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ISOCHRONOUS FAMILIES OF LIMIT CYCLES

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ABSTRACT. In this article we present a method for determining if the frequency of a family of periodic orbits remains constant when a parameter changes. Two-dimensional systems of ordinary and delayed differential equations are considered. Several examples are given.

1. INTRODUCTION

In dynamic systems with oscillations, it is of great interest to study the frequency of such oscillations. Clearly this knowledge will be of importance in the development of applications in engineering and other branches of technology.

For two-dimensional systems, the so-called isochronous centers have been studied in [4, 5, 10, 14, 13, 17]. These are centers for which all orbits have the same frequency. Related to isochronous centers are isochronous foci [9, 10, 16]. Here there are no periodic orbits and the time of return to a transversal cross-section to the orbit in the focus is studied.

The period of the emergent orbits of certain bifurcations has been studied in [8]. This is a particular case of a one-parameter dependent family of cycles. Such families have been studied for example in [6, 15]. Limit cycles of particular equations have also been studied, see [1].

In [3] it has been proved that, for certain second order equations with delay, bifurcation branches appearing in a neighborhood of Hopf bifurcation due to the delay are isochronous.

A central problem is to know when the frequency of the oscillation does not depend on a parameter of the system. In this work, a methodology is developed to determine if a periodic orbit in two-dimensional systems of differential equations and differential equations with delay changes its frequency by varying a parameter. We call isochronous orbit with respect to the parameter an orbit that behaves this way. We also call isochronous family of limit cycles a family of this type.

In a similar spirit to that used in applying the theory of Floquet in determining the stability of a cycle, the problem of determining whether a periodic orbit is isochronous is reduced to studying a linear system with periodic coefficients. In this case, the existence of certain periodic solutions of this new system, which we call auxiliary system, is equivalent to the isochronous character of the orbit.

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2. Auxiliary system

We begin by studying the case in which the periodic orbit is a limit cycle in a system of differential equations in the plane. Later, we will show the necessary modifications to treat the case of differential equations with delay.

Consider the following system of ordinary differential equations

$$\begin{aligned} \bar{x}_1' &= f_1(\bar{x}_1, \bar{x}_2, \mu), \\ \bar{x}_2' &= f_2(\bar{x}_1, \bar{x}_2, \mu). \end{aligned} \tag{2.1}$$

Here, μ is a real parameter, $\bar{x}_1$ and $\bar{x}_2$ are functions of the variable $\bar{t}$, and the real functions f_1 and f_2 are sufficiently differentiable. Suppose that this system has a limit cycle with frequency $\omega(\mu) > 0$. Calling $\bar{\mathbf{x}}_\mu = (\bar{x}_{1\mu}, \bar{x}_{2\mu})$ the cycle and $\mathbf{f} = (f_1, f_2)$ the field, we have

$$\bar{\mathbf{x}}_\mu' = \mathbf{f}(\bar{\mathbf{x}}_\mu, \mu), \quad \text{where} \quad \bar{\mathbf{x}}_\mu(\bar{t}) = \bar{\mathbf{x}}_\mu(\bar{t} + 2\pi/\omega(\mu)). \tag{2.2}$$

After a change of the independent variable to $t = \omega(\mu)\bar{t}$, (2.2) becomes

$$\omega(\mu)\mathbf{x}_\mu' = \mathbf{f}(\mathbf{x}_\mu, \mu), \tag{2.3}$$

where the 2π-periodic function $\mathbf{x}_\mu = (x_{1\mu}, x_{2\mu})$ is continuous in the new variable t.

If we call Ω the space of continuous functions from the unit circle S^1 to $\mathbb{R}^2$, then the orbit that is solution of system (2.3) provides a continuous application, $\mu \mapsto \mathbf{x}_\mu$, of some interval $I \subset \mathbb{R}$ into Ω. Without loss of generality, we can assume that the orbit surrounds the origin and suppose that it intersects the positive x_1-axis transversely at a point $\hat{x}_1(\mu)$. An additional condition must be set to define the $\mathbf{x}_\mu$ family due to the S^1-invariance of the periodic orbits. This can be done by imposing the condition that the orbits cross the x_1-axis at time $t = 0$ (of course for this construction we can take any straight line that passes through the origin). We then have

$$\mathbf{x}_\mu(0) = (\hat{x}_1(\mu), 0). \tag{2.4}$$

We assume that the function $\hat{x}_1(\mu)$ is differentiable with respect to μ. Thus, $\mathbf{x}_\mu$ is continuous in both μ and t.

Differentiating (2.3) with respect to μ and assuming that the frequency is constant; i.e., $\omega(\mu) = \omega_0$, and therefore $\partial\omega/\partial\mu = 0$, we obtain the following system of non-autonomous linear equations

$$\omega_0 \boldsymbol{\xi}_\mu' = \frac{\partial \mathbf{f}}{\partial \mathbf{x}_\mu} \boldsymbol{\xi}_\mu + \frac{\partial \mathbf{f}}{\partial \mu}, \tag{2.5}$$

where

$$\boldsymbol{\xi}_\mu(t) = (\xi_{1\mu}(t), \xi_{2\mu}(t)) = \Big(\frac{\partial x_{1\mu}}{\partial \mu}(t), \frac{\partial x_{2\mu}}{\partial \mu}(t)\Big), \tag{2.6}$$

and $\partial\mathbf{f}/\partial\mathbf{x}_\mu$ the Jacobian matrix of $\mathbf{f}$ with respect to $x_{1\mu}$ and $x_{2\mu}$. The coefficients are periodic functions of period 2π because they are evaluated on the periodic orbit $\mathbf{x}_\mu$. We call this system the auxiliary system.

