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EXISTENCE OF SOLUTIONS FOR A FRACTIONAL ELLIPTIC PROBLEM WITH CRITICAL SOBOLEV-HARDY NONLINEARITIES IN $\mathbb{R}^N$

LINGYU JIN, SHAOMEI FANG

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Abstract. In this article, we study the fractional elliptic equation with critical Sobolev-Hardy nonlinearity

$$(-\Delta)^\alpha u + a(x)u = \frac{|u|^{2_s^*-2}u}{|x|^s} + k(x)|u|^{q-2}u,\\ u \in H^\alpha(\mathbb{R}^N),$$

where $2 < q < 2^*$, $0 < \alpha < 1$, $N > 4\alpha$, $0 < s < 2\alpha$, $2_s^* = 2(N-s)/(N-2\alpha)$ is the critical Sobolev-Hardy exponent, $2^* = 2N/(N-2\alpha)$ is the critical Sobolev exponent, $a(x), k(x) \in C(\mathbb{R}^N)$. Through a compactness analysis of the functional associated, we obtain the existence of positive solutions under certain assumptions on $a(x), k(x)$.

1. Introduction

We consider the nonlinear elliptic equation

$$\begin{gathered}(-\Delta)^\alpha u + a(x)u = \frac{|u|^{2_s^*-2}u}{|x|^s} + k(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N,\\ u \in H^\alpha(\mathbb{R}^N),\end{gathered} \tag{1.1}$$

where $2 < q < 2^*$, $0 < \alpha < 1$, $0 < s < 2\alpha$, $N > 4\alpha$, $2_s^* = 2(N-s)/(N-2\alpha)$ is the critical Sobolev-Hardy exponent, $2^* = 2N/(N-2\alpha)$ is the critical Sobolev exponent, $a(x), k(x) \in C(\mathbb{R}^N)$.

Recently the fractional Laplacian and more general nonlocal operators of elliptic type have been widely studied, both for their interesting theoretical structure and concrete applications in many fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion and so on (see [4, 9, 13, 11, 8, 19, 20, 21]). In particular, many results have been obtained for elliptic equations with critical nonlinearity related to (1.1). Dipierro et al. [9] considered the critical problem with

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Hardy-Leray potential

$$\begin{gathered}(-\Delta)^{\alpha}u-\gamma\frac{u}{|x|^{2\alpha}}=|u|^{2^{*}-2}u,\quad x\in\mathbb{R}^{N},\\ u\in\dot{H}^{\alpha}(\mathbb{R}^{N}),\end{gathered}\tag{1.2}$$

where $\dot{H}^{\alpha}(\mathbb{R}^{N})$ is defined in (1.5). They proved existence, certain qualitative properties and asymptotic behavior of positive solutions to (1.2). Ghoussoub and Shakerian in [14] investigated the double critical problem in $\mathbb{R}^{N}$,

$$\begin{gathered}(-\Delta)^{\alpha}u-\gamma\frac{u}{|x|^{2\alpha}}=\frac{|u|^{2_s^{*}-2}u}{|x|^{s}}+|u|^{2^{*}-2}u,\quad x\in\mathbb{R}^{N},\\ u>0,\quad u\in\dot{H}^{\alpha}(\mathbb{R}^{N}),\end{gathered}\tag{1.3}$$

with $\gamma>0$. There through the non-compactness analysis of the Palais-Smale sequence of (1.3), they obtained the existence of the solutions. Also Yang etc. in [27], [25] consider a class of critical problems with a Hardy term for the fractional Laplacian in a bounded domain. For the two gathered of the spectral fractional Laplacian and of the regional fractional Laplacian, they obtained the existence of solutions respectively. In addition, the authors in [10] established a concentration-compactness result for a fractional Schrödinger equation with the subcritical nonlinearity $f(x,u)$. Motivated by [9, 14, 10, 27, 25] we consider the existence of positive solutions for problem (1.1) in $\mathbb{R}^{N}$. The main interest for this type of problems, in addition to the nonlocal fractional Laplacian is the presence of the singular potential $1/|x|^{s}$ related to the fractional Sobolev-Hardy's inequality. We recall the Sobolev-Hardy inequality

$$\Big(\int_{\mathbb{R}^{N}}\frac{|u(x)|^{2_s^{*}}}{|x|^{s}}dx\Big)^{2/2_s^{*}}\leq c\int_{\mathbb{R}^{N}}|(-\Delta)^{\alpha/2}u(x)|^{2}dx,\quad\forall u\in\dot{H}^{\alpha}(\mathbb{R}^{N}),\tag{1.4}$$

where c is a positive constant. The Sobolev embedding $\dot{H}^{\alpha}(\mathbb{R}^{N})\hookrightarrow L^{2_s^{*}}(|x|^{-s},\mathbb{R}^{N})$ is not compact, even locally, in any neighborhood of zero. As it is well known, the loss of the compactness of the embeddings is one of the main difficulties for elliptic problems with critical nonlinearities. Thus our problem has two factors, one is the critical Sobolev-Hardy term, the other is the unbounded domain. In [9] and [14], the authors can consider the solutions of critical problems in the homogeneous fractional Sobolev space $\dot{H}^{\alpha}(\mathbb{R}^{N})$, while we must deal with (1.1) in the nonhomogeneous fractional Sobolev space $H^{\alpha}(\mathbb{R}^{N})$ given the presence of low sub-critical terms in (1.1). This is why the methods in [9] and [14] can not be used directly to (1.1). As far as we know, the existence results for global problems for the fractional Lapalacian with a mixture of critical Sobolev-Hardy terms and subcritical terms are relatively new. To overcome the difficulties caused by the lack of compactness, we carry out a non-compactness analysis which can distinctly express all the parts which cause non-compactness. As a result, we are able to obtain the existence of nontrival solutions of the elliptic problem with the critical nonlinear term on an unbounded domain by getting rid of these noncompact factors. To be more specific, for the Palais-Smale sequences of the variational functional corresponding to (1.1) we first establish a complete noncompact expression which includes all the blowing up bubbles caused by the critical Sobolev-Hardy nonlinearity and by the unbounded domain. Then we derive the existence of positive solutions for (1.1). Our methods are based on some techniques of [5, 7, 10, 16, 17, 23, 24, 26, 28].

Notation and assumptions. Denote c and C as arbitrary constants which may change from line to line. Let $B(x,r)$ denote a ball centered at x with radius r and $B(x,r)^C = \mathbb{R}^N \setminus B(x,r)$.

Let $N \geq 1$, $u \in L^2(\mathbb{R}^N)$, let the Fourier transform of u be

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x} u(x)dx.$$

We define the operator $(-\Delta)^\alpha u$ by the Fourier transform

$$\widehat{(-\Delta)^\alpha} u(\xi) = |\xi|^{2\alpha}\hat{u}(\xi), \quad \forall u \in C_0^\infty(\mathbb{R}^N).$$

Let $\dot{H}^\alpha(\mathbb{R}^N)$ be the homogeneous fractional Sobolev space as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{\dot{H}^\alpha(\mathbb{R}^N)} = \||\xi|^\alpha \widehat{u}\|_{L^2(\mathbb{R}^N)}, \tag{1.5}$$

and denote by $H^\alpha(\mathbb{R}^N)$ the usual nonhomogeneous fractional Sobolev space with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \||\xi|^\alpha \widehat{u}\|_{L^2(\mathbb{R}^N)}. \tag{1.6}$$

For $0 < \alpha < 1$, a direct calculation (see e.g. [8, Proposition 4.4], or [9, Proposition 1.2], gives

$$c_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy = \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u(x)|^2 dx = \|u\|^2_{\dot{H}^\alpha(\mathbb{R}^N)},$$

where $c_{N,s} = 2^{2s-1}\pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}$.

Let $u^+ = \max\{u, 0\}$, $u^- = u^+ - u$. From the proof of (2.14) in [12], it follows

$$\|u^+\|_{\dot{H}^\alpha} \leq \|u\|_{\dot{H}^\alpha}. \tag{1.7}$$

We call $u \not\equiv 0$ in $\mathbb{R}^N$ if the measure of the set $\{x \in \mathbb{R}^N | u(x) \neq 0\}$ is positive.

Recall the definition of Morrey space. A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space with $p \in [1,\infty)$ and $\nu \in (0, N]$, if and only if

$$\|u\|^p_{L^{p,\nu}(\mathbb{R}^N)} = \sup_{r>0,\bar{x}\in\mathbb{R}^N} r^{\nu-N} \int_{B(\bar{x},r)} |u(x)|^p dx < \infty.$$

By Hölder inequality, we can verify (refer to [8])

$$L^{2^*}(\mathbb{R}^N) \hookrightarrow L^{p,\nu}(\mathbb{R}^N), \quad \text{for } 1 \leq p < 2^*, \tag{1.8}$$

and

$$L^{p,\frac{(N-2\alpha)p}{2}}(\mathbb{R}^N) \hookrightarrow L^{p_1,\frac{(N-2\alpha)p_1}{2}}(\mathbb{R}^N), \quad \text{for } 1 < p_1 < p < 2^*. \tag{1.9}$$

Moreover, we have $L^{p,\nu}(\mathbb{R}^N) \hookrightarrow L^{1,\frac{\nu}{p}}(\mathbb{R}^N)$.

