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OPTIMAL DESIGN OF MINIMUM MASS STRUCTURES FOR A GENERALIZED STURM-LIOUVILLE PROBLEM ON AN INTERVAL AND A METRIC GRAPH

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Abstract. We derive an optimal design of a structure that is described by a Sturm-Liouville problem with boundary conditions that contain the spectral parameter linearly. In terms of Mechanics, we determine necessary conditions for a minimum-mass design with the specified natural frequency for a rod of non-constant cross-section and density subject to the boundary conditions in which the frequency (squared) occurs linearly. By virtue of the generality in which the problem is considered other applications are possible. We also consider a similar optimization problem on a complete bipartite metric graph including the limiting case when the number of leafs is increasing indefinitely.

1. Introduction

The optimal design of an axially vibrating rod supporting a non-structural point mass was considered by Turner [13]. He determined an optimal cross-sectional mass distribution $m(x)$ such that a rod of given principal eigenvalue is designed with the least possible mass. Such an optimization allows for greater economy in a design that must meet certain minimum requirements for natural frequency. Due to a duality principle, Turner's technique can also be used to determine the optimal distribution $m(x)$ such that a rod of given total mass is made with the largest principal eigenvalue. Such an optimization would give the greatest resistance to resonance. Taylor [12] considered the same problem and proved that the design of Turner was indeed optimal. Taylor also clearly articulated the duality principle employed by Turner in a form that assists in generalizing the method.

We begin with a brief review of [13]. The axial displacement of a rod can be modeled by the wave equation

$$m\frac{\partial^2 u}{\partial t^2} - \frac{E}{\rho}\frac{\partial}{\partial x}\Big(m\frac{\partial u}{\partial x}\Big) = 0, \quad 0 < x < L. \tag{1.1}$$

Table 1. Physical Interpretation of Parameters

Quantity	Interpretation
E	Young's Modulus
$u(x,t)$	Axial Displacement
ρ	Density of Rod Material
$A(x)$	Cross-sectional Area
$m(x)$	Mass per Unit Length ($=\rho A(x)$)
γ^2	$\omega^2\rho/E$
ω	Angular Frequency
M_1	Non-Structural Mass Supported at the End of The Rod

Here and below we use the notation given in Table 1. After separating variables and removing the harmonic (in time) term, we come up with the following Sturm-Liouville optimization problem for a rod supporting a non-structural mass M_1.

Problem 1.1. Let $u(x)$ be a nontrivial solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(m\frac{\mathrm{d}u}{\mathrm{d}x}\big)+\gamma_1^2 mu=0,\quad 0<x<L, \tag{1.2}$$

for specified natural frequency $\omega_1(=\gamma_1\sqrt{E/\rho})$, subject to the boundary conditions

$$u(0)=0,\quad mu'(L)=\gamma_1^2 M_1 u(L). \tag{1.3}$$

Find the mass distribution $m(x)=m_{opt}(x)$ such that the total mass functional,

$$M_0[m]:=\int_0^L m dx \tag{1.4}$$

attains its minimum value. □

Since the problem is homogeneous, we may normalize the solution as follows

$$u(L)=1\quad\text{so that } mu'(L)=\gamma_1^2 M_1. \tag{1.5}$$

Note that the spectral parameter γ_1^2 appears linearly in the boundary condition. To determine a solution to this problem, Turner seeks to minimize the following mass functional in which the equations of motion and the boundary conditions are introduced as isoperimetric constraints [13, 12]:

$$\Phi[m,u]:=M_0[m]+\int_0^L \lambda(x)[(mu')'+\gamma_1^2 mu]dx+\lambda_1[\gamma_1^2 M_1-m(L)u'(L)]. \tag{1.6}$$

Here the λ's are Lagrange multipliers. Turner carries out an analysis using the techniques of the Calculus of Variations [8] to find that the optimal mass distribution $m_{opt}(x)$ is given by

$$m_{opt}(x)=m(L)\cosh^2(\gamma_1 L)/\cosh^2(\gamma_1 x) \tag{1.7}$$

where

$$m(L)=\gamma_1 M_1\tanh\gamma_1. \tag{1.8}$$

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The total mass for this design is then

$$M_0[m_{opt}] = M_1 \sinh^2(\gamma_1 L). \tag{1.9}$$

Formulas (1.7) and (1.8) represent the complete solution of Problem 1.

In this article, rather than working with Problem 1 which models the axial vibrations of a rod, we consider a general Sturm-Liouville problem with the spectral parameter that appears linearly in the boundary conditions. For the general theory of this problem see Hinton [10], Fulton [7, 6], and Walter [14]. This generalization results in some new phenomena, such as the occurrence of an additional critical point and some conditions of solvability, that did not occur in the models [13, 12, 3].

We adopt the notation from [10, 7, 6, 14], for dealing with this problem, that is, we consider

$$(p(x)y'(x))' - q(x)y(x) + \lambda p(x)r(x)y(x) = 0, \quad x \in (0,1), \tag{1.10}$$

$$\cos\alpha\, y(0) + \sin\alpha\, (p(0)y'(0)) = 0, \tag{1.11}$$

$$-\beta_1 y(1) + \beta_2 p(1)y'(1) = \lambda[\beta_1' y(1) - \beta_2' p(1)y'(1)], \tag{1.12}$$

$$\delta := \beta_1'\beta_2 - \beta_1\beta_2' > 0. \tag{1.13}$$

Here $\alpha \in [0,\pi)$, β_k, and β_k', $k = 1, 2$ and $r(x) > 0$ are the (known) parameters and function and the assumption that $\delta > 0$ is required for the problem to be self-adjoint [10], and therefore for all eigenvalues to be real and bounded below.

It is known (see [4, 1] and the references therein) that problems of this type arise in the study of many diverse physical models including oscillations of a rotating string, a Timoshenko-Mindlin beam with a tip mass, a rotating beam with a tip mass (which models a propeller), and a beam of non-uniform cross section with one end elastically restrained and the other end carrying a guided mass.

The consideration of the more general model was also motivated by the results of Hinton and McCarthy [9] where the authors consider oscillations of a string fixed at one end with a mass connected to a spring at the other end. This study also considered minimizing the principal eigenvalue subject to a fixed total mass constraint.

We also consider optimization problem on a graph. Our consideration of the differential equations on a metric graph was motivated by the known extensive study of the mechanical and electrical networks, such as circuit equations with distributed parameters, string equations with the tip masses, and systems of beam equations that model the structural constructions (see [15]). To our best knowledge, only the direct problem has been studied so far, but we consider optimization. Though we consider a simple graph, we believe that our research represents just the first step in this promising direction.