The variation of the limit cycle with μ is such that at $t = 0$ it moves horizontally with velocity

$$\boldsymbol{\xi}_\mu(0) = \Big(\frac{\partial \hat{x}_1(\mu)}{\partial \mu}, 0\Big). \tag{2.7}$$

Here, $\boldsymbol{\xi}_\mu(0)$ is on the horizontal axis by the particular choice (2.4) we have taken.

Next theorem ensures that the problem of studying the isochronous character of the family of limit cycles of equation (2.1) is transformed into finding periodic solutions, of period 2π, of system (2.5).

Theorem 2.1. *Suppose that system* (2.1) *has a limit cycle for all points of some interval $I \subset \mathbb{R}$ of the parameter μ. Also, suppose that it intersects the horizontal axis transversely at $t = 0$. Then, the family is isochronous with respect to μ; i.e., the frequency does not depend on the parameter in the interval, if and only if for some $\mu = \mu_0$ the limit cycle has frequency ω_0, and for all μ in I, system* (2.5) *has a periodic solution of period 2π, with initial condition* (2.7).

Proof. Suppose that system (2.1) has a family of isochronous cycles in some interval of the parameter μ; that is, it has a periodic solution $\bar{\mathbf{x}}_\mu = (\bar{x}_{1\mu}, \bar{x}_{2\mu})$ of constant frequency $\omega(\mu) = \omega_0$. Then, system (2.3) becomes

$$\omega_0 \mathbf{x}'_\mu = \mathbf{f}(\mathbf{x}_\mu, \mu), \tag{2.8}$$

with $\mathbf{x}_\mu = (x_{1\mu}, x_{2\mu})$ periodic of period 2π. Differentiating with respect to μ, we obtain that functions (2.6) verify equations (2.5). Therefore, $\boldsymbol{\xi}_\mu = \partial \mathbf{x}_\mu / \partial \mu$ is the periodic solution of period 2π sought.

Conversely, suppose that, for each μ, system (2.5) has a solution $\boldsymbol{\xi}_\mu$ of period 2π with initial condition (2.7). Take $\mu = \mu_0$ in the interval I and $\mathbf{x}_{\mu_0}$ the periodic solution corresponding to that parameter value. Define $\mathbf{x}_\mu$ as follows:

$$\mathbf{x}_\mu = \mathbf{x}_{\mu_0} + \int_{\mu_0}^{\mu} \boldsymbol{\xi}_\nu \, d\nu. \tag{2.9}$$

As can be easily verified, this $\mathbf{x}_\mu$ is solution of (2.3) with the appropriate initial conditions and with $\omega(\mu) = \omega_0$ constant. □

Once system (2.5) is constructed, the next step is to find the periodic solution of period 2π that verifies condition (2.7). To find this orbit we proceed to compute a line determined by the end points, at time $t = 2\pi$, of the trajectories beginning on the axis ξ_1. This line can provide us with a method to ensure that the desired limit cycle does not exist. If the cycle exists, this line helps us to find it for a fixed value of μ.

Linear system (2.5) can be written as

$$\boldsymbol{\xi}'(t) = A(t)\boldsymbol{\xi}(t) + \mathbf{b}(t). \tag{2.10}$$

If we know a principal fundamental matrix solution, $\Phi(t)$, of the associated homogeneous system we have the following expression for the solution

$$\boldsymbol{\xi}(t) = \Phi(t)\boldsymbol{c} + \Phi(t) \int_0^t \Phi^{-1}(s)\mathbf{b}(s) ds, \tag{2.11}$$

where $\boldsymbol{c}$ is a constant initial condition.

As a consequence of linearity, the orbits that begin at the axis ξ_1; that is,

$$\boldsymbol{\xi}(0) = \Phi(0)\boldsymbol{c} = \boldsymbol{c} = (\xi_{01}, 0), \tag{2.12}$$

are found in time $t = 2\pi$ on a same line in the plane ξ_1-ξ_2. Calling $(\xi_1(2\pi), \xi_2(2\pi)) = (\xi_1^*, \xi_2^*)$ the end point of the path in the plane ξ_1-ξ_2, these points obey the parametric equation as a function of ξ_{01},

$$\begin{aligned} \xi_1^* &= \Phi_{11}\xi_{01} + \gamma_1, \\ \xi_2^* &= \Phi_{21}\xi_{01} + \gamma_2, \end{aligned} \tag{2.13}$$

where

$$(\gamma_1,\gamma_2)^T = \Phi(2\pi)\int_0^{2\pi} \Phi^{-1}(s)\mathbf{b}(s)ds, \tag{2.14}$$

and Φ_{ij} are the components of the matrix $\Phi(2\pi)$.

Because all the trajectories of system (2.5) that start at the ξ_1-axis end at the same line at $t = 2\pi$, then by plotting two of them we can determine this line. In cases where it does not cut the ξ_1-axis, we can ensure that the family of cycles is not isochronous. This situation can occur when the line is horizontal at a nonzero value of the variable ξ_2 (see Figure 1). However, crossing the ξ_1-axis does not guarantee the existence of periodic orbits as shown in the following examples. In this case, it is enough to find the orbit that starts from that crossing point to check its periodicity. In this work, all the trajectories of the auxiliary system have been found by means of numerical simulations.

Remark 2.2. We note that in finding the periodic orbit in the ξ_1-ξ_2 plane we not only prove that the family of orbits is isochronous, but we also find how the cycle changes in the direction of the x_1-axis as μ varies. As stated above, this axis could be changed by anyone through the origin, with a suitable synchronization of the family of cycles. Moreover, a direction that does not pass through the origin can be chosen by using as a line of synchronization a differentiable curve that passes through the origin and divides the plane into exactly two regions.