Next we give the definition of the Palais-Smale sequence. Let X be a Banach space, $\Phi \in C^1(X,\mathbb{R})$, $c \in \mathbb{R}$, we call $\{u_n\} \subset X$ is a Palais-Smale sequence of Φ if

$$\Phi(u_n) \to c,\ \Phi'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{1.10}$$

In this article we assume that:

(H1) $a(x) \in C(\mathbb{R}^N)$, $k(x) \in C(\mathbb{R}^N)$;

(H2)

$$\lim_{|x|\to\infty} a(x) = \bar{a} > 0, \quad \lim_{|x|\to\infty} k(x) = \bar{k} > 0,$$

$$\inf_{x\in\mathbb{R}^N} a(x) = \hat{a} > 0, \quad \inf_{x\in\mathbb{R}^N} k(x) = \hat{k} > 0.$$

In this article, we assume that $a(x), k(x)$ always satisfy (H1) and (H2). The energy functional associated with (1.1) is for all $u \in H^\alpha(\mathbb{R}^N)$,

$$\begin{aligned} I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2}u(x)|^2 + a(x)|u(x)|^2\Big)\,dx \\ - \frac{1}{2_s^*}\int_{\mathbb{R}^N} \frac{(u^+(x))^{2_s^*}}{|x|^s}dx - \frac{1}{q}\int_{\mathbb{R}^N} k(x)(u^+(x))^q\,dx. \end{aligned}$$

Finally we present some problems associated to (1.1) as follows.

The limit equation of (1.1) involving subcritical terms is

$$\begin{gathered} (-\Delta)^\alpha u + \bar{a}u = \bar{k}|u|^{q-2}u, \\ u \in H^\alpha(\mathbb{R}^N), \end{gathered} \tag{1.11}$$

and its corresponding variational functional is

$$\begin{aligned} I^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2}u(x)|^2 + \bar{a}|u(x)|^2\Big)dx \\ - \frac{1}{q}\int_{\mathbb{R}^N} \bar{k}(u^+(x))^q dx, \quad u \in H^\alpha(\mathbb{R}^N). \end{aligned}$$

The limit equation of (1.1) involving the Sobolev-Hardy critical nonlinear term is

$$\begin{gathered} (-\Delta)^\alpha u = \frac{|u|^{2_s^*-2}u}{|x|^s}, \\ u \in \dot{H}^\alpha(\mathbb{R}^N), \end{gathered} \tag{1.12}$$

and the corresponding variational functional is

$$I_s(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u(x)|^2 dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} \frac{(u^+(x))^{2_s^*}}{|x|^s}dx, \quad u \in \dot{H}^\alpha(\mathbb{R}^N).$$

In [5] Chen and Yang proved that all the positive solutions of (1.12) are of the form

$$U^\varepsilon(x) := \varepsilon^{\frac{2\alpha-N}{2}}U(x/\varepsilon), \tag{1.13}$$

and $U(x)$ satisfies

$$\frac{C_1}{1+|x|^{N-2\alpha}} \le U(x) \le \frac{C_2}{1+|x|^{N-2\alpha}}, \tag{1.14}$$

where $C_2 > C_1 > 0$ are constants. These solutions are also minimizers for the quotient

$$S_{\alpha,s} = \inf_{u \in \dot{H}^\alpha(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u(x)|^2 dx}{\big(\int_{\mathbb{R}^N} \frac{|u(x)|^{2_s^*}}{|x|^s}dx\big)^{2/2_s^*}},$$

which is associated with the fractional Sobolev-Hardy inequality (1.4). Define

$$D_0 = \int_{\mathbb{R}^N} \Big(\frac{1}{2}|(-\Delta)^{\alpha/2}U(x)|^2 - \frac{1}{2_s^*}\frac{|U(x)|^{2_s^*}}{|x|^s}\Big)dx = \frac{2\alpha-s}{2(N-s)}S_{\alpha,s}^{\frac{N-s}{2\alpha-s}}, \tag{1.15}$$

$$\begin{aligned} \mathcal{N} = \Big\{u \in H^\alpha(\mathbb{R}^N)\setminus\{0\} : \int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2}u(x)|^2 + \bar{a}|u(x)|^2 \\ - \bar{k}(u^+(x))^q\Big)\,dx = 0\Big\}, \end{aligned} \tag{1.16}$$

$$J^\infty = \inf_{u \in \mathcal{N}} I^\infty(u). \tag{1.17}$$

It is known that $\mathcal{N} \neq \emptyset$ since problem (1.11) has at least one positive solution if $N > 2\alpha$ (see [18]) for $1 < q < 2^*$.

The main result of our paper is as follows.

Theorem 1.1. *Suppose $a(x)$, $k(x)$ satisfy* (H1) *and* (H2)*, $2 < q < 2^*$, $0 < \alpha < 1$, $N > 4\alpha$, $0 < s < 2\alpha$. Assume that $\{u_n\}$ is a positive Palais-Smale sequence of I at level $d \geq 0$, then there exist two sequences $\{R_n^i\} \subset \mathbb{R}^+ \,(1 \leq i \leq l_1)$ and $\{y_n^j\} \subset \mathbb{R}^N \,(1 \leq j \leq l_2)$, $u \in H^\alpha(\mathbb{R}^N)$, and $u_j \in H^\alpha(\mathbb{R}^N)\,(1 \leq j \leq l_2)$, $(l_1, l_2 \in \mathbb{N})$ such that up to a subsequence:*

$$d = I(u) + l_1 D_0 + \sum_{j=1}^{l_2} I^\infty(u_j);$$

$$\|u_n - u - \sum_{i=1}^{l_1} U^{R_n^i} - \sum_{j=1}^{l_2} u_j(x - y_n^j)\|_{H^\alpha(\mathbb{R}^N)} = o(1) \quad \text{as } n \to \infty \tag{1.18}$$

where u and u_j $(1 \leq j \leq l_2)$ satisfy

$$I'(u) = 0, \quad I^{\infty\prime}(u_j) = 0,$$

$$R_n^i \to 0 \ (1 \leq i \leq l_1), \quad |y_n^j| \to \infty \ (1 \leq j \leq l_2) \quad \text{as } n \to \infty.$$

In particular, if $u \not\equiv 0$, then u is a weakly solution of (1.1)*. Note that the corresponding sum in* (1.18) *will be treated as zero if $l_i = 0$ $(i = 1, 2)$.*

Remark 1.2. (1) Similar to [23, Corollary 3.3], one can show that any Palais-Smale sequence for I at a level which is not of the form $m_1 D_0 + m_2 J^\infty$, $m_1, m_2 \in \mathbb{N} \bigcup \{0\}$, gives rise to a non-trivial weak solution of equation (1.1).

(2) In our non-compactness analysis, we prove that the blowing up positive Palais-Smale sequences can bear exactly two kinds of bubbles. Up to harmless constants, they are either of the form

$$U^{R_n}(x), \quad |R_n| \to 0 \quad \text{as } n \to \infty,$$

or

$$u(x - y_n) \in H^\alpha(\mathbb{R}^N), \quad |y_n| \to \infty, \quad \text{as } n \to \infty,$$

where u is the solution of (1.11). For any Palais-Smale sequence u_n for I, ruling out the above two bubbles yields the existence of a non-trivial weak solution of equation (1.1).

(3) Because of the lower order terms $a(x)u$ and $k(x)|u|^{q-2}u$ in (1.1), we must deal with $u \in H^\alpha(\mathbb{R}^N)$ to ensure that the functional $I(u)$ is well defined. In fact, if $u \in H^\alpha(\mathbb{R}^N)$, by the Sobolev inequality, $u \in L^2(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N)$ for $2 < q < 2^*$. Noting that $\|u\|_{L^2}$ and $\|u\|_{L^q}$ only satisfy the translation invariance and $\int_{\mathbb{R}^N} \frac{(u^+(x))^{2_s^*}}{|x|^s} dx$ only satisfies the scaling invariance, then there exists a new limit equation (1.11) which causes some new structures for the Palais-Smale sequence of (1.1).

Using the compactness results and the Mountain Pass Theorem [3] we prove the following existence result.

Theorem 1.3. *Assume that $2 < q < 2^*$, $0 < \alpha < 1$, $0 < s < 2\alpha$, $N > 4\alpha$. If $a(x), k(x)$ satisfy* (H1), (H2) *and*

$$\bar{a} \geq a(x), \quad k(x) \geq \bar{k} > 0, \quad k(x) \not\equiv \bar{k}. \tag{1.19}$$

Then (1.1) *has a nontrivial solution* $u \in H^\alpha(\mathbb{R}^N)$ *which satisfies*

$$I(u) < \min\big\{\frac{2\alpha - s}{2(N-s)} S_{\alpha,s}^{\frac{N-s}{2\alpha-s}}, J^\infty\big\}.$$

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by carefully analyzing the features of a positive Palais-Smale sequence for I. Theorem 1.3 is proved in Section 3 by applying Theorem 1.1 and the Mountain Pass Theorem. Finally we put some preliminaries in the last section as an appendix.

2. Non-compactness analysis

In this section, we prove Theorem 1.1 by using the Concentration-Compactness Principle and a delicate analysis of the Palais-Smale sequences of I. Firstly we give the following Lemmas.

