^T \int_0^1 u_1 \psi_1\,\mathrm{d}x_1\,\mathrm{d}t,\end{aligned} \tag{6.2}$$

for any $\psi = (\psi_1(x_1),0,0)$, where $\psi_1 \in \mathcal{C}_0^\infty(0,T;\mathcal{C}^\infty(\langle 0,1\rangle))$ complies with $\psi_1(0,t) = \psi_1(1,t) = 0$, for all $t \in (0,T)$,

$$\begin{aligned}&\int_0^1 \hat\rho \frac{|u_1|^2}{2} + \hat\rho \ln(\hat\rho)\,\mathrm{d}x_1 + |S| \int_0^t \int_\Omega P(|\partial_1 u_1|)\,|\partial_1 u_1|^2\,\mathrm{d}x_1\,\mathrm{d}s \\ &+ |\partial S| h \int_0^t \int_0^1 |u_1|^2\,\mathrm{d}x_1\,\mathrm{d}s \\ &= \int_0^t \int_0^1 \widehat{\rho f_1} u_1\,\mathrm{d}x_1\,\mathrm{d}s + \int_0^1 \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\,\mathrm{d}x_1 + \int_0^1 \rho_0 \ln(\rho_0)\,\mathrm{d}x_1.\end{aligned} \tag{6.3}$$

Finally, we summarize our main result in the following theorem.

Theorem 6.1. *Let us assume that couples $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$, $\varepsilon \in (0,1)$, satisfying*

$$\begin{gathered}\rho_\varepsilon \in L^\infty(0,T;L_{\Phi_\gamma}(\Omega)), \\ \mathbf{v}_\varepsilon \in L^p(0,T;[W^{1,p}(\Omega)]^3) \cap L^2(0,T;[L^2(\partial\Omega)]^3),\end{gathered}$$

with $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon})$ and Ω being not axially symmetric, $\partial\Omega \in \mathcal{C}^{0,1}$, are weak solutions to equations (4.2)-(4.3) *and* (4.6) *with initial states $\rho_{0,\varepsilon} \in L_{\Phi_\gamma}(\Omega)$ and $\frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} \in L^1(\Omega)$ satisfying*

$$\int_S \rho_{0,\varepsilon} \ln(\rho_{0,\varepsilon})\,\mathrm{d}x_2\,\mathrm{d}x_3 \to \rho_0 \ln(\rho_0) \quad \text{in } L^1(0,1), \tag{6.4}$$

$$\int_S \Phi_\gamma(\rho_{0,\varepsilon})\,\mathrm{d}x_2\,\mathrm{d}x_3 \to \Phi_\gamma(\rho_0) \quad \text{in } L^1(0,1), \tag{6.5}$$

$$\int_S \frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\,\mathrm{d}x_2\,\mathrm{d}x_3 \to \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0} \quad \text{in } L^1(0,1), \tag{6.6}$$

for arbitrary but fixed $\gamma > 3$ and $p > 3$. In addition, we assume that Navier boundary conditions (3.1)-(3.3) *hold and $\omega_\varepsilon(\mathbf{u}_\varepsilon) \in [\tilde{L}_M(\Omega \times (0,T))]^9$.*

Further, we suppose that function P complies with conditions (3.6)-(3.10)*, $\mathbf{f}_\varepsilon \to \mathbf{f}$ in $[L^\infty(\Omega \times (0,T))]^3$, $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$, see* (3.1)*, and $q > 0$, see* (3.2)*. Then (passing to subsequences if necessary)*

$$\begin{gathered}
\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right), \\
\rho_\varepsilon \to \rho \quad \text{in } \mathcal{C}\left(\langle 0,T\rangle; [W^{1,p}(\Omega)]^*\right), \\
\omega_\varepsilon(\mathbf{u}_\varepsilon) \overset{N}{\rightharpoonup} \omega(\mathbf{u}) \\
u_{1,\varepsilon} \rightharpoonup u_1 \quad \text{in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right), \\
u_{\alpha,\varepsilon} \to 0 \quad \text{in } L_M(\Omega \times (0,T)),\ \alpha = 2,3.
\end{gathered}$$

In addition, the couple $(\hat{\rho}, u_1)$, where $u_1 = u_1(x_1)$ and $\hat{\rho} = \int_S \rho \,\mathrm{d}x_2\,\mathrm{d}x_3$, is a weak solution to the equations (6.1)-(6.2) *and complies with the energy equality* (6.3).

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Richard Andrášik
Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University Olomouc, tř. 17. listopadu 1192/12, 771 46 Olomouc, Czech Republic
E-mail address: andrasik.richard@gmail.com

Rostislav Vodák
Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University Olomouc, tř. 17. listopadu 1192/12, 771 46 Olomouc, Czech Republic
E-mail address: rostislav.vodak@upol.cz