Remark 2.3. The concept of isochronous family of limit cycles could be generalized, in principle, for the case where the cycles are in a center. In this case we would call them isochronous family of periodic orbits in a center. We must explain that this concept has nothing to do with that of isochronous center, where the different orbits, for a same value of the parameter, have the same frequency. In addition, a family of the type studied in this work can be built in almost any center, with the only condition that the frequency of the orbits varies from orbit to orbit, as will be seen below. This construction could be of importance in applications in control theory.

Suppose that a system as (2.1) has a center. We will also suppose that the center is structurally stable. Without loss of generality, we can assume that the orbit is around the origin. Then, we can suppose, as in the case of limit cycles, that at $t = 0$ a particular periodic orbit crosses the x_1-axis at the point $\hat{x}_1$. The frequency of this orbit depends on the parameter μ and on the point $\hat{x}_1$. We call $\bar{\mathbf{x}}_{\mu,\hat{x}_1}$ that orbit. It satisfies the equation

$$\bar{\mathbf{x}}'_{\mu,\hat{x}_1} = \mathbf{f}(\bar{\mathbf{x}}_{\mu,\hat{x}_1},\mu), \quad \text{where} \quad \bar{\mathbf{x}}_{\mu,\hat{x}_1}(\bar{t}) = \bar{\mathbf{x}}_{\mu,\hat{x}_1}(\bar{t}+2\pi/\omega). \tag{2.15}$$

Analogously to the case of limit cycles, if we call $\omega(\mu,\hat{x}_1)$ to the frequency, then we can write

$$\omega(\mu,\hat{x}_1)\mathbf{x}'_{\mu,\hat{x}_1} = \mathbf{f}(\mathbf{x}_{\mu,\hat{x}_1},\mu). \tag{2.16}$$

The frequency of this cycle depends on the value of $\mu = \mu_0$ and on the point $\hat{x}_1 = x_{10}$ where it crosses the x_1-axis. If the frequency changes by varying $\hat{x}_1$; that is,

$$\frac{\partial\omega}{\partial\hat{x}_1}(\mu_0,x_{10}) \neq 0, \tag{2.17}$$

then, in a neighborhood of μ_0, we can find a function $\hat{x}_1(\mu)$ satisfying

$$\omega(\mu,\hat{x}_1(\mu)) = \omega_0. \tag{2.18}$$

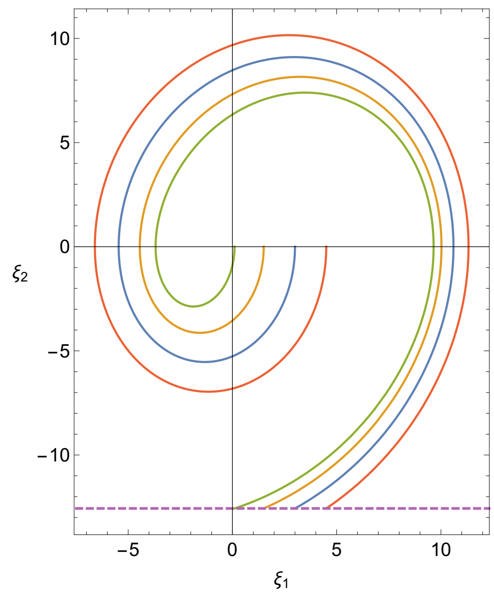

FIGURE 1. Orbits of system (2.22) for the harmonic oscillator with horizontal line $\xi_2 = -6.28$.

A family of isochronous periodic orbits can be defined by varying $\hat{x}_1$ with the function $\hat{x}_1(\mu)$. This family gives the following auxiliary equation

$$\omega_0 \boldsymbol{\xi}'_\mu = \frac{\partial \mathbf{f}}{\partial \mathbf{x}_{\mu,\hat{x}_1}} \boldsymbol{\xi}_\mu + \frac{\partial \mathbf{f}}{\partial \mu}. \tag{2.19}$$

Where

$$\boldsymbol{\xi}_\mu(t) = (\xi_{1\mu}(t), \xi_{2\mu}(t)) = \Big(\frac{\partial x_{1\mu,\hat{x}_1}}{\partial \mu}(t), \frac{\partial x_{2\mu,\hat{x}_1}}{\partial \mu}(t) \Big). \tag{2.20}$$

Two examples will suffice to clarify these points. First we consider the linear center in a harmonic oscillator of equation

$$\begin{aligned} x'_1 &= x_2, \\ x'_2 &= -\mu^2 x_1. \end{aligned} \tag{2.21}$$

The frequency depends on μ but not on the initial condition x_0, therefore there is no family of isochronous cycles. Equation (2.19) reduces to

$$\begin{aligned} \xi'_{1\mu} &= \xi_{2\mu}, \\ \xi'_{2\mu} &= -\mu^2 \xi_{1\mu} - 2x_{1\mu,x_0}\mu = -\mu^2 \xi_{1\mu} - 2x_0 \cos(\mu t). \end{aligned} \tag{2.22}$$

In Figure 1 we see that system (2.22) (with $\mu = 2$) has no periodic orbits, since the line is horizontal and does not cross the ξ_1-axis.

In contrast, if we consider the anharmonic oscillator of equation

$$\begin{aligned} x'_1 &= x_2, \\ x'_2 &= -x_1 - \mu x_1^3, \end{aligned} \tag{2.23}$$

equation (2.19) reduces to

$$\begin{aligned} \xi'_{1\mu} &= \xi_{2\mu}, \\ \xi'_{2\mu} &= -(1+3\mu x^2_{1\mu,x_0})\xi_{1\mu} - x^3_{1\mu,x_0}. \end{aligned} \tag{2.24}$$

In Figure 2 we can see the presence of a periodic orbit of period 2π starting from $\xi_1 = -1$. This negative value shows that the radius of the cycle decreases when μ increases.