The plan of the paper is as follows. In Section 2.1 we formulate the problem. In Section 2.2 we formulate our main result. The proof of it occupies Sections 2.3, 3 and 4. In Section 2.3 we use the methods of the Calculus of Variations to find critical points of the "mass" functional, i.e. functions $p(x)$ and also $y(x)$. These functions contain several arbitrary constants. In Section 3, we find some conditions on the parameters that guarantee that the function $y(x)$ satisfies the boundary conditions. In particular, we discover some zones of existence and non-existence of the parameters. We find an explicit formula for every critical point $p(x)$. In Section 4, we derive an explicit expression for the "mass" at each critical point and compare them. We also show that the result by [13] appears as a particular case of our general formulas. In Section 5 we consider the similar optimization problem on a complete bipartite metric graph (star). In Section 6 we derive the design and "mass" for a star with identical leafs and discuss the limiting case when the number of leafs is increasing indefinitely. Section 7 contains a discussion of the results.

2. Calculations

2.1. **Statement of the problem.** We reduce our consideration to the particular case $q(x) \equiv 0$. The reason for this is twofold. First, in many applications of problem (1.10)-(1.12), there is no term containing the function $q(x)$ (see [12, 13, 2, 3]). Second, the calculations of the optimal form for $q(x) \not\equiv 0$ seem to be intractable in the frame of an analytic approach. We briefly outline our plans for this case in Section 7.

Hence, we consider the Sturm-Liouville problem

$$(p(x)y'(x))' + \lambda p(x)r(x)y(x) = 0, \tag{2.1}$$

$$\cos\alpha\, y(0) + \sin\alpha\, p(0)y'(0) = 0, \tag{2.2}$$

$$-\beta_1 y(1) + \beta_2 p(1)y'(1) = \lambda[\beta_1' y(1) - \beta_2' p(1)y'(1)]. \tag{2.3}$$

Here and everywhere below (1.13) is implicitly assumed. Though we consider an abstract optimization problem, we prefer to use the physical terminology below, by interpreting the variables as in Table 2.

Table 2. Interpretation in the Notation in (2.1) - (2.3)

Quantity	Interpretation
$p(x)$	Cross-Sectional Area of Rod
$y(x)$	Axial Displacement
$r(x)$	Density of Rod Material
λ	ω^2/E
ω	Angular Frequency

As usual in the general theory of Sturm-Liouville problems, we will make the following assumption motivated by the physical restrictions of designing a rod.

(A1) The cross-sectional area $p(x)$ is continuous and strictly positive on $[0, 1]$. Only boundary parameters will be considered admissible which yield a positive $p(x)$.

Note the difference between (1.2) and (2.1) due to the loss of the assumption that the density is constant; this is, setting $\rho = r(x)$ does not reduce (1.2) to (2.1) since $r(x)$ can not be factored out and incorporated into the spectral parameter. We now formulate our problem.

Problem 2.1. Minimize the "mass" functional,

$$M[p] := \int_0^1 p(x)r(x)dx \tag{2.4}$$

associated with the Sturm-Liouville problem (2.1)-(2.3) if the principal eigenvalue, $\lambda_1 > 0$, of the problem is given. □

In view of (A1), the design $p(x)$ must be positive. Problem 2.1 is a generalization of the problems considered in [13, 12, 3].

2.2. **Formulation of the main result.** We now formulate our result on minimizing the "mass" functional (2.4).

Theorem 2.2. *For the Sturm-Liouville problem* (2.1)-(2.3) *subject to the condition* (1.13) *and* (A1)*,*

(a) *If $\alpha \neq \pi/2$, then the functional $M[p]$ has the critical point*

$$p_I(x) = \frac{B \sinh(2\sqrt{\lambda_1}\varrho(1) + \tanh^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)} \cosh^2(\sqrt{\lambda_1}\varrho(x) + \frac{1}{2}\tanh^{-1}(\zeta))}, \tag{2.5}$$

and if $\alpha \neq 0, \pi/2$, then this functional has a second critical point

$$p_{II}(x) = \frac{B \sinh(2\sqrt{\lambda_1}\varrho(1) + \tanh^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)} \sinh^2(\sqrt{\lambda_1}\varrho(x) + \frac{1}{2}\tanh^{-1}(\zeta))}. \tag{2.6}$$

Here

$$\varrho(x) := \int_0^x \sqrt{r(s)} ds, \tag{2.7}$$

$$B := \frac{\beta_1 + \lambda_1\beta_1'}{\beta_2 + \lambda_1\beta_2'}, \tag{2.8}$$

$$\zeta := -\frac{\sinh(2\sqrt{\lambda_1}\varrho(1))}{\frac{\hat{\alpha}}{B} + \cosh(2\sqrt{\lambda_1}\varrho(1))}, \tag{2.9}$$

$$\hat{\alpha} := \cot\alpha. \tag{2.10}$$

Here we assume that

$$\zeta \in (0,1). \tag{2.11}$$

(b) *For $\alpha \neq 0, \pi/2$, the "mass" of the design p_I is less than the "mass" of the design p_{II}.*

2.3. **Solution to Problem 2.1.** The proof of Theorem 2.2 is given in this Section and Sections 3 and 4.

Theorem 2.2 Part I. We follow the development of Turner [13] to find the critical points. Specifically, we formulate an isoperimetric problem in terms of the "mass" functional

$$\begin{aligned} F[y,p] := M[p] &+ \int_0^1 \Lambda_1(x)\Big((py')' + \lambda_1 pry\Big)dx \\ &+ \Lambda_2\Big(\cos\alpha\, y(0)) + \sin\alpha\, p(0)y'(0)\Big) \\ &+ \Lambda_3\Big([-\beta_1 y(1) + \beta_2\, p(1)y'(1)] - \lambda_1[\beta_1'\, y(1)) - \beta_2'\, p(1)y'(1)]\Big). \end{aligned} \tag{2.12}$$

Here $\Lambda_1(x)$, Λ_2, Λ_3 are Lagrange multipliers. Similarly to [13] (see also [8], [2], [3]) we compute the first variation of $F[y,p]$:

$$\begin{aligned} \delta F &= \big(\Lambda_1 y'\delta p\big)|_0^1 + \big(\Lambda_1 p\delta y'\big)|_0^1 - \big(\Lambda_1' p\delta y\big)|_0^1 \\ &\quad + \Lambda_2\Big(\cos\alpha\,\delta y(0) + \sin\alpha\, p(0)\delta y'(0) + \delta p(0) y'(0)\Big) \\ &\quad + \Lambda_3\Big(- \beta_1\delta y(1) + \beta_2(\delta p(1)y'(1) + \delta y'(1)p(1)) \\ &\quad - \lambda_1[\beta_1'\,\delta y(1) - \beta_2'(\delta p(1)y'(1) + \delta y'(1)\,p(1))]\Big) \\ &\quad + \int_0^1 \delta y\Big((\Lambda_1' p)' + \Lambda_1\lambda_1 rp\Big)dx \\ &\quad + \int_0^1 \delta p\Big(- \Lambda_1' y' + \Lambda_1\lambda_1 ry + r\Big)dx. \end{aligned} \tag{2.13}$$