Lemma 2.1. *Let* $0 < \alpha < N/2$, $0 < s < 2\alpha$, $r > 0$, $\{u_n\} \subset \dot{H}^\alpha(\mathbb{R}^N)$ *be a bounded sequence such that*

$$\inf_{n\in\mathbb{N}} \int_{B(0,r)} \frac{\big(u_n^+(x)\big)^{2_s^*}}{|x|^s} dx \geq c > 0. \tag{2.1}$$

Then, up to subsequence, there exist two sequences $\{r_n\} \subset \mathbb{R}^+$ *and* $\{x_n\} \subset B(0,2r)$ *such that*

$$\bar{u}_n \rightharpoonup w \not\equiv 0 \quad \text{in } \dot{H}^\alpha(\mathbb{R}^N), \tag{2.2}$$

where

$$\bar{u}_n(x) = \begin{cases} r_n^{\frac{N-2\alpha}{2}} u_n(r_n x) & \text{when } x_n/r_n \text{ is bounded}, \\ r_n^{\frac{N-2\alpha}{2}} u_n(r_n x + x_n) & \text{when } |x_n/r_n| \to \infty. \end{cases} \tag{2.3}$$

Proof. Let $0 \leq \eta(x) \leq 1$, $\eta(x) \in C_0^\infty(\mathbb{R}^N), \eta(x) \equiv 1$ on $B(0,r)$, $\eta(x) \equiv 0$ on $B(0,2r)^C$. From [8, Lemma 5.3], it follows that

$$\|\eta u_n\|_{\dot{H}^\alpha(\mathbb{R}^N)} \leq C\|u_n\|_{\dot{H}^\alpha(\mathbb{R}^N)}. \tag{2.4}$$

By [5, Theorem 1.2],

$$\Big(\int_{\mathbb{R}^N} \frac{|\eta(x)u_n(x)|^{2_s^*}}{|x|^s} dx\Big)^{1/2_s^*} \leq C\|\eta u_n\|_{\dot{H}^\alpha(\mathbb{R}^N)}^\theta \|\eta u_n\|_{L^{2,N-2\alpha}(\mathbb{R}^N)}^{1-\theta}, \tag{2.5}$$

where $\max\{\frac{N-2\alpha}{N-s}, \frac{2\alpha-s}{N-s}\} \leq \theta < 1$. From (2.4) and (2.5), it follows

$$\begin{aligned} c &\leq \Big(\int_{B(0,r)} \frac{(u_n^+(x))^{2_s^*}}{|x|^s} dx\Big)^{1/2_s^*} \leq \Big(\int_{\mathbb{R}^N} \frac{|\eta(x)u_n(x)|^{2_s^*}}{|x|^s} dx\Big)^{1/2_s^*} \\ &\leq C\|u_n\|_{\dot{H}^\alpha(\mathbb{R}^N)}^\theta \|\eta u_n\|_{L^{2,N-2\alpha}(\mathbb{R}^N)}^{1-\theta}. \end{aligned} \tag{2.6}$$

Then there exists a constant $c > 0$ such that

$$\|\eta u_n\|_{L^{2,N-2\alpha}(\mathbb{R}^N)}^2 = \sup_{\bar{x}\in\mathbb{R}^N,\, R\in\mathbb{R}^+} R^{-2\alpha} \int_{B(\bar{x},R)} |\eta(x)u_n(x)|^2 dx \geq c > 0. \tag{2.7}$$

From (2.7), we may find $r_n > 0$ and $x_n \in B(0,2r)$ such that for n large enough,

$$r_n^{-2\alpha} \int_{B(x_n,r_n)} |\eta(x)u_n(x)|^2 dx \geq \|\eta u_n\|_{L^{2,N-2\alpha}(\mathbb{R}^N)}^2 - \frac{c}{2n} \geq c/2 > 0. \tag{2.8}$$

Denote

$$\bar{u}_n(x) = \begin{cases} r_n^{\frac{N-2\alpha}{2}} u_n(r_n x) & \text{when } \frac{x_n}{r_n} \text{ is bounded,} \\ r_n^{\frac{N-2\alpha}{2}} u_n(r_n x + x_n) & \text{when } |\frac{x_n}{r_n}| \to \infty. \end{cases} \tag{2.9}$$

Since $\{u_n\}$ is bounded in $\dot{H}^\alpha(\mathbb{R}^N)$, from the scaling and translation invariance of $\dot{H}^\alpha(\mathbb{R}^N)$, we have $\{\bar{u}_n\}$ is bounded in $\dot{H}^\alpha(\mathbb{R}^N)$; therefore, up to a subsequence (still denoted by $\bar{u}_n$),

$$\bar{u}_n \rightharpoonup w \text{ in } \dot{H}^\alpha(\mathbb{R}^N), \quad \text{and} \quad \bar{u}_n \to w \text{ in } L^2_{\rm loc}(\mathbb{R}^N), \quad \text{as } n \to \infty.$$

If x_n/r_n is bounded, there exist a $\tilde{R} > 1$ such that $B(\frac{x_n}{r_n}, 1) \subset B(0, \tilde{R})$, then

$$c/2 < \int_{B(\frac{x_n}{r_n},1)} |\bar{u}_n(x)\eta(r_n x)|^2 dx \le \int_{B(0,\tilde{R})} |\bar{u}_n(x)|^2 dx \to \int_{B(0,\tilde{R})} |w(x)|^2 dx. \tag{2.10}$$

If $|\frac{x_n}{r_n}| \to \infty$, then

$$\begin{aligned} c/2 < \int_{B(0,1)} |\bar{u}_n(x)\eta(r_n x + x_n)|^2 dx &\le \int_{B(0,\tilde{R})} |\bar{u}_n(x)|^2 dx \\ \to \int_{B(0,\tilde{R})} |w(x)|^2 dx \end{aligned} \tag{2.11}$$

where $\tilde{R} > 1$. Obviously we have $w \not\equiv 0$. From (2.10) and (2.11), Lemma 2.1 is complete. □

Lemma 2.2. *Assume $N > 4\alpha, 0 < s < 2\alpha, 2 < q < 2^*, 0 < \alpha < 1$. Let $\{v_n\} \subset H^\alpha(\mathbb{R}^N)$ be a Palais-Smale sequence of I at level d_1 and $v_n \rightharpoonup 0$ in $H^\alpha(\mathbb{R}^N)$ as $n \to \infty$. If there exists a sequence $\{r_n\} \subset \mathbb{R}^+$, with $r_n \to 0$ as $n \to \infty$ such that $\bar{v}_n(x) := r_n^{\frac{N-2\alpha}{2}} v_n(r_n x)$ converges weakly in $\dot{H}^\alpha(\mathbb{R}^N)$ and almost everywhere to some $v_0 \in \dot{H}^\alpha(\mathbb{R}^N)$ as $n \to \infty$ with $v_0 \not\equiv 0$, then v_0 solves (1.12) and the sequence $z_n(x) := v_n(x) - v_0(\frac{x}{r_n}) r_n^{\frac{2\alpha-N}{2}}$ is a Palais-Smale sequence of I at level $d_1 - I_s(v_0)$.*

Proof. First, we prove that v_0 solves (1.12) and $I(z_n) = I(v_n) - I_s(v_0)$. Fix a ball $B(0,r)$ and a test function $\phi \in C_0^\infty(B(0,r))$. Since

$$\bar{v}_n \rightharpoonup v_0 \text{ in } \dot{H}^\alpha(\mathbb{R}^N), \tag{2.12}$$

applying Lemma 4.3, it implies

$$
\begin{aligned}
&\langle \phi, I_s'(v_0)\rangle + o(1) = \langle \phi, I_s'(\bar v_n)\rangle \\
&= c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\bar v_n(x)-\bar v_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \int_{\mathbb{R}^N}\frac{\big(\bar v_n^+(x)\big)^{2_s^*-1}\phi(x)}{|x|^s}\,dx \\
&= c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\bar v_n(x)-\bar v_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N}\frac{\big(\bar v_n^+(x)\big)^{2_s^*-1}\phi(x)}{|x|^s}\,dx + r_n^{2\alpha}\int_{\mathbb{R}^N} a(r_nx)\phi(x)\bar v_n(x)\,dx \\
&\quad - r_n^{N-\frac{N-2\alpha}{2}q}\int_{\mathbb{R}^N} k(r_nx)\phi(x)(\bar v_n^+(x))^{q-1}dx + o(1) \\
&= c_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v_n(x)-v_n(y))(\phi_n(x)-\phi_n(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N}\frac{(v_n^+(x))^{2_s^*-1}\phi_n(x)}{|x|^s}\,dx + \int_{\mathbb{R}^N} a(x)\phi_n(x)v_n(x)dx \\
&\quad - \int_{\mathbb{R}^N} k(x)\phi_n(x)(v_n^+(x))^{q-1}\,dx + o(1) \\
&= o(1) \quad \text{as } n\to\infty,
\end{aligned} \tag{2.13}
$$

where $\phi_n = r_n^{-\frac{N-2\alpha}{2}}\phi(\frac{x}{r_n})$. The last equality in (2.13) holds since

$$
\begin{gathered}
\int_{\mathbb{R}^N}|\phi_n(x)|^2dx = r_n^{2\alpha}\int_{\mathbb{R}^N}|\phi(x)|^2dx = o(1),\\
\|\phi\|_{\dot H^\alpha(\mathbb{R}^N)} = \|\phi_n\|_{\dot H^\alpha(\mathbb{R}^N)} = \|\phi_n\|_{H^\alpha(\mathbb{R}^N)} + o(1), \quad \text{as } n\to\infty.
\end{gathered}
$$

Thus v_0 is a nontrival critical point of I_s. By Lemma 4.6, (1.14) and the fact $N>4\alpha$, it follows

$$
\int_{\mathbb{R}^N}|v_0(x)|^pdx \le c\int_{\mathbb{R}^N}\frac{1}{(1+|x|^{N-2\alpha})^p}dx \le c, \quad \forall p\ge 2, \tag{2.14}
$$

which implies that $v_0\in L^2(\mathbb{R}^N)$. Let

$$
z_n(x) = v_n(x) - r_n^{\frac{2\alpha-N}{2}}v_0(\frac{x}{r_n}) \in H^\alpha(\mathbb{R}^N).
$$