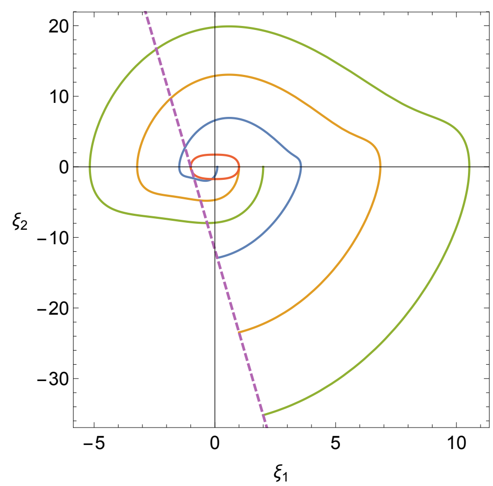

FIGURE 2. Orbits of system (2.24) for the anharmonic oscillator with line $\xi_2 = -11.726 - 11.726\xi_1$ and $\mu = 1$.

2.1. **Systems with angular velocity independent of the parameter.** One type of systems having a family of trivially isochronous cycles is one in which the angular velocity is constant and independent of μ. For example, system

$$\begin{aligned} x'_1 &= \mu x_1 - x_2 - x_1(2x_1^2 + x_2^2), \\ x'_2 &= x_1 + \mu x_2 - x_2(2x_1^2 + x_2^2), \end{aligned} \tag{2.25}$$

transformed to polar coordinates gives

$$\begin{aligned} r' &= \mu r - \Big(\frac{3}{2} + \cos(2\theta)\Big) r^3, \\ \theta' &= 1. \end{aligned} \tag{2.26}$$

In this case equation (2.5) becomes

$$\begin{aligned} \xi'_{1\mu} &= (-6x^2_{1\mu} - x^2_{2\mu} + \mu)\xi_{1\mu} - (1 + 2x_{1\mu}x_{2\mu})\xi_{2\mu} + x_{1\mu}, \\ \xi'_{2\mu} &= (1 - 4x_{1\mu}x_{2\mu})\xi_{1\mu} - (2x^2_{1\mu} + 3x^2_{2\mu} + \mu)\xi_{2\mu} + x_{2\mu}. \end{aligned} \tag{2.27}$$

In Figure 3 we see the cycle to which the trajectories that start from the ξ_1-axis converge ($\mu = 1$ was taken). As it can be seen, line (2.13) coincides with the ξ_1-axis. This occurs in all cases of systems with constant angular velocity.

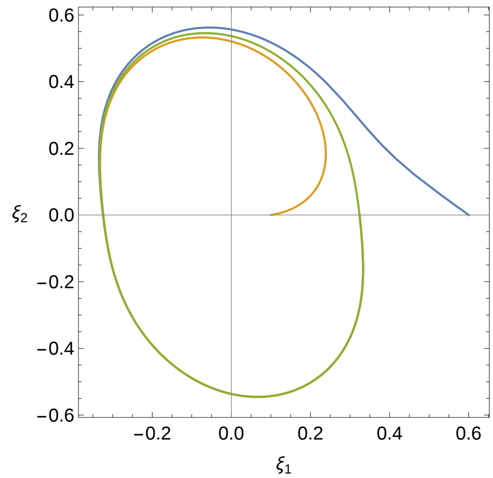

FIGURE 3. Orbits of system (2.5) for equation (2.25) with $\omega_0 = 1$ and $\mu = 1$.

2.2. **van der Pol and Rayleigh equations.** In this section we will show two examples of well-known equations whose families of cycles are not isochronous. However, the frequency of these cycles is known as a function of the parameter with arbitrary precision. Therefore, a change of variables can be made to obtain systems where the limit cycles are isochronous.

As it is known, the limit cycle of van der Pol equation

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -\mu(x_1^2 - 1)x_2 - x_1 \end{aligned} \tag{2.28}$$

is not isochronous with respect to μ. Here, equation (2.5) takes the form

$$\begin{aligned} \omega_0\xi_{1\mu}' &= \xi_{2\mu}, \\ \omega_0\xi_{2\mu}' &= -(1 + 2\mu x_{1\mu}x_{2\mu})\xi_{1\mu} + \mu(1 - x_{1\mu}^2)\xi_{2\mu} + (1 - x_{1\mu}^2)x_{2\mu}(t). \end{aligned} \tag{2.29}$$

Several orbits of system (2.29) are shown in Figure 4. Observing its behavior and the line given by equation (2.13) we see that the system does not have periodic orbits of period 2π. The green curve starts at the point where the line intersects the ξ_1-axis, at the value 0.0134, and clearly it is not a closed orbit of period 2π.

As stated before, the frequency $\omega(\mu)$ of the limit cycle is known with sufficient approximation in power series of the parameter μ [1]. Making a change on the variable t, we obtain

$$\begin{aligned} x_1' &= \omega(\mu)^{-1}x_2, \\ x_2' &= -\omega(\mu)^{-1}(\mu(x_1^2 - 1)x_2 - x_1), \end{aligned} \tag{2.30}$$

where we take the approximation

$$\omega(\mu) = 1 - \frac{1}{16}\mu^2 + \frac{17}{3072}\mu^4 + \frac{35}{884736}\mu^6 - \frac{678899}{5096079360}\mu^8 + \frac{28160413}{2293235712000}\mu^{10}, \tag{2.31}$$

and the names of the variables are preserved for simplicity. In this system the family of cycles is isochronous (within the approximation given by (2.31)). In Figure 5 we