To find the stationary points, we set $\delta F = 0$ and use the fundamental lemma of the Calculus of Variations to arrive at the following two differential equations

$$(p\Lambda_1')' + \lambda_1 rp\Lambda_1 = 0, \tag{2.14}$$

$$-\Lambda_1' y' + \Lambda_1\lambda_1 ry + r = 0. \tag{2.15}$$

Furthermore, we determine the following necessary conditions at the boundaries by considering the terms in which each of the independent variations ($\delta y(0)$, $\delta y'(0)$, $\delta p(0)$, $\delta y(1)$, $\delta y'(1)$, and $\delta p(1)$) appears. The boundary conditions are as follows:

$$\begin{gathered} \delta y(0) : \Lambda_2\cos\alpha - \Lambda_1'(0)p(0) = 0, \\ \delta y'(0) : p(0)(\Lambda_2\sin\alpha + \Lambda_1(0)) = 0, \\ \delta p(0) : y'(0)(\Lambda_2\sin\alpha + \Lambda_1(0)) = 0, \end{gathered} \tag{2.16}$$

$$\begin{gathered} \delta y(1) : \Lambda_1'(1)p(1) - \Lambda_3(\beta_1 + \lambda_1\beta_1') = 0, \\ \delta y'(1) : \Lambda_1(1)p(1) - \Lambda_3 p(1)(\beta_2 + \lambda_1\beta_2') = 0, \\ \delta p(1) : \Lambda_1(1)y'(1) - \Lambda_3 y'(1)(\beta_2 + \lambda_1\beta_2') = 0. \end{gathered} \tag{2.17}$$

From the set of equations (2.16), we can exclude Λ_2 to achieve (2.18) below and from the set (2.17), we can exclude Λ_3 to achieve (2.19),

$$\Lambda_1(0)\cos(\alpha) + \Lambda_1'(0)p(0)\sin\alpha = 0, \tag{2.18}$$

$$-\beta_1\Lambda_1(1) + \beta_2 p(1)\Lambda_1'(1) = \lambda_1[\beta_1'(\Lambda_1(1)) - \beta_2' p(1)\Lambda_1'(1)]. \tag{2.19}$$

We note that the boundary-value problem (2.14), (2.18), (2.19) is the same as (2.1)-(2.3). For this problem, it is well-known that the eigenspace is one dimensional. Therefore the multiplicity of the principal eigenvalue λ_1 is one, and we may conclude that $\Lambda_1(x) = ky(x)$ or $\Lambda_1(x) = -ky(x)$ (for a constant $k \in \mathbb{R}\setminus\{0\}$). Our necessary conditions (2.14) and (2.15) then become the original ODE (2.1):

$$(py')' + \lambda_1 pry = 0 \tag{2.20}$$

and one of the following non-linear differential equations:

$$-k(y')^2 + k\lambda_1 ry^2 + r = 0 \tag{2.21}$$

or

$$k(y')^2 - k\lambda_1 ry^2 + r = 0. \tag{2.22}$$

We observe that the sign of k is not important and assume further that $k > 0$. The solution of the equations (2.21) and (2.22) leads to valid critical points of the functional (2.12). We find respectively,

$$y_1(x) = \frac{1}{\sqrt{\lambda_1 k}} \sinh(\sqrt{\lambda_1}\varrho(x) + C_1) \tag{2.23}$$

and

$$y_2(x) = \frac{1}{\sqrt{\lambda_1 k}} \cosh(\sqrt{\lambda_1}\varrho(x) + C_2), \tag{2.24}$$

where $\varrho(x)$ is defined by (2.7).

Note that due to the non-linear nature of (2.21) and (2.22), linear combinations of these solutions are not necessarily solutions to (2.21) and (2.22).

The original differential equation (2.20) now becomes a first order linear differential equation for the unknown design $p(x)$. It may be rewritten in two ways depending on what function $y_j(x)$, $j = 1, 2$ is used,

$$(py_1')' + \lambda_1 pry_1 = 0, \tag{2.25}$$

$$(py_2')' + \lambda_1 pry_2 = 0. \tag{2.26}$$

Solving the differential equation (2.25) gives the design,

$$p_1(x) = C_3 \frac{\sqrt{r(0)} \cosh^2(C_1)}{\sqrt{r(x)} \cosh^2(\sqrt{\lambda_1}\varrho(x) + C_1)} \tag{2.27}$$

with the arbitrary constants C_3 and C_1. We note that by (A1) $C_3 > 0$.

Solving (2.26) gives the design

$$p_2(x) = C_4 \frac{\sqrt{r(0)} \sinh^2(C_2)}{\sqrt{r(x)} \sinh^2(\sqrt{\lambda_1}\varrho(x) + C_2)} \tag{2.28}$$

with the arbitrary constants C_4 and C_2. We note that by (A1) the design should be continuous and strictly positive. This requires that $C_4 > 0$ and $C_2 \in (-\infty, -\sqrt{\lambda_1}\varrho(1)) \cup (0, \infty)$. The condition on C_2 can be derived by enforcing that the arguments of the $\sinh^2$ functions in both the numerator and the denominator not be equal to zero. This derivation is as follows:

Observe that if $C_2 > 0$, (A1) is obviously satisfied (see the definition (2.7) of $\varrho(x)$). Similarly, if $C_2 = 0$, the denominator is equal to zero at $x = 0$. Further, if $C_2 < -\sqrt{\lambda_1}\varrho(1)$, the arguments of both $\sinh^2$ functions are negative and the design is strictly positive. If $0 > C_2 > -\sqrt{\lambda_1}\varrho(1)$, the argument has a unique zero at the point $x_0 \in (0, 1)$ where

$$\sqrt{\lambda_1} \int_0^{x_0} \sqrt{r(s)} ds = -C_2. \tag{2.29}$$

Therefore (A1) is satisfied when $C_2 \in (-\infty, -\sqrt{\lambda_1}\varrho(1)) \cup (0, \infty)$. Thus, we have two distinct stationary points of our variational problem. □

3. Boundary conditions: zones of existence and non-existence

Proof of Theorem 2.2 part II. We use the boundary conditions of our problem, (2.2) and (2.3), to determine arbitrary constants, as well conditions for which a solution exists. We discern three cases, shown in Table 3.