Obviously $z_n \rightharpoonup 0$ in $H^\alpha(\mathbb{R}^N)$ as $n\to\infty$. Now we prove that $\{z_n\}$ is a Palais-Smale sequence of I at level $d_1 - I_s(v_0)$. From (2.14), $v_0\in L^p(\mathbb{R}^N)$ for all $p\in[2,2^*)$. Then it follows that

$$
\int_{\mathbb{R}^N}|v_0(\frac{x}{r_n})r_n^{\frac{2\alpha-N}{2}}|^pdx = r_n^{N-p\frac{(N-2\alpha)}{2}}\|v_0\|^p_{L^p(\mathbb{R}^N)} \to 0 \tag{2.15}
$$

as $n\to\infty$ for all $2\le p<2^*$. By the Brézis-Lieb Lemma and the weak convergence, similar to Lemma 4.7, we can prove that

$$
\begin{gathered}
I(z_n) = I(v_n) - I_s(v_0),\\
\langle I'(z_n),\phi\rangle = o(1)
\end{gathered}
$$

as $n\to\infty$. This completes the proof. □

Proof of Theorem 1.1. By Lemma 4.4 in the appendix, we can assume that $\{u_n\}$ is bounded. Up to a subsequence, let $n \to \infty$, and assume that

$$u_n \rightharpoonup u \quad \text{in } H^\alpha(\mathbb{R}^N), \tag{2.16}$$

$$u_n \to u \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N) \text{ for } 2 \le p < 2^*, \tag{2.17}$$

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \tag{2.18}$$

Denote $v_n(x) = u_n(x) - u(x)$, then $\{v_n\}$ is a Palais-Smale sequence of I and

$$v_n \rightharpoonup 0 \quad \text{in } H^\alpha(\mathbb{R}^N), \tag{2.19}$$

$$v_n \to 0 \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N) \text{ for } 2 \le p < 2^*, \tag{2.20}$$

$$v_n \to 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{2.21}$$

Then by Lemma 4.7 we know that

$$I(v_n) = I(u_n) - I(u) + o(1), \text{ as } n \to \infty, \tag{2.22}$$

$$I'(v_n) = o(1), \text{ as } n \to \infty, \tag{2.23}$$

$$\|v_n\|_{H^\alpha(\mathbb{R}^N)} = \|u_n\|_{H^\alpha(\mathbb{R}^N)} - \|u\|_{H^\alpha(\mathbb{R}^N)} + o(1), \text{ as } n \to \infty. \tag{2.24}$$

Without loss of generality, we assume that

$$\|v_n\|^2_{H^\alpha(\mathbb{R}^N)} \to l > 0 \quad \text{as } n \to \infty.$$

In fact if $l = 0$, Theorem 1.1 is proved for $l_1 = 0, l_2 = 0$.

Step 1. Getting rid of the blowing up bubbles caused by the Sobolev-Hardy term. Suppose there exists $0 < \delta < \infty$ such that

$$\inf_{n\in\mathbb{N}} \int_{|x|<R} \frac{\big(v_n^+(x)\big)^{2^*_s}}{|x|^s}\, dx \ge \delta > 0, \quad \text{for some } 0 < R < \infty. \tag{2.25}$$

It follows from Lemma 2.1 that there exist two sequences $\{r_n\} \subset \mathbb{R}^+$ and $\{x_n\} \subset B(0, 2R)$, such that

$$\bar v_n(x) \rightharpoonup v_0 \not\equiv 0 \quad \text{in } \dot H^\alpha(\mathbb{R}^N), \tag{2.26}$$

where

$$\bar v_n(x) = \begin{cases} r_n^{\frac{N-2\alpha}{2}} v_n(r_n x) & \text{when } \frac{x_n}{r_n} \text{ is bounded,} \\ r_n^{\frac{N-2\alpha}{2}} v_n(r_n x + x_n) & \text{when } |\frac{x_n}{r_n}| \to \infty. \end{cases} \tag{2.27}$$

Now we claim that $r_n \to 0$ as $n \to \infty$. In fact there exists a $R_1 > 0$ such that

$$\int_{B(0,R_1)} |v_0(x)|^p dx = \delta_1 > 0, \quad \text{for } 2 \le p < 2^*. \tag{2.28}$$

From the Sobolev compact embedding, (2.17), (2.26) and (2.28), we have that for all $r > 0$,

$$v_n \to 0 \quad \text{in } L^p(B(0, r)) \text{ for all } 2 \le p < 2^*,$$

$$\bar v_n \to v_0 \quad \text{in } L^p(B(0, r)) \text{ for all } 2 \le p < 2^*,$$

$$\begin{aligned} 0 &\neq \|v_0\|^2_{L^2(B(0,R_1))} + o(1) \\ &= \int_{B(0,R_1)} |\bar{v}_n(x)|^2 dx \\ &= \begin{cases} r_n^{-2\alpha} \int_{B(0,r_nR_1)} |v_n(x)|^2 dx, & \text{if } \frac{x_n}{r_n} \text{ is bounded,} \\ r_n^{-2\alpha} \int_{B(x_n,r_nR_1)} |v_n(x)|^2 dx & \text{if } |\frac{x_n}{r_n}| \to \infty. \end{cases} \end{aligned} \tag{2.29}$$

If $r_n \to r_0 > 0$, then

$$\begin{gathered} r_n^{-2\alpha} \int_{B(0,r_nR_1)} |v_n(x)|^2 dx \le c r_0^{-2\alpha} \|v_n\|^2_{L^2(B(0,cR_1))} \to 0; \\ r_n^{-2\alpha} \int_{B(x_n,r_nR_1)} |v_n(x)|^2 dx \le c r_0^{-2\alpha} \|v_n\|^2_{L^2(B(0,cR_1+4R))} \to 0. \end{gathered} \tag{2.30}$$

If $r_n \to \infty$, then

$$\begin{gathered} r_n^{-2\alpha} \int_{B(0,r_nR_1)} |v_n(x)|^2 dx \le r_n^{-2\alpha} \|v_n\|^2_{H^\alpha(\mathbb{R}^N)} \to 0, \\ r_n^{-2\alpha} \int_{B(x_n,r_nR_1)} |v_n(x)|^2 dx \le r_n^{-2\alpha} \|v_n\|^2_{H^\alpha(\mathbb{R}^N)} \to 0. \end{gathered} \tag{2.31}$$

A contradiction to (2.29). Thus we have $r_n \to 0$.

Next we claim that x_n/r_n is bounded. Indeed, if on the contrary, $|\frac{x_n}{r_n}| \to \infty$, fix a ball $B(0,r)$ and a test function $\phi \in C_0^\infty(B(0,r))$, then

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{(\bar{v}_n^+(x))^{2_s^*-1}\phi(x)}{|x+\frac{x_n}{r_n}|^s}\, dx &= \int_{B(0,r)} \frac{(\bar{v}_n^+(x))^{2_s^*-1}\phi(x)}{|x+\frac{x_n}{r_n}|^s}\, dx \\ &\le \frac{c}{|\frac{x_n}{r_n}|} \int_{B(0,r)} (\bar{v}_n^+(x))^{2_s^*-1}\phi(x)\, dx \to 0, \end{aligned} \tag{2.32}$$

similar to (2.13), it follows that

$$(-\Delta)^\alpha u = 0, \quad x \in \mathbb{R}^N \tag{2.33}$$

which implies that $\|v_0\|_{\dot{H}^\alpha(\mathbb{R}^N)} = 0$. By the Sobolev inequality and the Hölder inequality it follows

$$\|v_0\|_{L^p(B(0,R_1))} \le c\|v_0\|_{L^{2^*}(B(0,R_1))} \le c\|v_0\|_{L^{2^*}(\mathbb{R}^N)} \le C\|v_0\|_{\dot{H}^\alpha(\mathbb{R}^N)} = 0 \tag{2.34}$$

for $2 \le p < 2^*$. This contradicts (2.28). So we can deduce that x_n/r_n is bounded and $\bar{v}_n(x) = r_n^{\frac{N-2\alpha}{2}} v_n(r_n x)$.

Define $z_n(x) = v_n(x) - v_0(\frac{x}{r_n}) r_n^{\frac{2\alpha-N}{2}}$, then $z_n \rightharpoonup 0$ in $H^\alpha(\mathbb{R}^N)$. It follows from Lemma 2.2 that $\{z_n\}$ is a Palais-Smale sequence of I satisfying

$$I(z_n) = I(v_n) - I_s(v_0) + o(1), \quad \text{as } n \to \infty. \tag{2.35}$$

Since v_0 satisfies (1.12), from Lemma 4.6, (1.13) and (1.15) there exists $\varepsilon_1 > 0$ such that

$$v_0(x) = \varepsilon_1^{\frac{2\alpha-N}{2}} U(\frac{x}{\varepsilon_1}), \quad I_s(v_0) = D_0. \tag{2.36}$$

Let $R_n^1 = r_n\varepsilon_1$, from (2.36), it follows

$$r_n^{\frac{2\alpha-N}{2}} v_0(\frac{x}{r_n}) = (R_n^1)^{\frac{2\alpha-N}{2}} U(\frac{x}{R_n^1}) = U^{R_n^1}(x), \tag{2.37}$$

with $R_n^1 \to 0$. Then from (2.22) it follows that

$$\begin{gathered} z_n(x) = v_n(x) - U^{R_n^1}(x) = u_n(x) - u(x) - U^{R_n^1}(x), \\ I(z_n) = I(v_n) - D_0 + o(1) = I(u_n) - I(u) - D_0 + o(1) \end{gathered} \tag{2.38}$$

with $R_n^1 \to 0$. From Lemma 4.8, letting $a = v_n, b = U^{R_n^1}$, it follows

$$\begin{aligned} \int_{|x|<R} \frac{\big(z_n^+(x)\big)^{2_s^*}}{|x|^s} dx &= \int_{\tilde{B}(0,R)} \frac{\big(z_n(x)\big)^{2_s^*}}{|x|^s} dx \\ &\le \int_{\tilde{B}(0,R)} \frac{\big(v_n(x)\big)^{2_s^*} - (U^{R_n^1}(x))^{2_s^*}}{|x|^s} dx \\ &= \int_{\tilde{B}(0,R)} \frac{\big(v_n^+(x)\big)^{2_s^*}}{|x|^s} dx - C \\ &\le \int_{|x|<R} \frac{\big(v_n^+(x)\big)^{2_s^*}}{|x|^s} dx - C \end{aligned} \tag{2.39}$$

where $\tilde{B}(0,R) = \{x | z_n(x) \ge 0\} \cap B(0,R)$.