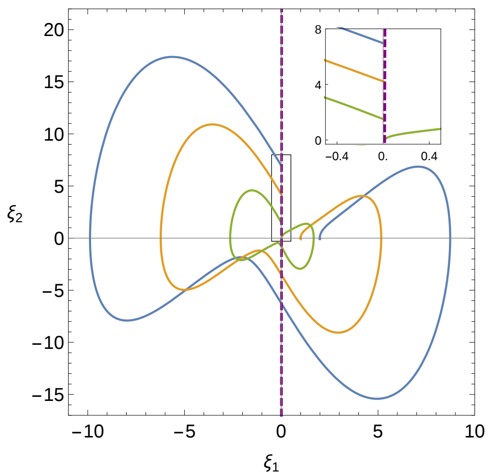

FIGURE 4. Orbits of system (2.29) for van der Pol equation with $\mu = 1$. Straight line $\xi_2 = 3174.35\xi_1 - 42.58$ is shown.

observe the limit cycle in the auxiliary system for $\mu = 1$, it is a periodic orbit of period 2π.

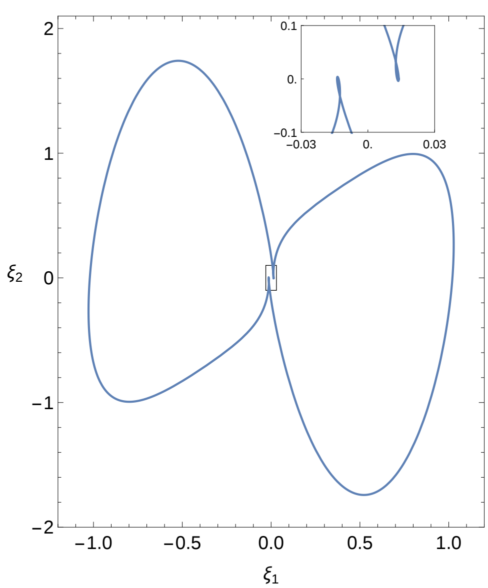

FIGURE 5. Cycles of the auxiliary system for van der Pol equation (2.30), with $\mu = 1$.

Similarly, the limit cycles of the Rayleigh oscillator,

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -\mu(x_2^2 - 1)x_2 - x_1, \end{aligned} \tag{2.32}$$

are not isochronous either. In Figure 6 we can see the orbits of the auxiliary system for this case and the corresponding line given by equation (2.13). This line intersects the ξ_1-axis in the value 0.1915, but in the graph it is observed that the orbit starting from that point, the green curve, is not periodic with period 2π.

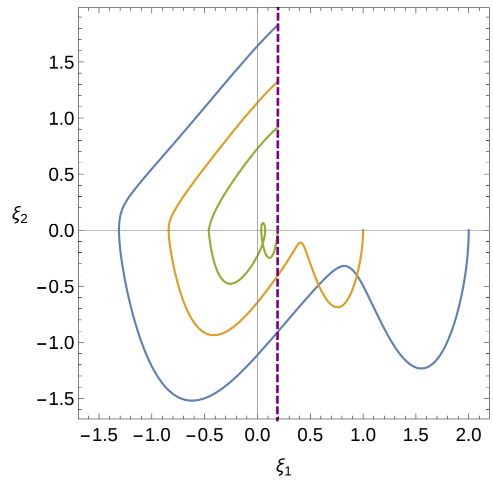

FIGURE 6. Orbits of the auxiliary system for Rayleigh equation (2.32) with $\mu = 1$. Straight line $\xi_2 = -112.5 + 587.59\xi_1$ is shown.

In the same way as with van der Pol equation, system (2.32) becomes

$$\begin{aligned} x_1' &= \omega(\mu)^{-1} x_2, \\ x_2' &= -\omega(\mu)^{-1}(\mu(x_2^2 - 1)x_2 - x_1). \end{aligned} \tag{2.33}$$

The frequency of these limit cycles can be calculated by the method of Poincaré-Lindstedt [18]. We have the fact that the formula for $\omega(\mu)$ is the same as (2.31). Again the obtained system is isochronous. We see in Figure 7 the cycle of period 2π of the auxiliary system for $\mu = 1$.

3. Delay differential equations

In [3] the problem of isochronous cycles in delayed systems of a particular type was considered. The differential equations with delay are a special case of functional differential equations [11]. The latter are of the form

$$\mathbf{x}'(t) = F(t, \mathbf{x}_t), \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{x}_t \in C([-\tau, 0], \mathbb{R}^n)$ is the function $\mathbf{x}_t(\theta) = \mathbf{x}(t + \theta)$ and $F : D \subset \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous.

In the case of differential equations with delay, the functional F is of the form $F(t, \mathbf{x}_t) = h(t, \mathbf{x}(t), \mathbf{x}(t - \tau_1), \dots, \mathbf{x}(t - \tau_k))$, where $h \colon U \subset \mathbb{R} \times \mathbb{R}^{nk} \to \mathbb{R}^n$ is continuous and there is a constant τ such that $0 < \tau_i \le \tau$, $i = 1, \cdots, k$.

In [3] equations of form

$$x'' + g(x, \beta) = \gamma f(x - x^\tau) \tag{3.2}$$

are considered, where $x(t)$, β and $\gamma \in \mathbb{R}$, and $x^\tau(t) = x(t - \tau)$ (we use the superscript to avoid confusion with the function $\mathbf{x}_t$). Function g is C^1 and f is real analytic. In

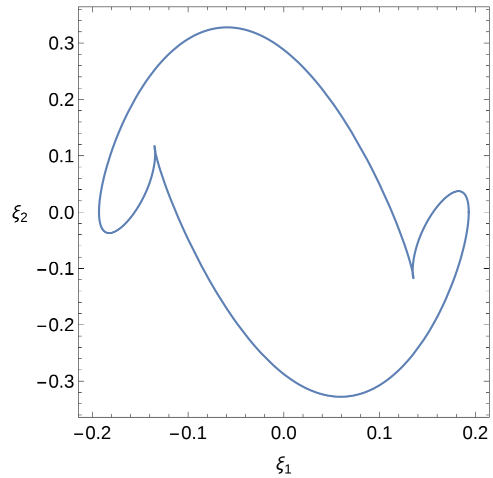

FIGURE 7. Cycles of the auxiliary system for Rayleigh equation (2.33), with $\mu = 1$

some cases a Hopf bifurcation occurs when any of the parameters γ or β changes. In the above work it was proved that the generic situation is that the frequency of the emergent limit cycles is constant in a neighborhood of the bifurcation.