First, we consider the solutions stemming from p_1.

Table 3. Summary of Cases

$p(x)$	α	Case for Constants and Existence	Final Design
$p_1(x)$	0	Case(1)	(3.5)
	$\pi/2$	Case(2)	Does Not Exist
	$\neq 0, \pi/2$	Case(3)	(3.13)
$p_2(x)$	0	Case(4)	Does Not Exist
	$\pi/2$	Case(5)	Does Not Exist
	$\neq 0, \pi/2$	Case(6)	(3.22)

Case (1) In this case $y = y_1$ as given by (2.23), $p = p_1$ as given by (2.27), and $\alpha = 0$. The boundary condition (2.2) immediately implies

$$C_1 = 0. \tag{3.1}$$

The boundary condition (2.3), after the long but simple algebraic manipulations leads to the following

$$C_3 = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1))}{2\sqrt{\lambda_1 r(0)}}. \tag{3.2}$$

Since it is required that $p(x) > 0$, a solution exists when

$$B > 0 \tag{3.3}$$

or equivalently

$$\beta_1'\beta_2'\Big(\lambda_1 + \frac{\beta_1}{\beta_1'}\Big)\Big(\lambda_1 + \frac{\beta_2}{\beta_2'}\Big) > 0. \tag{3.4}$$

Here the final design p_1 is

$$p_{1;1}(x) = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}\varrho(x))}. \tag{3.5}$$

Case (2) Note that for p_1, the solution does not exist when $\alpha = \pi/2$. To see this, consider that when $\alpha = \pi/2$, (2.2), together with (2.23), (2.7), and (2.27) becomes

$$C_3\sqrt{r(0)}\cosh(C_1) = 0. \tag{3.6}$$

Due to the condition that $C_3 > 0$ (which follows from (A1)), this boundary condition cannot be satisfied.

Case (3) In this case $y = y_1$ as given by (2.23), $p = p_1$ as given by (2.27), and $\alpha \notin \{0, \pi/2\}$. The boundary condition (2.2) immediately implies

$$C_3 = -\frac{\hat\alpha\tanh(C_1)}{\sqrt{\lambda_1\, r(0)}}. \tag{3.7}$$

Isolating C_3 from the boundary condition (2.3) (see also (2.23) and (2.27)) and equating the result with (3.7) gives the equation

$$\frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + 2C_1)}{2\sqrt{\lambda_1 r(0)}\cosh^2(C_1)} = C_3 = -\frac{\hat\alpha\tanh(C_1)}{\sqrt{\lambda_1 r(0)}}. \tag{3.8}$$

After some algebraic manipulations and utilization of the notation (2.9) we arrive at

$$\tanh(2C_1) = \zeta. \tag{3.9}$$

This results in the following formulas

$$C_1 = \frac{1}{2}\tanh^{-1}(\zeta), \tag{3.10}$$

$$C_3 = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + 2C_1)}{2\cosh^2(C_1)\sqrt{\lambda_1 r(0)}} = p_1(0), \tag{3.11}$$

the first of which is well-defined since $\zeta \in (0,1)$ by (2.11).

Here a solution exists as long as the resulting design $p(x)$ is positive definite. The representation (2.27) shows that this is equivalent to the inequality $C_3 > 0$, or by (3.7), $\hat{\alpha}C_1 < 0$, or by (3.10) $\hat{\alpha}\zeta < 0$, or by (2.11),

$$\hat{\alpha} < 0. \tag{3.12}$$

The final design is given by

$$p_{1;3}(x) = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + \tanh^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}\varrho(x) + \frac{1}{2}\tanh^{-1}(\zeta))}. \tag{3.13}$$

We now consider the solution stemming from p_2.

Case (4) We note that for $\alpha = 0$ the solution does not exist. Indeed, for $\alpha = 0$, (2.2), together with (2.24) implies

$$\cosh(C_2) = 0 \tag{3.14}$$

which is a contradiction.

Case (5) Likewise, for $\alpha = \pi/2$, (2.2) implies

$$C_4\sqrt{r(0)}\sinh(C_2) = 0. \tag{3.15}$$

If $C_2 = 0$, then $p_2(x) = 0$ for all $x \in (0,1)$ which contradicts (A1). If $C_4 = 0$, then the same contradiction of (A1) is seen; therefore (3.15) cannot be satisfied, and the solution does not exist.

Case (6) In this case $y = y_2$ as given by (2.24), $p = p_2$ as given by (2.28), and $\alpha \notin \{0, \pi/2\}$. The boundary condition (2.2) immediately implies that

$$C_4 = \frac{-\hat{\alpha}\coth(C_2)}{\sqrt{\lambda_1 r(0)}}. \tag{3.16}$$

Isolating C_4 from the boundary condition (2.3) (see also (2.24) and (2.28)) and equating the result with (3.16) gives the equation

$$\frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + 2C_2)}{2\sqrt{\lambda_1 r(0)}\sinh^2(C_2)} = -\frac{\hat{\alpha}\coth(C_2)}{\sqrt{\lambda_1 r(0)}}. \tag{3.17}$$

After some algebraic manipulations and utilization of the notation (2.9), we arrive at

$$\tanh(2C_2) = \zeta. \tag{3.18}$$

This results in the formulas

$$C_2 = \frac{1}{2}\tanh^{-1}(\zeta), \tag{3.19}$$

$$C_4 = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + 2C_2)}{2\sinh^2(C_2)\sqrt{\lambda_1 r(0)}} = p_2(0) \tag{3.20}$$

provided that $\zeta \in (-1,0) \cup (0,1)$. Note that formula for C_2 in this case coincides with the formula for C_1 in Case(3). A solution exists in this case as long as the resulting design is positive definite, again this means that from (2.28), $C_4 > 0$. By (3.16) $\hat{\alpha} C_2 < 0$ or by (3.18) $\hat{\alpha}\zeta < 0$, or by (2.9),

$$\frac{\hat{\alpha}}{\frac{\hat{\alpha}}{B} + \cosh(2\sqrt{\lambda_1}\varrho(1)} > 0. \tag{3.21}$$

Note that this condition is exactly the same as (3.12). The final design is given by

$$p_{2;6}(x) = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + \tanh^{-1}(\zeta))}{2\sqrt{\lambda_1 r(x)}\sinh^2(\sqrt{\lambda_1}\varrho(x) + \frac{1}{2}\tanh^{-1}(\zeta))}. \tag{3.22}$$