If still there exists a $\bar{\delta} > 0$ such that

$$\int_{|x|<R} \frac{\big(z_n^+(x)\big)^{2_s^*}}{|x|^s} dx \ge \bar{\delta} > 0,$$

then we repeat the previous argument. From (2.39) and the fact

$$\int_{|x|<R} \frac{\big(v_n^+(x)\big)^{2_s^*}}{|x|^s} dx \le \|v_n\|_{H^\alpha}^{2_s^*} \le c,$$

we deduce that the iteration must stop after finite times. That is to see, there exist a positive constant l_1 and a new Palais-Smale sequence of I, (without loss of generality) denoted by $\{v_n\}$, such that as $n \to \infty$,

$$d = I(v_n) + I(u) + l_1 D_0, \quad v_n(x) = u_n(x) - u(x) - \sum_{i=1}^{l_1} U^{R_n^i}(x), \tag{2.40}$$

with $R_n^i \to 0$,

$$\int_{|x|<R} \frac{\big(v_n^+(x)\big)^{2_s^*}}{|x|^s} dx = o(1) \quad \text{for any } 0 < R < \infty, \tag{2.41}$$

$$v_n \rightharpoonup 0 \quad \text{in } H^\alpha(\mathbb{R}^N). \tag{2.42}$$

Step 2. Getting rid of the blowing up bubbles caused by unbounded domains. Suppose there exists $0 < \delta < \infty$ such that

$$\Big(\int_{\mathbb{R}^N} (v_n^+(x))^q dx\Big)^{2/q} \ge \delta > 0, \quad \text{for } 2 < q < 2^*. \tag{2.43}$$

By the interpolation inequality, it follows that

$$\|v_n\|_{L^q} \le \|v_n\|_{L^2}^\lambda \|v_n\|_{L^{2^*}}^{1-\lambda}, \quad \text{for } 2 < q < 2^*$$

where $0 < \lambda < 1$. Thus there exist $\tilde{\delta} > 0$ such that

$$\|v_n\|_{L^2}^2 \geq \tilde{\delta} > 0.$$

By Lemma 4.1, there exists a subsequence still denoted by $\{v_n\}$, such that one of the following two gathered occurs.

(i) Vanish occurs: for all $0 < R < \infty$,

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |v_n(x)|^2 dx \to 0 \quad \text{as } n \to \infty.$$

By Lemma 4.2, (4.10) and Sobolev inequality, it follows

$$\int_{\mathbb{R}^N} (v_n^+(x))^q dx \to 0 \quad \text{as } n \to \infty,\ \forall\, 2 < q < 2^*,$$

which contradicts (2.43).

(ii) Nonvanish occurs: there exist $\beta > 0$, $0 < \bar{R} < \infty$, $\{y_n\} \subset \mathbb{R}^N$, such that

$$\liminf_{n\to\infty} \int_{y_n + B_{\bar{R}}} |v_n(x)|^2 dx \geq \beta > 0. \tag{2.44}$$

We claim that $|y_n| \to \infty$ as $n \to \infty$. Otherwise, if there exists a constant $M > 0$ such that $|y_n| \leq M$, then we can choose a $R_2 > 0$ large enough such that

$$\int_{y_n + B_{\bar{R}}} |v_n(x)|^2 dx \leq \|v_n\|_{L^2(B(0,R_2))}^2 \to 0 \text{ as } n \to \infty. \tag{2.45}$$

which contradicts (2.44).

To proceed, we first construct the Palais-Smale sequences of I^∞. Denote $\bar{v}_n(x) = v_n(x + y_n)$. Since $\|\bar{v}_n\|_{H^\alpha(\mathbb{R}^N)} = \|v_n\|_{H^\alpha(\mathbb{R}^N)} \leq c$, without loss of generality, we assume that as $n \to \infty$,

$$\begin{gathered} \bar{v}_n \rightharpoonup v_0 \quad \text{in } H^\alpha(\mathbb{R}^N), \\ \bar{v}_n \to v_0 \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N), \text{ for any } 1 < p < 2^*. \end{gathered} \tag{2.46}$$

By (2.41), we have that for all $\phi \in C_0^\infty(\mathbb{R}^N)$ as $n \to \infty$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{(\bar{v}_n^+(x))^{2_s^*-1}\phi(x)}{|x+y_n|^s} dx \\ &= \int_{\mathbb{R}^N} \frac{(v_n^+(x))^{2_s^*-1}\phi_n(x)}{|x|^s} dx \\ &= \int_{|x|>r} \frac{(v_n^+(x))^{2_s^*-1}\phi_n(x)}{|x|^s} dx + o(1) \\ &\leq \frac{1}{r^s}\Big(\int_{\mathbb{R}^N} |v_n(x)|^{2^*} dx\Big)^{\frac{2_s^*-1}{2^*}} \Big(\int_{\mathbb{R}^N} |\phi_n(x)|^{q_1} dx\Big)^{1/q_1} + o(1), \end{aligned} \tag{2.47}$$

where $\phi_n = \phi(x - y_n)$ and $q_1 = \frac{2^*}{2^*+1-2_s^*}$. Obviously

$$\int_{\mathbb{R}^N} |\phi_n(x)|^{q_1} dx = \int_{\mathbb{R}^N} |\phi(x)|^{q_1} dx \leq c, \quad \int_{\mathbb{R}^N} |v_n(x)|^{2^*} dx \leq c. \tag{2.48}$$

Let $r \to \infty$, from (2.47) and (2.48), we have

$$\int_{\mathbb{R}^N} \frac{(\bar{v}_n^+(x))^{2_s^*-1}\phi(x)}{|x+y_n|^s} dx = o(1) \quad \text{as } n \to \infty. \tag{2.49}$$

Similarly we have

$$\int_{\mathbb{R}^N} \frac{(\bar v_n^+(x))^{2_s^*}}{|x+y_n|^s}dx = o(1) \quad \text{as } n\to\infty. \tag{2.50}$$

Since $v_n \rightharpoonup 0$ in $H^\alpha(\mathbb{R}^N)$ and $\lim_{n\to\infty} a(x+y_n)=\bar a$, we have as $n\to\infty$,

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^N} a(x)v_n(x)\phi_n(x)\,dx \\ &= \int_{\mathbb{R}^N} \bar a\bar v_n(x)\phi(x)\,dx + \int_{\mathbb{R}^N}[a(x+y_n)-\bar a]\bar v_n(x)\phi(x)\,dx \end{aligned}$$

and

$$|\int_{\mathbb{R}^N}[a(x+y_n)-\bar a]\bar v_n(x)\phi(x)\,dx| \le c(\int_{\mathbb{R}^N}|a(x+y_n)-\bar a|^2\phi(x)^2dx)^{1/2} = o(1);$$

that is,

$$\int_{\mathbb{R}^N} \bar a\bar v_n(x)\phi(x)\,dx = o(1) = \int_{\mathbb{R}^N} a(x)v_n(x)\phi_n(x)\,dx \quad \text{as } n\to\infty. \tag{2.51}$$

Similarly we have

$$\int_{\mathbb{R}^N} k(x)(v_n^+(x))^{q-1}\phi_n(x)\,dx = \int_{\mathbb{R}^N}\bar k(\bar v_n^+(x))^{q-1}\phi(x)\,dx = o(1) \tag{2.52}$$

as $n\to\infty$. Recall that v_n is a Palais-Smale sequence of I, by (2.46) and (2.49)-(2.52) we have

$$o(1) = \langle I'(v_n),\phi_n\rangle = \langle {I^\infty}'(\bar v_n),\phi\rangle + o(1) = \langle {I^\infty}'(v_0),\phi\rangle + o(1), \tag{2.53}$$

as $n\to\infty$. This shows that v_0 is a weak solution of (1.11).

We claim that $v_0\not\equiv 0$. From (2.43), we may assume that there exists a sequence $\{y_n\}$ satisfying (2.44) and

$$\int_{B(y_n,R)} (v_n^+(x))^q dx = b + o(1) > 0, \quad \text{as } n\to\infty, \tag{2.54}$$

where $b>0$ is a constant. If $v_0\equiv 0$, we have

$$\int_{B(0,R)} (\bar v_n^+(x))^q dx = \int_{B(y_n,R)} (v_n^+(x))^q dx = o(1) \text{ as } n\to\infty \text{ for } 0<R<\infty$$

which contradicts (2.54).