In this work we consider differential equations with delay in the plane, of the type

$$\bar{\mathbf{x}}' = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{x}}^{\tau}, \mu). \tag{3.3}$$

As in Section 2, we assume that system has a limit cycle, $\bar{\mathbf{x}}_\mu$, with frequency $\omega(\mu)$. Then by suitable change of variables it can be written as

$$\omega(\mu)\mathbf{x}'_\mu = \mathbf{f}(\mathbf{x}_\mu, \mathbf{x}_\mu^{\omega\tau}, \mu). \tag{3.4}$$

Differentiating with respect to μ and assuming that the frequency does not depend on μ, we get the following delayed auxiliary system

$$\omega_0 \boldsymbol{\xi}'_\mu = \frac{\partial \mathbf{f}}{\partial \mathbf{x}_\mu}\boldsymbol{\xi}_\mu + \frac{\partial \mathbf{f}}{\partial \mathbf{x}_\mu^{\omega_0\tau}}\boldsymbol{\xi}_\mu^{\omega_0\tau} + \frac{\partial \mathbf{f}}{\partial \mu}, \tag{3.5}$$

where the variations $\boldsymbol{\xi}_\mu$ are given again by equations (2.6) and $\partial\mathbf{f}/\partial\mathbf{x}_\mu$, $\partial\mathbf{f}/\partial\mathbf{x}_\mu^{\omega_0\tau}$ are the Jacobians with respect to the variables and retarded variables, respectively.

Of course the interpretation of solutions of system (3.5) is much more involved than the case of an ordinary differential equation. The difference essentially lies in that the initial condition is in an infinite dimensional space and it is not a point in $\mathbb{R}^2$. We use a heuristic approach to study this problem.

Suppose that system (3.3) has a unique periodic solution through the point $(\hat{x}_1, 0)$. For example, from a Hopf bifurcation. Let us further suppose that with a small variation of μ another cycle is obtained. Suppose that these cycles are stable, then if we integrate equation (3.4) with a constant function in the interval $[-\tau, 0]$, at the point $(\hat{x}_1, 0)$, then the orbit will probably converge to the new cycle. It is interesting to know whether, with this variation of μ, the frequency of the new cycle remains constant.

System (3.5) helps us to answer this question. If the points on the ξ_1-axis are used as the origin of orbits with a constant initial condition during the delay time, then these orbits can converge to periodic orbits of period 2π. If this happens we may consider it as an indication that the system is isochronous. On the other hand, if it fails to find cycles of period 2π with constant initial conditions of the type here considered, it is a strong evidence that the family of limit cycles is not isochronous. The difficulty in interpreting the solutions of the auxiliary system in delayed equations is that it is impossible to cover all the initial conditions to test the existence of periodic orbits.

An interesting situation we have found is that there may be infinite orbits of period 2π, as will be seen in the examples below. This happens as follows. Suppose that $\boldsymbol{\xi}_\mu$ is a solution of period 2π of the auxiliary system. Then the family $\boldsymbol{\xi}_\mu + \boldsymbol{\eta}_\mu$ will also be a solution of period 2π if $\boldsymbol{\eta}_\mu$ satisfies the homogeneous equation

$$\omega_0 \boldsymbol{\eta}'_\mu = \frac{\partial \mathbf{f}}{\partial \mathbf{x}_\mu} \boldsymbol{\eta}_\mu + \frac{\partial \mathbf{f}}{\partial \mathbf{x}_\mu^{\omega_0 \tau}} \boldsymbol{\eta}_\mu^{\omega_0 \tau}. \tag{3.6}$$

If $\omega_0 \tau$ is a multiple of π, then this equation has infinite proportional solutions of period 2π. One of them is the one that allows us to write $\mathbf{x}_\mu$ as a function of $\mathbf{x}_{\mu_0}$ using equation (2.9); that is,

$$\mathbf{x}_\mu = \mathbf{x}_{\mu_0} + \int_{\mu_0}^{\mu} \boldsymbol{\xi}_\nu \, d\nu.$$

We state the following theorem.

Theorem 3.1. *Suppose that system* (3.3) *has a limit cycle for all points of some interval $I \subset \mathbb{R}$ of the parameter μ. Also, suppose that it crosses the horizontal axis transversely at $t = 0$. Then the family is isochronous with respect to μ; i.e., the frequency does not depend on the parameter in the interval, if and only if for some $\mu = \mu_0$ the limit cycle has frequency ω_0, and for all μ in I system* (3.5) *has a periodic solution of period 2π which is equal to the derivative of $\mathbf{x}_\mu$ with respect to μ for $t \in [-\omega_0 \tau, 0]$.*

Remark 3.2. The presence of an infinite number of cycles of period 2π in the auxiliary system can also occur in systems of ordinary differential equations. Here $\boldsymbol{\eta}_\mu$ must verify equation

$$\omega_0 \boldsymbol{\eta}'_\mu = \frac{\partial \mathbf{f}}{\partial \mathbf{x}_\mu} \boldsymbol{\eta}_\mu. \tag{3.7}$$

According to Floquet's theory [7] the fundamental matrix of solutions of this system is of the form $\boldsymbol{P}(t) e^{t\boldsymbol{R}}$, where $\boldsymbol{P}$ is a 2π-periodic matrix and $\boldsymbol{R}$ is a constant matrix. That is, there may be periodic solutions of period 2π only if any of the eigenvalues of $\boldsymbol{R}$ (characteristic values) is $2\pi i$.