So far, the proof does not establish that $\lambda_1 > 0$ is actually the principal eigenvalue. We establish this with the help of the zero properties of the first eigenfunction, see [11, Theorem 1, p. 445]. According to this theorem, the first (and only first) eigenfunction has no zeros in $(0, 1)$. We now analyze the eigenfunctions (2.23) and (2.24). Obviously the eigenfunction $y_2(x) > 0$. The eigenfunction $y_1(x) > 0$ in $(0, 1)$ if $C_1 \geq 0$ which takes place because either (3.1) for Case (1) or (3.10) and (2.11) for Case (3), and this completes the proof of Theorem 2.2 part (a). □

4. "Mass" functional

We now compare the total "mass" of each design (critical point), i.e. (3.13) and (3.22) for $\alpha \neq \{0, \pi/2\}$, when both designs exist. Hence, we compare both

$$M[p_{1;3}] = \frac{C_3\cosh^2(C_1)\sqrt{r(0)}}{\sqrt{\lambda_1}}[\tanh(\sqrt{\lambda_1}\varrho(1) + C_1) - \tanh(C_1)], \tag{4.1}$$

and

$$M[p_{2;6}] = \frac{C_4\sinh^2(C_2)\sqrt{r(0)}}{\sqrt{\lambda_1}}[\coth(C_2) - \coth(\sqrt{\lambda_1}\varrho(1) + C_2)], \tag{4.2}$$

where based on previous considerations

$$\begin{gathered}
C_1 = \frac{1}{2}\tanh^{-1}(\zeta), \quad C_2 = \frac{1}{2}\tanh^{-1}(\zeta),\\
C_3 = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + 2C_1)}{2\cosh^2(C_1)\sqrt{\lambda_1 r(0)}},\\
C_4 = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + 2C_2)}{2\sinh^2(C_2)\sqrt{\lambda_1 r(0)}}.
\end{gathered}$$

Then it follows that

$$M[p_{1;3}] = \frac{B\sinh(\sqrt{\lambda_1}\varrho(1) + C_1)\sinh(\sqrt{\lambda_1}\varrho(1))}{\lambda_1\cosh(C_1)}, \tag{4.3}$$

$$M[p_{2;6}] = \frac{-B\cosh(\sqrt{\lambda_1}\varrho(1) + C_2)\sinh(\sqrt{\lambda_1}\varrho(1))}{\lambda_1\sinh(C_2)}. \tag{4.4}$$

At this point, we note that the total "mass" for design $p_{2;6}(x)$, formally speaking, may be negative for some combination of parameters. Rather than discuss when this "mass" is positive, we consider the following quotient

$$\Big|\frac{M[p_{1;3}]}{M[p_{2;6}]}\Big| = \Big|\frac{-\sinh(\sqrt{\lambda_1}\varrho(1) + C_1)\sinh(C_2)}{\cosh(\sqrt{\lambda_1}\varrho(1) + C_2)\cosh(C_1)}\Big|. \tag{4.5}$$

Noting that $C_1 = C_2$, we have

$$\Big|\frac{M[p_{1;3}]}{M[p_{2;6}]}\Big| = \Big| -\tanh(\sqrt{\lambda_1}\varrho(1) + C_1)\tanh(C_1)\Big| < 1. \tag{4.6}$$

So regardless of when $p_{2;6}(x)$ has a positive "mass", we conclude that the design corresponding to $p_{1;3}(x)$ will always have less "mass" than the one corresponding to $p_{2;6}(x)$, and this completes the proof of part (b), and hence the proof of Theorem 2.2.

Remark 4.1. We analyze the design (3.13) as the function of α. It is easy to check that if $\alpha \to 0$, i.e. $\hat{\alpha} \to \infty$, then $\zeta \to 0$, and the design (3.13) approaches the design (3.5). Similarly, if $\alpha \to \pi/2$, i.e. $\hat{\alpha} \to 0$, then $\zeta \to -\tanh(2\sqrt{\lambda_1}\varrho(1))$, so that $2\sqrt{\lambda_1}\varrho(1) + \tanh^{-1}(\zeta) \to 0$, and the design (3.13) is not positive (see Case (2) above).

Remark 4.2. If $\alpha = 0$ then

$$M[p_{1;1}] = \frac{C_3\cosh^2(C_1)\sqrt{r(0)}}{\sqrt{\lambda_1}}[\tanh(\sqrt{\lambda_1}\varrho(1) + C_1) - \tanh(C_1)], \tag{4.7}$$

where $C_1 = 0$ as in (3.1) and C_3 is given by (3.2). Substituting in these values and simplifying gives

$$M[p_{1;1}] = \frac{B\sinh^2(\sqrt{\lambda_1}\varrho(1))}{\lambda_1} = \frac{\beta_1 + \lambda_1\beta_1'}{\beta_2 + \lambda_1\beta_2'}\frac{\sinh^2(\sqrt{\lambda_1}\varrho(1))}{\lambda_1}.$$

In this case, we can recover the result of Turner [13]. To see this, set $\beta_1 = \beta_2 = \beta_2' = 0$, $\beta_1' = M_1$ and $r(x) = \rho$. This gives

$$M[p_{1;1}] = M_1\sinh^2(\sqrt{\lambda_1}\int_0^1 \sqrt{\rho}dx) = M_1\sinh^2(\sqrt{\lambda_1\rho}). \tag{4.8}$$

Recall from Table 1 and Table 2 that $\lambda = \frac{\omega^2}{E}$ and $\gamma^2 = \frac{\omega^2\rho}{E}$. From these two equations, it follows that

$$\sqrt{\lambda_1} = \frac{\omega_1}{\sqrt{E}} \tag{4.9}$$

and

$$\gamma_1 = \frac{\omega_1\sqrt{\rho}}{\sqrt{E}}. \tag{4.10}$$

We see by substituting (4.9) into (4.10) that we have

$$M[p_{1;1}] = M_1\sinh^2(\gamma_1). \tag{4.11}$$

We see complete agreement with the result of Turner in (1.9) since for our problem $L = 1$.

5. Optimization problem on a metric graph

We now consider the similar optimization problem on a complete bipartite metric graph $K_{1,n}$, $n > 1$ that we will call the *star* for brevity. We denote $J := \{1, \dots, n\}$ and equip every leaf e_j, $j \in J$ of the graph with the coordinate $x_j \in [0, a_j]$, where $x_j = 0$ is the common vertex of all leafs. The wave type partial differential equations on a metric graph appear naturally in engineering problems relating to mechanical and electrical networks [15]. One of the models is a system of strings (or rods) with the tip masses. After separating variables and removing the harmonic (in time) factor, we come up with a Sturm-Liouville problem on the system of strings (see

Fig. 1). We assume that the displacements are continuous at the common point of all string and this point is attached to an elastic string, so that Hook's law is satisfied. Further, we assume that some masses are attached to the other end points of the strings (see the boundary condition (1.3)). Hence, we come to the following problem.