Denote $z_n(x) = v_n(x) - v_0(x-y_n)$. Since

$$\begin{aligned} I(v_n) &= \frac12\int_{\mathbb{R}^N}\Big(|(-\Delta)^{\alpha/2}v_n(x)|^2 + a(x)|v_n(x)|^2\Big)dx \\ &\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^N}\frac{(v_n^+(x))^{2_s^*}}{|x|^s}dx - \frac1q\int_{\mathbb{R}^N}k(x)(v_n^+(x))^q\,dx \\ &= \frac12\int_{\mathbb{R}^N}\Big(|(-\Delta)^{\alpha/2}\bar v_n(x)|^2 + a(x+y_n)|\bar v_n(x)|^2\Big)dx \\ &\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^N}\frac{(\bar v_n^+(x))^{2_s^*}}{|x+y_n|^s}dx - \frac1q\int_{\mathbb{R}^N}k(x+y_n)(\bar v_n^+(x))^q\,dx \\ &= \frac12\int_{\mathbb{R}^N}\Big(|(-\Delta)^{\alpha/2}\bar v_n(x)|^2 + \bar a|\bar v_n(x)|^2\Big)dx \\ &\quad - \frac1q\int_{\mathbb{R}^N}\bar k(\bar v_n^+(x))^q\,dx + o(1), \end{aligned}$$

where the last equality is a result of (2.50), therefore, as $n \to \infty$,

$$\|z_n\|_{H^\alpha(\mathbb{R}^N)} = \|\bar{v}_n\|_{H^\alpha(\mathbb{R}^N)} - \|v_0\|_{H^\alpha(\mathbb{R}^N)} + o(1), \tag{2.55}$$

$$I(z_n) = I^\infty(\bar{v}_n) - I^\infty(v_0) + o(1) = I(v_n) - I^\infty(v_0) + o(1). \tag{2.56}$$

Hence $z_n \rightharpoonup 0$ in $H^\alpha(\mathbb{R}^N)$ as $n \to \infty$, and z_n is a Palais-Smale sequence of I. From (4.10) in Lemma 4.5, it follows $\|v_0^-\|_{H^\alpha} = 0$, that is $v_0 \geq 0$ a.e. in $\mathbb{R}^N$. Then by Brezis-Lieb Lemma and (4.10), there exists a constant $c > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} (z_n^+(x))^q dx &= \int_{\mathbb{R}^N} (v_n^+(x))^q dx - \int_{\mathbb{R}^N} (v_0^+(x))^q dx + o(1) \\ &\leq \int_{\mathbb{R}^N} (v_n^+(x))^q dx - c \end{aligned} \tag{2.57}$$

where the last inequality follows from the fact $v_0 \not\equiv 0$. If $\|z_n\|_{L^q(\mathbb{R}^N)} \to \delta_2 > 0$ as $n \to \infty$, from (2.57) and the boundedness of $\|v_n\|_{L^q}$, then one can repeat Step 2 for finite times (l_2 times). Thus from (2.40) and Step 2, we obtain a new Palais-Smale sequence of I, without loss of generality still denoted by v_n, such that

$$d = I(u) + I(v_n) + l_1 D_0 + \sum_{j=1}^{l_2} I^\infty(u_j) + o(1), \tag{2.58}$$

$$v_n(x) = u_n(x) - u(x) - \sum_{i=1}^{l_1} U^{R_n^i}(x) - \sum_{j=1}^{l_2} u_j(x - y_n^j), \quad \text{with } R_n^i \to 0, \tag{2.59}$$

$$\|v_n^+\|_{L^q(\mathbb{R}^N)} \to 0, \quad \int_{\mathbb{R}^N} \frac{\big(v_n^+(x)\big)^{2_s^*}}{|x|^s} dx \to 0 \tag{2.60}$$

as $n \to \infty$. Then from the fact $< I'(v_n), v_n >= o(1)$, it follows that

$$\begin{aligned} \|v_n\|^2_{H^\alpha(\mathbb{R}^N)} &\leq c \int_{\mathbb{R}^N} (|\big(-\Delta\big)^{\alpha/2} v_n(x)|^2 + a(x)|v_n(x)|^2\big) dx \\ &= c\Big(\int_{\mathbb{R}^N} k(x)(v_n^+(x))^q dx + \int_{\mathbb{R}^N} \frac{\big(v_n^+(x)\big)^{2_s^*}}{|x|^s} dx\Big) \to 0 \end{aligned} \tag{2.61}$$

as $n \to \infty$. From (2.60) and (2.61), it gives

$$I(v_n) = o(1). \tag{2.62}$$

From (2.58)-(2.62), the proof is complete. □

3. Proof of Theorem 1.3

To this end we use the mountain pass theorem [3] and Theorem 1.1.

Proof of Theorem 1.3. From

$$\begin{aligned} I(tu) = \frac{t^2}{2} \Big[\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u(x)|^2 dx + \int_{\mathbb{R}^N} a(x)|u(x)|^2 \, dx \Big] \\ - \frac{|t|^{2_s^*}}{2_s^*} \int_{\mathbb{R}^N} \frac{(u^+(x))^{2_s^*}}{|x|^s} dx - \frac{|t|^q}{q} \int_{\mathbb{R}^N} k(x)(u^+(x))^q \, dx, \end{aligned}$$

we deduce that for a fixed $u \not\equiv 0$ in $H^\alpha(\mathbb{R}^N)$, $I(tu) \to -\infty$ if $t \to \infty$. Since

$$\int_{\mathbb{R}^N} (u^+(x))^q \, dx \leq C\|u\|^q_{H^\alpha(\mathbb{R}^N)}, \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{(u^+(x))^{2_s^*}}{|x|^s} \, dx \leq C\|u\|^{2_s^*}_{H^\alpha(\mathbb{R}^N)},$$

we have

$$I(u) \geq c\|u\|^2_{H^\alpha(\mathbb{R}^N)} - C(\|u\|^q_{H^\alpha(\mathbb{R}^N)} + \|u\|^{2^*_s}_{H^\alpha(\mathbb{R}^N)}).$$

Hence, there exists $r_0 > 0$ small such that $I(u)\Big|_{\partial B(0,r_0)} \geq \rho > 0$ for q, $2^*_s > 2$.

As a consequence, $I(u)$ satisfies the geometry structure of Mountain-Pass Theorem. Now, we define

$$c^* =: \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^\alpha(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = \psi_0 \in H^\alpha(\mathbb{R}^N)\}$ with $I(t\psi_0) \leq 0$ for all $t \geq 1$.

To complete the proof of Theorem 1.3, we need to verify that $I(u)$ satisfies the local Palais-Smale conditions. According to By 1.2, we only need to verify that

$$c^* < \min\{\frac{2\alpha - s}{2(N-s)} S_{\alpha,s}^{\frac{N-s}{2\alpha-s}}, J^\infty\}. \tag{3.1}$$

Set

$$v_\varepsilon(x) = \frac{U_\varepsilon(x)}{(\int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^{2^*_s}}{|x|^s} dx)^{1/2^*_s}}.$$

We claim that

$$\max_{t>0} I(tv_\varepsilon) < \frac{2\alpha - s}{2(N-s)} S_{\alpha,s}^{\frac{N-s}{2\alpha-s}}. \tag{3.2}$$

In fact, from (1.14) it is easy to calculate the following estimates

$$\|v_\varepsilon\|^2_{\dot{H}^\alpha(\mathbb{R}^N)} = S_{\alpha,s}, \tag{3.3}$$

$$\int_{\mathbb{R}^N} (v_\varepsilon(x))^2 dx \leq c\varepsilon^{2\alpha-N} \int_{\mathbb{R}^N} \frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{N-2\alpha}} dx \leq O(\varepsilon^{2\alpha}), \quad \text{for } N > 4\alpha, \tag{3.4}$$

$$\int_{\mathbb{R}^N} (v_\varepsilon(x))^q dx = O(\varepsilon^{\frac{(2\alpha-N)q}{2}+N}). \tag{3.5}$$

Since $2^* > q > 2$, we have

$$O(\varepsilon^{2\alpha}) = o(\varepsilon^{\frac{(2\alpha-N)q}{2}+N}). \tag{3.6}$$

Denote by t_ε the attaining point of $\max_{t>0} I(tv_\varepsilon)$, similar to the proof of [6, Lemma 3.5] we can prove that t_ε is uniformly bounded. In fact, we consider the function

$$\begin{aligned} h(t) &= I(tv_\varepsilon) \\ &= \frac{t^2}{2}(\|(-\Delta)^{\alpha/2} v_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2\, dx) \\ &\quad - \frac{t^{2^*_s}}{2^*_s} \int_{\mathbb{R}^N} \frac{(v_\varepsilon(x))^{2^*_s}}{|x|^s} dx - \frac{t^q}{q} \int_{\mathbb{R}^N} (k(x)v_\varepsilon(x))^q dx \\ &\geq \frac{ct^2}{2}\|v_\varepsilon\|^2_{H^\alpha(\mathbb{R}^N)} - \frac{Ct^{2^*_s}}{2^*_s}\|v_\varepsilon\|^{2^*_s}_{H^\alpha(\mathbb{R}^N)} - \frac{Ct^q}{q}\|v_\varepsilon\|^q_{H^\alpha(\mathbb{R}^N)}. \end{aligned} \tag{3.7}$$

Since $\lim_{t\to+\infty} h(t) = -\infty$ and $h(t) > 0$ when t is closed to 0, it follows that $\max_{t>0} h(t)$ is attained for $t_\varepsilon > 0$. From the fact $\int_{\mathbb{R}^N} \frac{(v_\varepsilon(x))^{2_s^*}}{|x|^s} dx = 1$, we have

$$\begin{aligned} h'(t_\varepsilon) &= t_\varepsilon(\|(-\Delta)^{\alpha/2} v_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2\,dx) \\ &\quad - t_\varepsilon^{2_s^*-1} - t_\varepsilon^{q-1}\int_{\mathbb{R}^N} k(x)(v_\varepsilon(x))^q dx = 0. \end{aligned} \tag{3.8}$$

Since $k(x) > 0$, from (3.3) and (3.4) for ε sufficiently small, we have

$$t_\varepsilon^{2_s^*-2} \le \|(-\Delta)^\alpha v_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2\,dx < 2S_{\alpha,s}. \tag{3.9}$$

Then

$$\begin{aligned} &\|(-\Delta)^{\alpha/2} v_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2\,dx \\ &= t_\varepsilon^{2_s^*-2} + t_\varepsilon^{q-2}\int_{\mathbb{R}^N} k(x)(v_\varepsilon(x))^q dx \\ &\le t_\varepsilon^{2_s^*-2} + (2S_{\alpha,s})^{\frac{q-2}{2_s^*-2}}\int_{\mathbb{R}^N} k(x)(v_\varepsilon(x))^q dx. \end{aligned} \tag{3.10}$$

Choosing $\varepsilon > 0$ small enough, by (3.3)-(3.5), there exists a constant $\mu > 0$ such that $t_\varepsilon > \mu > 0$. Combining this with (3.9), it implies that t_ε is bounded for $\varepsilon > 0$ small enough. Then, for $\varepsilon > 0$ small,

$$\begin{aligned} \max_{t>0} I(tv_\varepsilon) &= I(t_\varepsilon v_\varepsilon) \\ &\le \max_{t>0}\Big\{\frac{t^2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} v_\varepsilon(x)|^2 dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N} \frac{(v_\varepsilon(x))^{2_s^*}}{|x|^s} dx\Big\} \\ &\quad - O(\varepsilon^{\frac{(2\alpha-N)q}{2}+N}) + O(\varepsilon^{2\alpha}), \\ &< \frac{2\alpha-s}{2(N-s)} S_{\alpha,s}^{\frac{N-s}{2\alpha-s}} \quad \text{(by (3.6))}. \end{aligned}$$

This proves (3.2). By the definition of c^*, we have $c^* < \frac{2\alpha-s}{2(N-s)} S_{\alpha,s}^{\frac{N-s}{2\alpha-s}}$.