In what follows we will show three examples. In the first one, it is known that the limit cycles are not isochronous [2]. For the other two, it was proved in [3] that the branches of Hopf bifurcation are isochronous.

3.1. **van der Pol with delayed feedback.** We consider the equation studied in [2],

$$x'' + (x^2 - 1)x' + x = \mu x^\tau. \tag{3.8}$$

When the parameter μ is zero, we have a van der Pol equation. For μ sufficiently small, the limit cycle is maintained but it is deformed, generating a family which is not isochronous. These cycles of frequency $\omega(\mu)$ give rise to cycles of frequency 1 of system

$$\begin{aligned}\omega(\mu)x'_{1\mu} &= x_{2\mu},\\ \omega(\mu)x'_{2\mu} &= -(x_{1\mu}^2-1)x_{2\mu} - x_{1\mu} + \mu x_{1\mu}^{\omega(\mu)\tau}.\end{aligned} \tag{3.9}$$

The auxiliary system takes the form

$$\begin{aligned}\omega_0\xi'_{1\mu} &= \xi_{2\mu},\\ \omega_0\xi'_{2\mu} &= -(1+2x_{1\mu}x_{2\mu})\xi_{1\mu} + (1-x_{1\mu}^2)\xi_{2\mu} + \mu\xi_{1\mu}^{\omega_0\tau} + x_{1\mu}^{\omega_0\tau}.\end{aligned} \tag{3.10}$$

Hopf bifurcations also appear in this system by varying the parameter μ. These branches are not isochronous either.

As can be seen in Figure 8 the orbits of the auxiliary system (3.10) are not bounded, because they move away from the origin. This is an indication of the non-isochronous character of the family. The orbit of the figure was calculated with an initial condition constant during the delay time at a point in the ξ_1-axis.

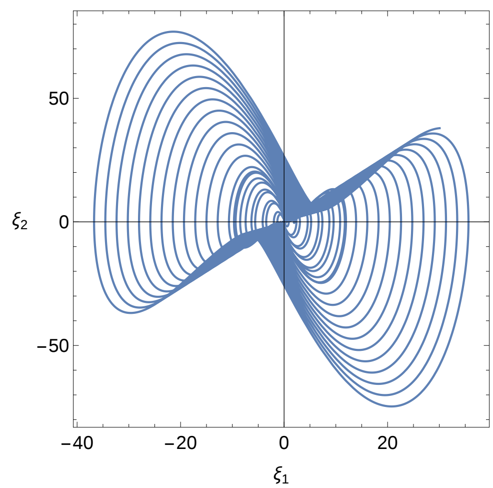

FIGURE 8. Orbit of the auxiliary system (3.10) for $\mu = 0.1$, $\tau = 9$ and $\omega_0 = 0.99315$.

3.2. **Rotating pendulum with delay.** In [3] the following differential equation with delay was studied:

$$x'' + (\beta - \cos x)\sin x = \mu \sin(x - x^\tau). \tag{3.11}$$

It represents the movement of a pendulum restricted to move on the surface of a vertical plane that rotates with constant angular velocity. A delayed feedback, which may be interpreted as a torque acting on the pivot, was added. This equation has two equilibria, one around $x = 0$ and another at $x^* = \arccos\beta$. It was proved that the branches of Hopf bifurcations that appear due to the delay are isochronous, i.e., the emergent cycles of this bifurcation around the nonzero equilibrium have constant frequency. Hopf bifurcations occur at frequency $\omega = (2n+1)\pi/\tau$ for $n = 0, 1, \dots$.

To apply the method developed in the present work we rewrite equation (3.11) around the non-zero equilibrium. After the change of independent variable, the 2π-periodic orbits verify the equation

$$\begin{aligned} &\omega(\mu)x'_{1\mu} = x_{2\mu},\\ &\omega(\mu)x'_{2\mu} = -(\beta - \cos(x_{1\mu} - x^*))\sin(x_{1\mu} - x^*) + \mu\sin(x_{1\mu} - x_{1\mu}^{\omega(\mu)\tau}). \end{aligned} \tag{3.12}$$

The auxiliary system for this problem is

$$\begin{aligned} \omega_0\xi'_{1\mu} &= \xi_{2\mu},\\ \omega_0\xi'_{2\mu} &= (\cos 2(x_{1\mu} - x^*) - \beta\cos(x_{1\mu} - x^*) + \mu\cos(x_{1\mu} - x^*))\xi_{1\mu}\\ &\quad + \mu\cos(x_{1\mu} - x^*)\xi_{1\mu}^{\omega_0\tau}. \end{aligned} \tag{3.13}$$

The orbits of system (3.13) converge to cycles of period 2π for initial conditions constant during the delay time at a point in the ξ_1-axis. This provides a verification of the isochronous character of the Hopf cycles in this system. In this case the cycles are not unique, as explained in Section 3. As the delay is a multiple of π, infinite solutions of period 2π of equation (3.6) are obtained.

If we plot the cycles to which the solutions of (3.13) converge and in a third axis the initial value of the coordinate ξ_1, where the integration starts, we obtain Figure 9. The surface determined by all these cycles is a ruled surface as also shown in Figure 9, where the lines are represented by points on the lines joining the corresponding cycles. It is a consequence of the linearity of the equations. If, instead of taking initial conditions during the delay time as mentioned above, other conditions are taken (for example a non-constant function of time or starting from another line), the same cycles are obtained.