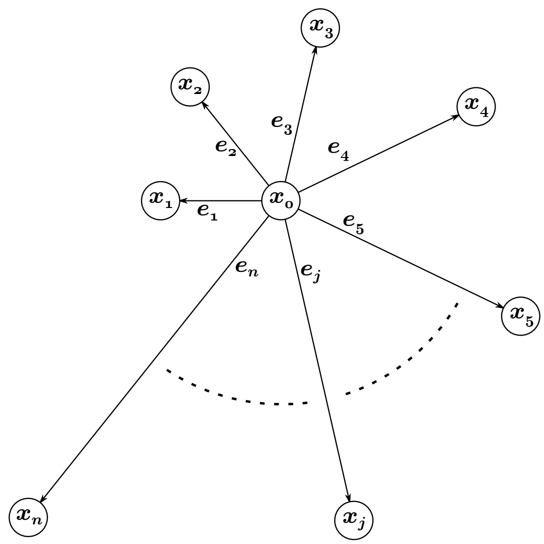

FIGURE 1. Graph $K_{1,n}$

We consider the Sturm-Liouville problem on the metric graph

$$(p_j(x)y_j'(x))' + \lambda_1 p_j(x)r_j(x)y_j(x) = 0, \quad 0 < x_j < a_j,\ j \in J; \tag{5.1}$$

$$y_j(0) = y_k(0) \quad \text{for all } j,\ k \in J,\ j \neq k; \tag{5.2}$$

$$\cos\alpha\, y_j(0) + \sin\alpha \sum_J (p_j(0)y_j'(0)) = 0, \tag{5.3}$$

$$-\beta_{1,j}y_j(a_j) + \beta_{2,j}p_j(a_j)y_j'(a_j) = \lambda_1[\beta_{1,j}'y_j(a_j) - \beta_{2,j}'p_j(a_j)y_j'(a_j)], \quad j \in J. \tag{5.4}$$

Here and below we use the abbreviation $\sum_J := \sum_{j\in J}$.

The boundary condition (5.2) allows us to let $y_j(0) := 1$, $j \in J$. We note that the condition (5.3) has the meaning of Hook's law, and that allows us to view the graph $K_{1,n}$ as a mechanical construction. Hence, it is natural to introduce the following simplifying assumption

$$p_j(0) = p_k(0) := p(0), \quad r_j(0) = r_k(0) := r(0) \quad \forall j,k \in J, \tag{5.5}$$

which means that the cross-sectional area of the rods and their densities are continuous at the common knot $x = 0$.

Our goal is to optimize the "mass" functional

$$M := \int_{K_{1,n}} rp\, dx. \tag{5.6}$$

We introduce the functional similar to (2.12),

$$\begin{aligned} F[p_1,\dots,p_n,y_1,\dots,y_n] := \int_{K_{1,n}} &\Big(rp+\Lambda_1\Big(((py')'+\lambda_1 rpy\Big)\Big)dx \\ &+\sum_J \Lambda_2\Big(-\beta_1 y+\beta_2 py'-\lambda_1[\beta_1' y-\beta_2' py']\Big). \end{aligned} \tag{5.7}$$

Here and below we use the abbreviations

$$\int_{K_{1,n}} f dx := \sum_J \int_0^{a_j} f_j dx, \quad \sum_J f := \sum_J f_j(a_j).$$

As in Section 2, we use the optimality condition $\delta F = 0$. We skip cumbersome calculations that are philosophically similar to once in Section 2 and allow us to find two types of critical point on each of the leafs e_j,

$$\begin{aligned} y_j^+(x) &= \frac{\cosh(\sqrt{\lambda_1}\varrho_j(x)+C_j^+)}{\cosh(C_j^+)}, \quad p_j^+(x) = \frac{C\sqrt{r(0)}\sinh^2(C_j^+)}{\sqrt{r_j(x)}\sinh^2(\sqrt{\lambda_1}\varrho_j(x)+C_j^+)}, \\ y_j^-(x) &= \frac{\sinh(\sqrt{\lambda_1}\varrho_j(x)+C_j^-)}{\sinh(C_j^-)}, \quad p_j^-(x) = \frac{C\sqrt{r(0)}\cosh^2(C_j^-)}{\sqrt{r_j(x)}\cosh^2(\sqrt{\lambda_1}\varrho_j(x)+C_j^-)}, \end{aligned} \tag{5.8}$$

where

$$\varrho_j(x) := \int_0^x \sqrt{r_j(s)} ds \tag{5.9}$$

and $C_j^\pm$, C are arbitrary constants, so that $C = p_j^\pm(0)$. We note that the constant C is indeed the same for all j in view of (5.5).

Since any of the critical points may be chosen on the $j-$th leaf, the total number of critical points for the optimization problem on the graph $K_{1,n}$ is 2^n. We denote the set of leafs where the point $(y_j^\pm, p_j^\pm)$ is chosen on the $j-$th leaf as $J^\pm$, so that $J^+\cup J^- = J$. We do not exclude that one of the sets $J^\pm$ is empty. The boundary condition (5.3) implies

$$\cos\alpha + \sin\alpha C\sqrt{\lambda_1 r(0)}\Big(\sum_{J^+}\tanh(C_j^+) + \sum_{J^-}\coth(C_j^-)\Big) = 0, \tag{5.10}$$

so that

$$C = -\frac{\hat\alpha}{\sqrt{\lambda_1 r(0)}\Big(\sum_{J^+}\tanh(C_j^+)+\sum_{J^-}\coth(C_j^-)\Big)}, \tag{5.11}$$

where $\hat\alpha$ is defined by (2.10). The boundary conditions at the vertices $x = a_j$, $j\in J$ of the leafs lead to the equations

(a) If $j \in J^+$, then

$$\begin{aligned} &-\beta_{1,j}\cosh\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^+\big)+\beta_{2,j}C\sqrt{\lambda_1 r(0)}\frac{\sinh^2(C_j^+)}{\sinh\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^+\big)} \\ &= \lambda_1\Big[\beta_{1,j}'\cosh\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^+\big)-\beta_{2,j}'C\sqrt{\lambda_1 r(0)}\frac{\sinh^2(C_j^+)}{\sinh\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^+\big)}\Big]. \end{aligned} \tag{5.12}$$