Next we verify that

$$c^* < J^\infty. \tag{3.11}$$

Let $\{u_0\}$ be the minimizer of J^∞, $I^\infty(u_0) = J^\infty$ and

$$\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2} u_0(x)|^2 + \bar{a}|u_0(x)|^2\Big) dx = \int_{\mathbb{R}^N} \bar{k}(u_0^+(x))^q\,dx.$$

Let

$$\begin{aligned} g(t) &= I^\infty(tu_0) \\ &= \frac{1}{2}t^2\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2} u_0(x)|^2 + \bar{a}|u_0(x)|^2\Big) dx - \frac{t^q}{q}\int_{\mathbb{R}^N} \bar{k}(u_0^+(x))^q\,dx, \\ g'(t) &= t\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2} u_0(x)|^2 + \bar{a}|u_0(x)|^2\Big) dx - t^{q-1}\int_{\mathbb{R}^N} \bar{k}(u_0^+(x))^q\,dx. \end{aligned}$$

Thus $g'(t) \ge 0$ if $t \in (0,1)$; $g'(t) \le 0$ if $t \ge 1$. Then

$$g(1) = I^\infty(u_0) = \max_l I^\infty(u), \tag{3.12}$$

where $l = \{tu_0, t \ge 0\}$ for a fixed u_0.

Since there exists a $t_0 > 0$ such that $\sup_{t\geq 0} I(tu_0) = I(t_0u_0)$, from (3.12) and the assumptions on $a(x)$ and $k(x)$, we have

$$J^\infty = I^\infty(u_0) \geq I^\infty(t_0u_0) > I(t_0u_0) = \sup_{t\geq 0} I(tu_0).$$

This proves (3.11). By (3.2) and (3.11) we have (3.1). Then the proof is completed. □

4. Appendix

In this section, we give some lemmas and detailed proofs for the convenience of the reader.

Lemma 4.1 ([28, Lemma 2.1])**.** *Let $\{\rho_n\}_{n\geq 1}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geq 0 \ \text{on } \mathbb{R}^N, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} \rho_n(x)dx = \lambda > 0, \tag{4.1}$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $\{\rho_{n_k}\}$ satisfying one of the following two possibilities:

(1) *(Vanishing):*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x)\, dx = 0, \quad \text{for all } R < +\infty. \tag{4.2}$$

(ii) *(Nonvanishing): there exist $\alpha > 0$, $R < +\infty$ and $\{y_k\} \subset \mathbb{R}^N$ such that*

$$\liminf_{k\to+\infty} \int_{y_k+B_R} \rho_{n_k}(x)dx \geq \alpha > 0.$$

Lemma 4.2 ([12, Lemma 2.2])**.** *If $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^N)$ and for some $R > 0$, we have*

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_n(x)|^2\, dx \to 0 \quad \text{as } n \to \infty.$$

Then $u_n \to 0$ in $L^q(\mathbb{R}^N)$, for $2 < q < \frac{2N}{N-2\alpha}$.

Lemma 4.3. *Suppose that $0 < s < 2\alpha$ and $N > 2\alpha$. Then there exists $C > 0$ such that for any $u \in \dot{H}^\alpha(\mathbb{R}^N)$,*

$$\Big(\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^s} dx\Big)^{2/p} \leq C\|u\|^2_{\dot{H}^\alpha(\mathbb{R}^N)}, \tag{4.3}$$

i.e., $\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N, |x|^{-s})$ is continuous. In addition, the inclusion

$$\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^p_{\rm loc}(\mathbb{R}^N, |x|^{-s}),$$

is compact if $2 \leq p < 2^*_s$.

Proof. The proof of (4.3) is similar to that of [26, Lemma 3.1]. Now we prove the compact impeding if $2 \leq p < 2^*_s$. Let $\{u_n\}$ be a bounded sequence in $\dot{H}^\alpha(\mathbb{R}^N)$, then up to a subsequence (still denoted by $\{u_n\}$),

$$u_n \rightharpoonup u \text{ in } \dot{H}^\alpha(\mathbb{R}^N).$$

Denote $v_n(x) = u_n(x) - u(x)$, then for any $B(0,r)$,

$$v_n \rightharpoonup 0 \text{ in } \dot{H}^\alpha(\mathbb{R}^N), \quad v_n \to 0 \text{ in } L^q(B(0,r)), \quad 2 \leq q < 2^* = \frac{2N}{N-2\alpha}.$$

Fixing $r > 0$, since $(p - \frac{s}{\alpha})(\frac{2\alpha}{2\alpha - s}) < 2^*$, it follows that

$$
\begin{aligned}
&\int_{B(0,r)} \frac{|v_n(x)|^p}{|x|^s} dx \\
&= \int_{B(0,r)} \frac{|v_n(x)|^{s/\alpha}}{|x|^s} |v_n(x)|^{p-s/\alpha} dx \\
&\le \Big(\int_{B(0,r)} \frac{|v_n(x)|^2}{|x|^{2\alpha}} dx \Big)^{\frac{s}{2\alpha}} \Big(\int_{B(0,r)} |v_n(x)|^{(p-\frac{s}{\alpha})(\frac{2\alpha}{2\alpha-s})} dx \Big)^{\frac{2\alpha-s}{2\alpha}} \\
&\le c \|(-\Delta)^{\alpha/2} v_n(x)\|_{L^2(\mathbb{R}^N)}^{\frac{s}{2\alpha}} \Big(\int_{B(0,r)} |v_n(x)|^{(p-\frac{s}{\alpha})(\frac{2\alpha}{2\alpha-s})} dx \Big)^{\frac{2\alpha-s}{2\alpha}} \to 0.
\end{aligned} \tag{4.4}
$$

Then we have

$$
u_n \to u \text{ in } L^p_{\rm loc}(\mathbb{R}^N, |x|^{-s}),
$$

which completes the proof. □

Lemma 4.4. *Let $\{u_n\}$ be a Palais-Smale sequence of I at level $d \in \mathbb{R}$. Then $d \ge 0$ and $\{u_n\} \subset H^\alpha(\mathbb{R}^N)$ is bounded. Moreover, every Palais-Smale sequence for I at a level zero converges strongly to zero.*

Proof. Since $a(x) \ge 0$, $\bar{a} > 0$ and $\inf_{x\in\mathbb{R}^N} a(x) = \hat{a} > 0$, we have

$$
\|u_n\|^2_{\dot{H}^\alpha(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a(x)|u_n(x)|^2 dx \ge c\|u_n\|^2_{H^\alpha(\mathbb{R}^N)},
$$

and hence for $q \le 2^*_s$,

$$
\begin{aligned}
d + 1 + o(\|u_n\|) &\ge I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\
&= (\frac{1}{2} - \frac{1}{q}) \int_{\mathbb{R}^N} \big(|(-\Delta)^{\alpha/2} u_n(x)|^2 + a(x)|u_n(x)|^2\big) dx \\
&\quad + (\frac{1}{q} - \frac{1}{2^*_s}) \int_{\mathbb{R}^N} \frac{(u_n^+(x))^{2^*_s}}{|x|^s} dx \\
&\ge C\|u_n\|^2_{H^\alpha(\mathbb{R}^N)}.
\end{aligned} \tag{4.5}
$$

For $2^*_s < q < 2^*$,

$$
\begin{aligned}
d + 1 + o(\|u_n\|) &\ge I(u_n) - \frac{1}{2^*_s}\langle I'(u_n), u_n\rangle \\
&= (\frac{1}{2} - \frac{1}{2^*_s}) \int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2} u_n(x)|^2 + a(x)|u_n(x)|^2\Big) dx \\
&\quad + (\frac{1}{2^*_s} - \frac{1}{q}) \int_{\mathbb{R}^N} k(x)(u_n^+(x))^q \, dx \\
&\ge C\|u_n\|^2_{H^\alpha(\mathbb{R}^N)}.
\end{aligned} \tag{4.6}
$$

It follows from (4.5) and (4.6) that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^N)$ for $2 < q < 2^*$. Since

$$
d = \lim_{n\to\infty} I(u_n) - \max\{\frac{1}{q}, \frac{1}{2^*_s}\}\langle I'(u_n), u_n\rangle \ge C \limsup_{n\to\infty} \|u_n\|^2_{H^\alpha(\mathbb{R}^N)},
$$

we have $d \ge 0$. Suppose now that $d = 0$, we obtain from the above inequality that

$$
\lim_{n\to\infty} \|u_n\|_{H^\alpha(\mathbb{R}^N)} = 0.
$$

□

Lemma 4.5. *Let $\{u_n\}$ be a Palais-Smale sequence of I at level $d \in \mathbb{R}$ and $u_n^+ = \max\{u_n, 0\}$. Then $\{u_n^+\}$ is also a Palais-Smale sequence of I at level d.*

Proof. From the definition of I we have that as $n \to \infty$,

$$\begin{aligned} I(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2}u_n(x)|^2 + a(x)|u_n(x)|^2\Big)\,dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} \frac{(u_n^+(x))^{2_s^*}}{|x|^s}dx \\ - \frac{1}{q}\int_{\mathbb{R}^N} k(x)(u_n^+(x))^q\,dx \to d, \end{aligned}$$

and

$$\begin{aligned} &\langle I'(u_n), \phi\rangle \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} a(x)u_n(x)\phi(x)\,dx \\ &\quad - \int_{\mathbb{R}^N} \frac{(u_n^+(x))^{2_s^*-1}\phi(x)}{|x|^s}dx - \int_{\mathbb{R}^N} k(x)(u_n^+(x))^{q-1}\phi(x)dx \to 0, \end{aligned} \tag{4.7}$$

for all $\phi \in H^\alpha(\mathbb{R}^N)$.