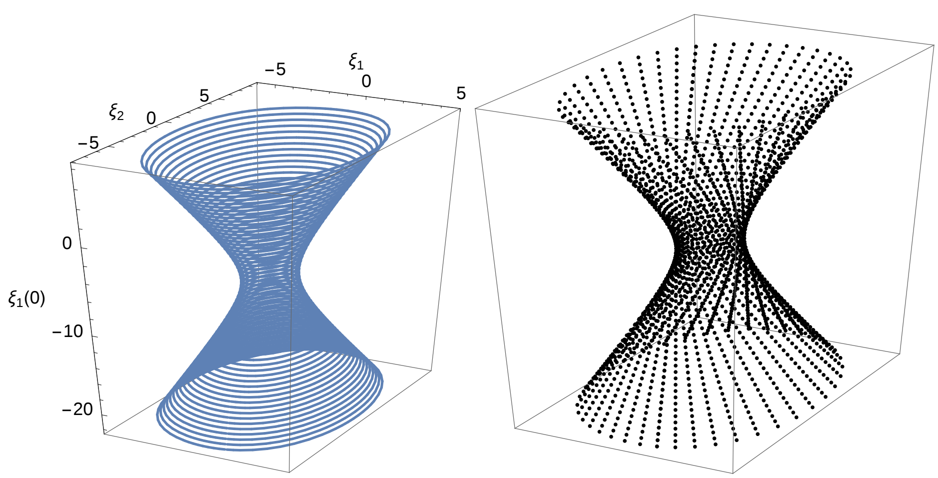

FIGURE 9. Left: cycles of the auxiliary system (3.13) for delayed rotatory pendulum, with $\mu = -1$, $\beta = 0.5$, $\tau = 2$ and $\omega_0 = \pi/\tau$. Right: dotted right lines joining different cycles

3.3. **Anharmonic oscillator with delay.** Another similar example is shown in [3]. It is given by the following differential equation with delay

$$x'' + x + \beta x^3 = \mu(x - x^\tau). \tag{3.14}$$

As in the previous example this equation undergoes Hopf bifurcations when the parameter μ varies. The emergent cycles have frequency $\omega = (2n+1)\pi/\tau$ for $n = 0, 1, \dots$. After reparametrization, equation (3.14) gives

$$\begin{aligned} \omega(\mu)x_1' &= x_2, \\ \omega(\mu)x_2' &= -\beta x_1^3 - x_1 + \mu(x_1 - x_1^{\omega(\mu)\tau}), \end{aligned} \tag{3.15}$$

and the auxiliary equation is

$$\begin{aligned} \omega_0\xi_{1\mu}' &= \xi_{2\mu}, \\ \omega_0\xi_{2\mu}' &= (\mu - 3\beta x_{1\mu}^2 - 1)\xi_{1\mu} - \mu\xi_{1\mu}^{\omega_0\tau} + x_1 - x_1^{\omega_0\tau}. \end{aligned} \tag{3.16}$$

The behavior of this system is very similar to the previous one. In Figure 10 we observe the cycles obtained by integrating equation (3.16) from different points on the ξ_1-axis (which is represented on the vertical axis). The lines connecting the cycles are also observed as dotted lines. Again this shows that the cycles in this system are isochronous.

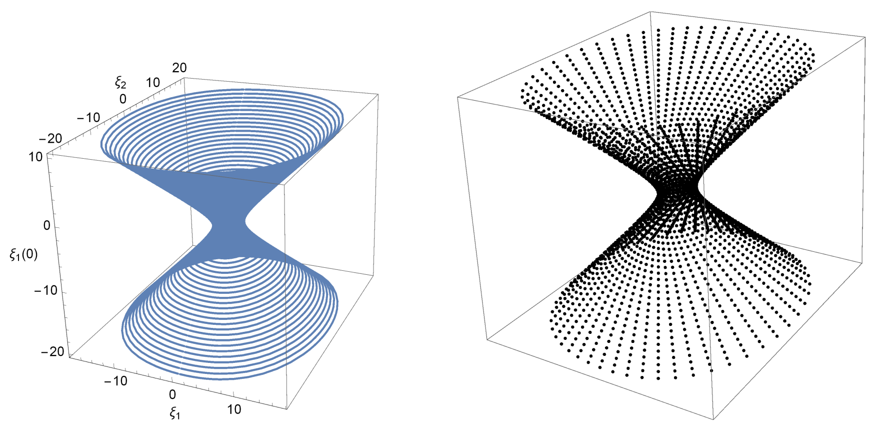

FIGURE 10. Left: cycles of the auxiliary system (3.16) for delayed anharmonic oscillator, with $\mu = -0.4$, $\beta = -1$, $\tau = 2.5$ and $\omega_0 = \pi/\tau$. Right: dotted right lines joining different cycles.

4. Conclusions

We have developed a methodology to study the frequency behavior of a family of limit cycles in differential equations and differential equations with delay in the plane. In particular, this methodology allows us to know if the family has constant frequency when the parameter changes. The problem is reduced to study the periodic solutions of non-homogeneous and non-autonomous linear differential equations.

While the method presented in this article works for systems in the plane, it could be extended to higher dimensions. However, its utility would be compromised by the fact that the search for periodic solutions of the auxiliary system should be done in one plane.

For the case of differential equations with delay the search for periodic cycles of the auxiliary equation is much more involved than in ordinary equations. In particular, the straight line through which the solutions pass after integration for time 2π corresponds to a linear subvariety in the Banach space of continuous functions and therefore it is not accessible with the graphic methods developed here. The problem would deserve further study, perhaps with the application of topological techniques such as degree theory [12].

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