(b) If $j \in J^-$, then

$$
\begin{aligned}
&-\beta_{1,j}\sinh\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^-\big)+\beta_{2,j}C\sqrt{\lambda_1 r(0)}\frac{\cosh^2(C_j^-)}{\cosh\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^-\big)}\\
&=\lambda_1\Big[\beta_{1,j}'\sin\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^+\big)-\beta_{2,j}'C\sqrt{\lambda_1 r(0)}\frac{\cosh^2(C_j^-)}{\cosh\big(\sqrt{\lambda_1}\varrho_j(a_i)+C_j^-\big)}\Big].
\end{aligned}
\tag{5.13}
$$

Using notation similar to (2.8),

$$
B_j := \frac{\beta_{1,j}+\lambda_1\beta_{1,j}'}{\beta_{2,j}+\lambda_1\beta_{2,j}'}, \quad j \in J \tag{5.14}
$$

in equations (5.12) and (5.13) we find two expressions for C:

$$
\begin{aligned}
C &= \frac{B_j}{2\sqrt{\lambda_1 r(0)}}\frac{\sinh(2\sqrt{\lambda_1}\varrho_j(a_j)+2C_j^+)}{\sinh^2(C_j^+)} \quad \text{if } j\in J^+;\\
C &= \frac{B_j}{2\sqrt{\lambda_1 r(0)}}\frac{\sinh(2\sqrt{\lambda_1}\varrho_j(a_j)+2C_j^-)}{\sinh^2(C_j^-)} \quad \text{if } j\in J^+.
\end{aligned}
\tag{5.15}
$$

Combining the formulas (5.11), (5.15) we get the system of n equations for n constants $C_j^{\pm}$, $j \in J$ similar to (3.8) and (3.17),

$$
\begin{aligned}
&\frac{B_j\sinh(2\sqrt{\lambda_1}\varrho_j(a_j)+2C_j^+)}{\sinh^2(C_j^+)}\\
&= -\frac{\hat{\alpha}}{\sum_{J^+}\tanh(C_j^+)+\sum_{J^-}\coth(C_j^-)} \quad \text{if } j\in J^+;\\
&\frac{B_j\sinh(2\sqrt{\lambda_1}\varrho_j(a_j)+2C_j^-)}{\cosh^2(C_j^-)}\\
&= -\frac{\hat{\alpha}}{\sum_{J^+}\tanh(C_j^+)+\sum_{J^-}\coth(C_j^-)} \quad \text{if } j\in J^-.
\end{aligned}
\tag{5.16}
$$

It may be shown that if $J^+=\emptyset$, $J^-=\{1\}$ or $J^-=\emptyset$, $J^+=\{1\}$, the system (5.16) results in (3.8) or (3.17). We are not optimistic though about possibility to solve the system (5.16) in the general case.

6. Optimization problem on a star graph with identical leafs. The limiting case

Instead of the general star, we consider a particular case when all leafs are of the same length and the p_j, r_j, B_j are the same on all leafs, so that we may skip the index j. We also let $a_j := 1$ as in Sections 2-4. We assume $J^+=\emptyset$. This choice is based on the observation that the design $p_{1;3}$ in Theorem 2.2 (Section 4) results in the minimal "mass". Correspondingly, we assume $\alpha \neq \pi/2$. Following these Sections, we denote $C_j^- := C_1$.

Theorem 6.1. *For a star graph with identical leafs, the following statements hold.*

(a) *The "mass" (5.6) has the form*

$$
M = \frac{nB\sinh(\sqrt{\lambda_1}\varrho(1)+C_1)\sinh(\sqrt{\lambda_1}\varrho(1))}{\lambda_1\cosh(C_1)}. \tag{6.1}
$$

(b) *If the parameter $\kappa := nB$ is large and $\alpha \in (0, \pi/2)$, then the asymptotic representation holds*

$$M = \frac{\hat{\alpha}}{\lambda_1} \sinh^2(\sqrt{\lambda_1}\varrho(1)) + O\Big(\frac{1}{\kappa}\Big). \tag{6.2}$$

Proof. Instead of the system (5.16) we have

$$\frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + 2C_1)}{2\cosh^2(C_1)} = -\frac{\hat{\alpha}}{n\coth C_1}, \tag{6.3}$$

so that similarly to (3.10) and (2.9),

$$C_1 = \frac{1}{2}\tanh^{-1}(\zeta_n) \tag{6.4}$$

where

$$\zeta_n := -\frac{\sinh(2\sqrt{\lambda_1}\varrho(1))}{\frac{\hat{\alpha}}{Bn} + \cosh(2\sqrt{\lambda_1}\varrho(1))}. \tag{6.5}$$

Here $\varrho(x)$ is defined as in (2.7) and (5.9).

The design (5.8) implies

$$p(x) = \frac{B\sinh(2\sqrt{\lambda_1}\varrho(1) + \tanh^{-1}(\zeta_n))}{2\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}\varrho(x) + \frac{1}{2}\tanh^{-1}(\zeta_n))}, \tag{6.6}$$

which is similar to (3.13).

We finally evaluate the "mass" (5.6),

$$M = \int_{K_{1,n}} rpdx = n\int_0^1 rpdx = \frac{nB\sinh(\sqrt{\lambda_1}\varrho(1) + C_1)\sinh(\sqrt{\lambda_1}\varrho(1))}{\lambda_1\cosh(C_1)} \tag{6.7}$$

and this completes the proof of Theorem 6.1 (a). We note that the representation (6.7) is similar to (4.3).

We further consider the limiting case $n \to \infty$, that may be interpreted as optimization problem for a star with infinitely many leafs. More specifically, we assume that the parameter $\kappa = Bn$ is large, i.e.

$$\kappa := nB \gg 1. \tag{6.8}$$

Our goal is to find the leading terms of the asymptotic representation for the "mass" as $\kappa \to \infty$. Firstly, we find from (6.5)

$$\zeta_n = -\tanh(2\sqrt{\lambda_1}\varrho(1)) + \frac{\hat{\alpha}}{\kappa}\frac{\sinh(2\sqrt{\lambda_1}\varrho(1))}{\cosh^2(2\sqrt{\lambda_1}\varrho(1))} + O\Big(\frac{1}{\kappa^2}\Big). \tag{6.9}$$

Further, from (6.4) we derive

$$2C_1 = \tanh^{-1}(\zeta_n) = -2\sqrt{\lambda_1}\varrho(1) + \frac{\hat{\alpha}}{\kappa}\cdot\sinh(2\sqrt{\lambda_1}\varrho(1)) + O\Big(\frac{1}{\kappa^2}\Big), \tag{6.10}$$

so that

$$\sqrt{\lambda_1}\varrho(1) + C_1 = \frac{\hat{\alpha}}{\kappa}\frac{\sinh(2\sqrt{\lambda_1}\varrho(1))}{2} + O\Big(\frac{1}{\kappa^2}\Big).$$