Taking $\phi = -u_n^- = \min\{u_n, 0\}$, from

$$u_n(x) = u_n^+(x) - u_n^-(x), \quad u_n^+(x)u_n^-(x) = 0, \tag{4.8}$$

we have

$$\begin{aligned} \langle I'(u_n), -u_n^-\rangle &= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &\quad - \int_{\mathbb{R}^N} a(x)u_n(x)u_n^-(x)\,dx + \int_{\mathbb{R}^N} \frac{(u_n^+(x))^{2_s^*-1}u_n^-(x)}{|x|^s}dx \\ &\quad + \int_{\mathbb{R}^N} k(x)(u_n^+(x))^{q-1}u_n^-(x)dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n^-(x)-u_n^-(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_n^+(x)u_n^-(y)+u_n^+(y)u_n^-(x)}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N} a(x)(u_n^-(x))^2\,dx \to 0\,. \end{aligned} \tag{4.9}$$

From (4.9), $u_n^+(x) \ge 0$, $u_n^-(x) \ge 0$ and $a(x) > 0$, it follows that

$$\|u_n^-\|_{H^\alpha} \to 0, \tag{4.10}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{2(u_n^+(x)-u_n^+(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \to 0. \tag{4.11}$$

Then from (4.8) and (4.10)-(4.11), we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big((u_n^+(x)-u_n^+(y))^2+(u_n^-(x)-u_n^-(y))^2 \\ &\quad -2(u_n^+(x)-u_n^+(y))(u_n^-(x)-u_n^-(y))\Big)/|x-y|^{N+2\alpha}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n^+(x)-u_n^+(y))^2}{|x-y|^{N+2\alpha}}\,dx\,dy+o(1). \end{aligned} \tag{4.12}$$

That is

$$\|u_n\|_{\dot H^\alpha} = \|u_n^+\|_{\dot H^\alpha}+o(1). \tag{4.13}$$

Therefore

$$\lim_{n\to\infty} I(u_n^+) = \lim_{n\to\infty} I(u_n) = d, \\ I'(u_n^+,\phi) = I'(u_n,\phi)\to 0$$

as $n\to\infty$. This complete the proof. □

Lemma 4.6. *All nontrivial critical points of I_s are positive solutions of* (1.12).

Proof. Let $u\not\equiv 0$ and $u\in H^\alpha(\mathbb{R}^N)$ be a nontrivial critical point of I_s. First, arguing as in the proof of Lemma 4.5 (similar to (4.9) and (4.10)), we can obtain that $\|u^-\|_{H^\alpha}=0$ which gives that

$$u\ge 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{4.14}$$

Then for any $x_0\in\mathbb{R}^N$,

$$\begin{gathered} (-\Delta)^\alpha u = \frac{|u|^{2_s^*-2}u}{|x|^s}\ge 0, \quad \text{a.e. in } B(x_0,1), \\ \int_{\mathbb{R}^N}\frac{|u(x)|}{1+|x|^{N+2\alpha}}dx\le c\|u\|_{L^2}\le c\,. \end{gathered} \tag{4.15}$$

From [22, Proposition 2.2.6], we have u is lower semicontinuous in $B(x_0,1)$. Combining this with (4.14), it follows $u(x_0)\ge 0$. Then $u(x)\ge 0$ pointwise in $\mathbb{R}^N$.

Next we claim that $u>0$ in $\mathbb{R}^N$. Otherwise there exist $x_1\in\mathbb{R}^N$ such that $u(x_1)=0$. Then u is lower semicontinuous in $\overline{B(x_1,1/2)}$. From [22, Proposition 2.2.8], it follows $u\equiv 0$ in $\mathbb{R}^N$. This contradicts the assumption u is nontrivial. □

Let $\{u_n\}$ be a Palais-Smale sequence at level d. Up to a subsequence, we assume that

$$u_n\rightharpoonup u \quad \text{in } H^\alpha(\mathbb{R}^N) \text{ as } n\to\infty.$$

Obviously, we have $I'(u)=0$. Let $v_n(x)=u_n(x)-u(x)$, from Lemma 4.3 as $n\to\infty$,

$$v_n\rightharpoonup 0 \quad \text{in } H^\alpha(\mathbb{R}^N), \tag{4.16}$$

$$v_n\to 0 \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N,|x|^{-s}) \text{ for all } 2\le p<2_s^*, \tag{4.17}$$

$$v_n\to 0 \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N) \text{ for all } 2<q<2^*, \tag{4.18}$$

$$v_n\to 0, \quad \text{a.e. in } \mathbb{R}^N. \tag{4.19}$$

As a consequence, we have the following Lemma.

Lemma 4.7. *$\{v_n\}$ is a Palais-Smale sequence for I at level $d_0 = d - I(u)$.*

Proof. For $\phi(x) \in C_0^\infty(\mathbb{R}^N)$, there exists a $B(0,r)$ such that supp$\phi \subset B(0,r)$. Then as $n \to \infty$,

$$\Big|\int_{\mathbb{R}^N} k(x)(v_n^+(x))^{q-1}\phi(x)\,dx\Big| \le c\Big|\int_{B(0,r)} (v_n^+(x))^{q-1}\phi(x)dx\Big| = o(1), \tag{4.20}$$

and from Lemma 4.3,

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} \frac{(v_n^+(x))^{2_s^*-1}\phi(x)}{|x|^s}dx\Big| &\le \Big|\int_{|x|\le r} \frac{(v_n^+(x))^{2_s^*-1}\phi(x)}{|x|^s}dx\Big| \\ &\le c\int_{|x|\le r} \frac{(v_n^+(x))^{2_s^*-1}}{|x|^s}dx = o(1). \end{aligned} \tag{4.21}$$

By (4.16), (4.20) and (4.21), we have $\langle \phi, I'(v_n)\rangle = o(1)$ as $n \to \infty$. Then similar to (4.10), we have

$$\|v_n^-\|_{\dot H^\alpha} \to 0, \|u^-\|_{\dot H^\alpha} = 0. \tag{4.22}$$

By Sobolev inequality, (4.10) and (4.22) it follows that

$$\|u_n\|_{L^q} = \|u_n^+\|_{L^q} + o(1),\ \|v_n\|_{L^q} = \|v_n^+\|_{L^q} + o(1), \|u\|_{L^q} = \|u^+\|_{L^q}.$$

Then by the Brézis-Lieb Lemma in [3] as $n \to \infty$, we have

$$\int_{\mathbb{R}^N} (v_n^+(x))^q dx = \int_{\mathbb{R}^N} (u_n^+(x))^q dx - \int_{\mathbb{R}^N} (u^+(x))^q dx + o(1) \tag{4.23}$$

for all $2 \le q \le 2_s^*$. Similarly

$$\int_{\mathbb{R}^N} \frac{\big(v_n^+(x)\big)^{2_s^*}}{|x|^s}dx = \int_{\mathbb{R}^N} \frac{\big(u_n^+(x)\big)^{2_s^*}}{|x|^s}dx - \int_{\mathbb{R}^N} \frac{(u^+(x))^{2^*(s)}}{|x|^s}dx + o(1), \tag{4.24}$$

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(v_n(x)+u(x))-(v_n(y)+u(y))|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(|v_n(x)-v_n(y)|^2 + |u(x)-u(y)|^2 \\ &\quad + 2(v_n(x)-v_n(y))(u(x)-u(y))\Big)/|x-y|^{N+2\alpha}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy + o(1). \end{aligned} \tag{4.25}$$

Then from (4.23)-(4.25), it follows that $I(v_n) = I(u_n) - I(u) + o(1) = d - I(u) + o(1)$. □

Lemma 4.8. *Assume $t \ge b > 0$ and $q > 1$, then $t^q - (t-b)^q \ge b^q$.*

Proof. Let $f(t) = t^q - (t-b)^q$, it follows

$$f'(t) = qt^{q-1} - q(t-b)^{q-1} > 0 \quad \text{for } t \ge b > 0,\ q > 1.$$

Then $f(t) = t^q - (t-b)^q \ge f(b) = b^q$. □

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Lingyu Jin

Department of Mathematics, South China Agricultural University, Guangzhou 510642, China

E-mail address: `jinlingyu300@126.com`

Shaomei Fang

Department of Mathematics, South China Agricultural University, Guangzhou 510642, China

E-mail address: `fangshaomeidz90@126.com`