Based on (5.8), (5.15), (6.10), we now can find the asymptotic representation for $y(x)$ and $p(x)$. We skip calculations and only give the results

$$\begin{gathered} y(x)=\frac{\sinh(\sqrt{\lambda_1}(\varrho(1)-\varrho(x)))}{\sinh(\sqrt{\lambda_1}\varrho(1))}+O\Big(\frac{1}{\kappa}\Big), \\ p(x)=\frac{\sinh(2\sqrt{\lambda_1}\varrho(1))\hat{\alpha}}{2n\sqrt{\lambda_1 r(x)}\cosh^2(\sqrt{\lambda_1}(\varrho(1)-\varrho(x)))}+O\Big(\frac{1}{\kappa^2}\Big). \end{gathered} \tag{6.11}$$

The asymptotic representation for the "mass" (6.7) appears based on the asymptotic representation (6.4). After some algebraic manipulations we find

$$M=\frac{\hat{\alpha}}{\lambda_1}\sinh^2(\sqrt{\lambda_1}\varrho(1))+O\Big(\frac{1}{\kappa}\Big). \tag{6.12}$$

The answer makes sense if $\alpha\in(0,\pi/2)$. This completes the proof of part (b), and hence the proof of Theorem 6.1 is complete. □

Remark 6.2. Comparison of the formulas (6.5) and (2.9) shows that the formulas for one interval and the star with identical leafs are quite similar except the parameter $\hat{\alpha}$ is changed for $\hat{\alpha}/n$.

Remark 6.3. (a) The leading terms of the asymptotic representation for $y(x), p(x)$ and M do not depend on the parameters β_k, β'_k, $k=1,2$.

(b) It is rather easy to check that the leading terms of the asymptotic representation (6.11) for $y(x)$ and $p(x)$ satisfy the boundary conditions at the vertex $x=0$ exactly and the boundary conditions at the vertices $x_j=1$ within an error $O\big(\frac{1}{\kappa}\big)$.

Remark 6.4. We suggest the following interpretation of our asymptotic formulas.
(a) The boundary condition (5.3) at the vertex $x=0$ of the graph has the form

$$\cos\alpha\, y(0)+\sin\alpha\cdot n\, p(0)y'(0)=0. \tag{6.13}$$

It may be viewed as "almost" Neumann condition for the Sturm-Liouville problem on a single interval $(0,1)$,

$$p(0)y'(0)=-\frac{\hat{\alpha}}{n} \tag{6.14}$$

where we use our usual normalization $y(0)=1$ and the notation $\hat{\alpha}=\cot\alpha$ (2.10). Hence, for $n\to\infty$, the star is split into n disconnected leafs with the boundary condition at $x=0$ that "approaches" Neumann condition. It is not too complex to check the following. If we take the formula for the optimal "mass" $M[p]$ for the Sturm-Liouville problem on one interval (4.3) with $\hat{\alpha}$ formally substituted by $\hat{\alpha}/n$ (see Remark 6.2), then multiply this "mass" by n, and find the first term of asymptotic representation as $n\to\infty$, then we get the leading term of the formula for the "mass" of the star (6.12). We skip this calculation since it almost repeats calculation (6.9)-(6.12).
(b) It is interesting to note that if we consider the limiting case of the boundary condition (6.14), i.e. $y'(0)=0$, then, in terms of the boundary condition (1.11), we need to let $\alpha=\pi/2$, and, as we show in Section 3, the corresponding critical point does not exist. Simultaneously, the leading term of the asymptotic representation (6.12) for the "mass" of the star vanishes. That shows that indeed the value of the parameter $\alpha=\pi/2$ should be excluded from consideration.

7. Conclusion and discussion

We consider the optimal design problem modeled by a Sturm-Liouville problem on an interval or a complete bipartite graph and find the explicit formulas for the optimal design. We analyze the intervals of the parameters where such a design exists. We are motivated by the known publications on (a) the Sturm-Liouville problem with the spectral parameter that appears in the boundary conditions linearly; (b) optimization problem for an elastic rod with an attached mass; (c) differential equations describing mechanical and electrical networks.

1. There are two surprises the authors discovered in this study. (a) The existence of a solution corresponding to the design $p_2(x)$, not only to $p_1(x)$ as in [13], [12], [3]. (b) The existence of the limit of the "mass" functional for the star as the number of leafs increases indefinitely. As for (a), this other solution was not expected, though the fact that it does not exist for either $\alpha = 0$ or $\alpha = \pi/2$ explains, to some extent, why it was elusive. In this work, it appears unfruitful since it does not lead to a minimum "mass" design, yet we feel it is important to include since this critical point might be of interest for other optimization problems. It is also intriguing that this solution does not exist for both $\alpha = 0$ and $\alpha = \pi/2$. As for (b), we interpret this phenomenon in terms of the split of the star into disconnected leafs with "almost" Neumann condition at the vertex $x = 0$. Generally speaking, for a mechanical construction, disconnection of the leafs may result in a destruction of this construction. Both phenomena (a) and (b) may give an interesting chance for further studies.

2. In the case $\alpha = 0$, the functional $M[p]$ has only one critical point, and based on the duality that was derived in [12], we may expect that the following two problems have the same optimal solution $p(x)$:

(I) Given $r(x)$, β_1, β_2, β_1', β_2', and λ_1, find $p(x)$ such that $M[p] \to \min$.

(II) Given $r(x)$, β_1, β_2, β_1', β_2', and M, find $p(x)$ such that $\lambda_1[p] \to \max$.

We have solved Problem (I) but may hope that the optimal $p(x)$ from solving (I) is the same as the optimal $p(x)$ from solving (II). The validity of this duality in the case of multiple critical points should be studied further.

3. We have made some restrictions on the data in the process of the construction of the optimal solution. Removing them would represent a challenging problem. (a) We assumed $q(x) \equiv 0$. The reason for this is twofold. First, in many applications of the problem (1.10)-(1.12), there is no term containing the function $q(x)$. Second, the calculations of the optimal form for $q(x) \not\equiv 0$ seem to be intractable in the frame of an analytic approach. Yet, the complete analysis here is probably possible at the numerical level. For example, an alternative approach for a similar but simpler problem based on the discretization is developed in [5], [2], [3]. (b) We assumed $r(x) > 0$. Removing this condition is non-trivial since even to analyze the Sturm-Liouville problem itself, before solving optimization problem, it is necessary to work in a space with indefinite metric [16].

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